RECURRENCE AND TRANSIENCE OF RANDOM DIFFERENCE EQUATIONS IN THE CRITICAL CASE

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For i.i.d. random vectors $(M_1, Q_1), (M_2, Q_2), \ldots$ such that $M > 0$ a.s., $Q \geq 0$ a.s. and $\mathbb{P}(Q = 0) < 1$, the random difference equation $X_n = M_n X_{n-1} + Q_n$, $n = 1, 2, \ldots$, is studied in the critical case when the random walk with increments $\log M_1, \log M_2$ is oscillating. We provide conditions for the null-recurrence and transience of the Markov chain $(X_n)_{n\geq 0}$ by inter alia drawing on techniques developed in the related article [1] for another case exhibiting the null-recurrence/transience dichotomy.

1. Introduction. Let $(M_1, Q_1), (M_2, Q_2), \ldots$ be i.i.d. $\mathbb{R}_+^2$-valued random vectors with generic copy (M, Q), where $\mathbb{R}_+ := [0, \infty)$. The purpose of this article is to continue recent work [1] on the recurrence/transience properties of the Markov chain $(X_n)_{n\geq 0}$ which is recursively defined by the random difference equation (RDE)

$$X_n \;:=\; M_n X_{n-1} + Q_n, \quad n \in \mathbb{N} \tag{1}$$

and called *RDE-chain with associated random vector* (M, Q) hereafter. If $X_0 = x$, we also write X_n^x for X_n, and it is generally understood that X_0 and the (M_n, Q_n) are independent. Basic assumptions throughout this work are that

$$\mathbb{P}(M = 0) \;=\; 0, \quad \mathbb{P}(Q = 0) \;<\; 1, \tag{2}$$

and, most importantly,

$$\liminf_{n\to-\infty} \Pi_n \;=\; 0 \quad \text{and} \quad \limsup_{n\to\infty} \Pi_n \;=\; +\infty \quad \text{a.s.} \tag{3}$$

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where

$$\Pi_0 := 0 \quad \text{and} \quad \Pi_n := \prod_{k=1}^{n} M_k \quad \text{for } n \in \mathbb{N}.$$

Condition (3), which particularly holds true if

$$\mathbb{E} \log M = 0 \quad \text{and} \quad \mathbb{P}(M=1) < 1, \tag{4}$$

is often referred to as the *critical case* because it marks the interface between two quite different situations: the *contractive case* $\lim_{n\to\infty} \Pi_n = 0$ a.s. when the RDE-chain is positive recurrent under some mild additional conditions on (M,Q), see [17, Thm. 2.1], and the *divergent case* $\lim_{n\to\infty} \Pi_n = +\infty$ a.s. when the chain is typically transient. The latter can be seen from representation (23) given below.

Let us also point out that, as M,Q are nonnegative, (2) and (3) further imply the nondegeneracy condition

$$\mathbb{P}(Mc + Q = c) < 1 \text{ for all } c \in \mathbb{R}. \tag{5}$$

For a proof, notice that (3) entails $\mathbb{P}(M<1) > 0$ and $\mathbb{P}(M>1) > 0$. Therefore, $Mc + Q = c$ for some $c \in \mathbb{R}$ would lead to the impossible conclusion that either $Q = 0$ a.s., which is ruled out by (2), or

$$\mathbb{P}(Q<0) = \mathbb{P}(c(1-M)<0) = \begin{Bmatrix} \mathbb{P}(M>1), & \text{if } c>0 \\ \mathbb{P}(M<1), & \text{if } c<0 \end{Bmatrix} > 0.$$

As usual, we put $\log_+ x := \log(x \vee 1)$ and $\log_- x := -\log(x \wedge 1)$ for $x > 0$. Assuming (4) and, furthermore,

$$\mathbb{E} \log^{2+\epsilon} M < \infty \quad \text{and} \quad \mathbb{E} \log_+^{2+\epsilon} Q < \infty \tag{6}$$

for some $\epsilon > 0$, Babillot et al. [2] showed more than twenty years ago that $(X_n)_{n\ge 0}$ is null recurrent and possesses a unique (up to scalars) stationary Radon measure. Both intuitively and from their provided proof, one can expect that Condition (6) is far from being necessary. In view of the large number of publications on RDE's during the last decade, see the recent monographs by Buraczewski et al. [9] and Iksanov [18] for surveys, it appears to be surprising that the result has apparently not been improved until today. Such improvements are now provided by Theorems 1.1 and 1.3, which are our main results and stated below after some further notation and relevant information.

Put $S_0 := 0$ and

$$S_n \;:=\; \log \Pi_n \;=\; \sum_{k=1}^{n} \log M_k \quad \text{for } n \in \mathbb{N}.$$

In the critical case, $(S_n)_{n\geq 0}$ forms an ordinary *oscillating* random walk, i.e.

$$\liminf_{n\to-\infty} S_n \;=\; -\infty \quad \text{and} \quad \limsup_{n\to\infty} S_n \;=\; +\infty \quad \text{a.s.}$$

The associated strictly descending ladder epochs, defined by $\sigma_0^< := 0$ and, recursively,

$$\sigma_n^< \;:=\; \inf\left\{k > \sigma_{n-1}^< : S_k - S_{\sigma_{n-1}^<} < 0\right\}, \quad n \in \mathbb{N} \tag{7}$$

are then a.s. finite with infinite mean, i.e. $\mathbb{E}\sigma^< = \infty$ for $\sigma^< := \sigma_1^<$. Regarding the associated first ladder height $S_{\sigma^<}$, we note that $\mathbb{E}\log_-^{p+1} M < \infty$ for $p > 0$ ensures

$$\mathbb{E}|S_{\sigma^<}|^p \;<\; \infty, \tag{8}$$

see [12, p. 250]. In particular, $\mathbb{E}\log_-^2 M < \infty$ is sufficient for

$$\kappa \;:=\; \mathbb{E}|S_{\sigma^<}| \;<\; \infty. \tag{9}$$

Recall that $(S_n)_{n\geq 0}$ satisfies the *Spitzer condition* if, for some $0 \leq \rho \leq 1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(S_k < 0) \;=\; \rho. \tag{10}$$

The limit exists also when replacing $\mathbb{P}(S_k < 0)$ with $\mathbb{P}(S_k \leqslant 0)$, $\mathbb{P}(S_k > 0)$, or $\mathbb{P}(S_k \geqslant 0)$. Moreover, as shown by Doney [14] for $0 < \rho < 1$ and by Bertoin and Doney [4] for $\rho \in \{0, 1\}$, (10) always implies the stronger convergence

$$\lim_{n\to\infty} \mathbb{P}(S_n < 0) \;=\; \rho.$$

For our purposes, a more important consequence of (10) for $0 < \rho < 1$ is

$$\mathbb{E}\log \sigma^< \;<\; \infty \tag{11}$$

which follows directly from the stronger tail property

$$\mathbb{P}(\sigma^< > n) \;\sim\; \frac{\ell_\rho^<(n)}{\Gamma(1-\rho)\, n^\rho} \quad \text{as } n \to \infty, \tag{12}$$

valid in this case, see [5, Thm. 8.9.12]. Here Γ denotes the Eulerian Gamma function and $\ell_\rho^<$ a slowly varying function which may be chosen as

$$\ell_\rho^<(s) \ = \ \exp\left(\sum_{n\ge1}\frac{(1-s^{-1})^n}{n}\big(\rho-\mathbb{P}(S_n<0)\big)\right), \quad s\in(1,\infty), \tag{13}$$

and thus in fact as a constant if

$$\sum_{n\ge1}\frac{1}{n}\big(\rho-\mathbb{P}(S_n<0)\big) \tag{14}$$

is convergent.

THEOREM 1.1. *Given an RDE-chain with associated random vector (M,Q) in $\mathbb{R}_+^2$ such that (2), (3), and (10) for some $\rho\in(0,1)$ hold, suppose that*

$$\mathbb{P}(\log Q>t|M) \ \le \ \overline{F}(t) \quad a.s. \tag{15}$$

for all sufficiently large t and a survival function $\overline{F}$ satisfying

$$\lim_{t\to\infty} t\,\overline{F}(t)^\rho\,\ell_\rho^<(1/\overline{F}(t)) \ = \ 0. \tag{16}$$

Then the chain is null-recurrent and possesses an essentially unique invariant Radon measure.

REMARK 1.2. For a comparison of this result with the one by Babillot et al. mentioned earlier, we need to compare their condition (6), which entails $\rho=\frac{1}{2}$ because $(S_n)_{n\ge0}$ satisfies the central limit theorem, with our condition (16) for this ρ. The comparison becomes easier when noting that

$$\mathbb{E}\big(\log_+^{2+\varepsilon}Q|M\big) \ \le \ K \quad \text{a.s.} \tag{17}$$

for some constants $\varepsilon,K>0$ provides a sufficient condition for (16). Then one can see that, in essence, our result does not need an extra moment condition on $\log M$ while their result makes no assumption on the joint law of (M,Q).

THEOREM 1.3. *Given an RDE-chain with associated random vector (M,Q) in $\mathbb{R}_+^2$ satisfying (2), (3), (9), and (10) for some $\rho\in(0,1)$, the following assertions hold true:*

(a) If Q satisfies (15) for all sufficiently large t and a survival function $\overline{F}$ such that

$$s^*(\overline{F}) \ := \ \limsup_{t\to\infty} t\,\overline{F}(t)^\rho\,\ell_\rho^<(1/\overline{F}(t)) \ < \ \kappa, \tag{18}$$

then the chain is null-recurrent and possesses an essentially unique invariant Radon measure.

(b) If Q satisfies

$$\mathbb{P}(\log Q > t|M) \;\geq\; \overline{G}(t) \quad a.s. \tag{19}$$

for all sufficiently large t and a survival function $\overline{G}$ such that

$$s_*(\overline{G}) \;:=\; \liminf_{t\to\infty} t\,\overline{G}(t)^\rho\,\ell_\rho^<(1/\overline{G}(t)) \;>\; \kappa, \tag{20}$$

then the chain is transient.

(c) If Q satisfies both (15) and (19) for all sufficiently large t and survival functions $\overline{F},\overline{G}$ such that $0 < s_*(\overline{G}) \leq s^*(\overline{F}) < \infty$, then there exists a critical exponent

$$p_0 \;\in\; \left[\frac{\kappa}{s^*(\overline{F})}, \frac{\kappa}{s_*(\overline{G})}\right]$$

such that an RDE-chain with associated random vector (M, Q^p) is recurrent for $0 \leq p < p_0$ and transient for $p_0 < p < \infty$.

For later use, we note that the survival functions $\overline{F}$ and $\overline{G}$ above may easily be modified in such a way that (15) and (19) remain valid while (18) and (20) hold in the stronger form

$$s(\overline{F}) \;:=\; \lim_{t\to\infty} t\,\overline{F}(t)^\rho\,\ell_\rho^<(1/\overline{F}(t)) \;<\; \kappa \quad \text{and} \quad s(\overline{G}) \;>\; \kappa,$$

respectively. The latter entails that $\overline{F}(t)$ and $\overline{G}(t)$ are in fact regularly varying at ∞ with index $-1/\rho \in (-\infty, -1)$ (see [5, Prop. 1.5.15]), and we may also assume that they are smooth, convex and with monotone derivatives for sufficiently large t. In particular, $\overline{F}'(t)$ and $\overline{G}'(t)$ are negative, increasing, concave and regularly varying with index $-(1+\rho)/\rho \in (-\infty, -2)$ for large t, see [5, Thms. 1.8.2 and 1.6.3]).

It is also clear that the *conditional* tail conditions (15) and (19) turn into ordinary unconditional ones if M and Q are independent. As for Theorem 1.3, it is worthwile to give an explicit formulation of the result in this case including an improvement when $t\,\mathbb{P}(\log Q > t)^\rho \ell_\rho^<(1/\mathbb{P}(\log Q > t))$ converges as $t \to \infty$.

COROLLARY 1.4. *Given the situation of Theorem 1.3, suppose further that M, Q are independent and that $\overline{F}(t) := \mathbb{P}(\log Q > t)$, the survival function of $\log Q$, satisfies $s_*(\overline{F}) = s^*(\overline{F}) =: s(\overline{F}) \in [0, \infty]$. Then the critical exponent p_0 equals $\kappa/s(\overline{F})$, in other words, an RDE-chain with associated random vector (M, Q^p) is recurrent for $0 \leq p < \kappa/s(\overline{F})$ and transient for $\kappa/s(\overline{F}) < p < \infty$. In particular, an RDE-chain with associated (M, Q) is recurrent if $s(\overline{F}) < \kappa$ and transient if $s(\overline{F}) > \kappa$.*

Besides the basic assumption $\Pi_n \to 0$ a.s. (negative divergence of $(S_n)_{n\geq 0}$), the conditions provided by Goldie and Maller [17, (2.1) of Thm. 2.1] for the positive recurrence and in [1, Thms. 3.1 and 3.2] for the null-recurrence or transience of an RDE-chain with associated random vector (M,Q) do only involve the unconditional distributions of M and Q. It is therefore natural to ask whether substitutes of that kind for (15) and (19) may also be given here. Our last two theorems provide answers that are rather pointing in another direction. In essence, the first one provides null-recurrence under a strong condition on the relation between M and Q but no tail condition beyond, while the second result shows that both transience and null-recurrence may occur when the laws of M and Q are fixed (here to be equal) but the dependence between them varies.

THEOREM 1.5. *An RDE-chain with associated random vector (M,Q) in $\mathbb{R}_+^2$ satisfying (2), (3), (11), and*

$$Q \leq aM + b \quad \text{for some } a,b>0 \tag{21}$$

is null-recurrent and possesses an essentially unique invariant Radon measure.

THEOREM 1.6. *Let $(X_n)_{n\geq 0}$ be an RDE-chain with associated random vector (M,Q) in $\mathbb{R}_+^2$ satisfying (2),(4), and $Q \overset{d}{=} M$, where $\overset{d}{=}$ means equality in law. Suppose also that $\mathbb{E}\log_-^2 M < \infty$ and that the function $L(t) := t^{1/\rho}\,\mathbb{P}(\log M > t)$ is slowly varying for some $\rho \in (\frac{1}{2},1)$ with*

$$\lim_{t\to\infty} L(t) \;=\; \infty.$$

Then the chain is null-recurrent if $Q = M$, but it is transient if M and Q are independent.

The proofs of these results are presented in Section 6. They combine techniques from [1] and [2], as for the latter, the most notable being the use of an embedded contractive RDE-chain obtained by observing the original one at the descending ladder epochs $\sigma_n^<$ of $(S_n)_{n\geq 0}$, see Sections 3–5.

2. Theoretical background and prerequisites. Defining the random linear functions $\Psi_n(x) := Q_n + M_n x$ for $n \in \mathbb{N}$, the RDE-chain $(X_n)_{n\geq 0}$ defined by (1) may also be viewed as the *forward iterated function system*

$$X_n \;=\; \Psi_n(X_{n-1}) \;=\; \Psi_n \circ \ldots \circ \Psi_1(X_0), \quad n \in \mathbb{N},$$

where $\circ$ denotes as usual composition of maps, and opposed to its closely related counterpart of *backward iterations*

$$\widehat{X}_0 \ := \ X_0 \quad \text{and} \quad \widehat{X}_n \ := \ \Psi_1 \circ \ldots \circ \Psi_n(X_0), \quad n \in \mathbb{N}.$$

The relation is established by the obvious fact that X_n has the same law as $\widehat{X}_n$ for each n, regardless of the law of X_0. Moreover, $\Psi_1 \cdots \Psi_n$ is used as shorthand for $\Psi_1 \circ \ldots \circ \Psi_n$ hereafter.

Put $\mathbb{R}_* := \mathbb{R}\backslash\{0\}$. Since the set of affine transformations $x \mapsto ax + b$, $(a, b) \in \mathbb{R}_* \times \mathbb{R}$, endowed with $\circ$ as composition law forms a non-Abelian group, which is in fact isomorphic to the group $(\mathbb{G}, \cdot) = (\mathbb{R}_* \times \mathbb{R}, \cdot)$ upon defining

$$(a_1, b_1) \cdot (a_2, b_2) \ := \ (a_1a_2, a_1b_2 + b_1)$$

for all $(a_1, b_1), (a_2, b_2) \in \mathbb{G}$, we see that $(X_n)_{n\geq 0}$ may also be interpreted as a (left) multiplicative random walk on $\mathbb{G}$.

Yet another sequence associated with $(X_n)_{n\geq 0}$ and called its *dual* hereafter is defined by ${}^{\#}X_0 := X_0$ and

$$(22) \qquad {}^{\#}X_n \ := \ \frac{1}{M_n}{}^{\#}X_{n-1} \ + \ \frac{Q_n}{M_n}.$$

for $n \in \mathbb{N}$. Plainly, $({}^{\#}X_n)_{n\geq 0}$ is an RDE-chain with associated $(M^{-1}, M^{-1}Q)$ and properly defined on $\mathbb{R}$ whenever $\mathbb{P}(M = 0) = 0$ which is guaranteed by Condition (2). The associated backward iterations ${}^{\#}\widehat{X}_n := {}^{\#}\Psi_1 \cdots {}^{\#}\Psi_n(X_0)$ for $n \in \mathbb{N}$, where ${}^{\#}\Psi(x) := M^{-1}x + M^{-1}Q$, are given by

$${}^{\#}X_n \ = \ \Pi_n^{-1}X_0 + \sum_{k=1}^{n}\Pi_k^{-1}Q_k \ = \ e^{-S_n}X_0 + \sum_{k=1}^{n}e^{-S_k}Q_k,$$

so that in particular ${}^{\#}\widehat{X}_n^0 = \sum_{k=1}^{n}\Pi_k^{-1}Q_k$. We then have the obvious relation

$$(23) \qquad X_n \ = \ \Pi_n(X_0 \ + {}^{\#}\widehat{X}_n^0)$$

which will be used below to provide a very simple argument for local contractivity of $(X_n)_{n\geq 0}$.

Recall that $(X_n)_{n\geq 0}$ is called *locally contractive* if, for any compact set K and all $x, y \in \mathbb{R}$,

$$(24) \qquad \lim_{n\to\infty}\left|X_n^x - X_n^y\right| \cdot \mathbf{1}_{\{X_n^x \in K\}} \ = \ 0 \quad \text{a.s.}$$

For critical RDE-chains with associated general $\mathbb{R}^2$-valued (M, Q), the notion was introduced by Babillot et al. [2, p. 479] and called *global stability at* *finite distance*. Later, Benda, in his PhD thesis [3], used it more systematically in the framework of general stochastic dynamical systems, see also the recent article by Peigné and Woess [19] for further information. Regarding RDE-chains, the notion plays an important role also in [6, 7, 8, 1].

The subsequent three results summarize the main properties of locally contractive Markov chains and have also been stated (and partially proved) in [1]. The first one is actually quoted from [19, Lemma 2.2] and states that a locally contractive chain is either transient or visits a large interval infinitely often (i.o.).

LEMMA 2.1. *If $(X_n)_{n\geq 0}$ is locally contractive, then the following dichotomy holds: either*

$$\mathbb{P}\left(\lim_{n\to\infty} |X_n^x - x| = \infty\right) \;=\; 0 \quad \textit{for all } x \in \mathbb{R} \tag{25}$$

or

$$\mathbb{P}\left(\lim_{n\to\infty} |X_n^x - x| = \infty\right) \;=\; 1 \quad \textit{for all } x \in \mathbb{R}. \tag{26}$$

The chain is called *recurrent* if there exists a nonempty closed set $L \subset \mathbb{R}$ such that $\mathbb{P}(X_n^x \in U \text{ i.o.}) = 1$ for every $x \in L$ and every open set U that intersects L. A proof of the next lemma can be found in [3, Thm. 5.8] and [19, Prop. 2.7 and Thm. 2.13], see also [2, Thm. 3.3].

LEMMA 2.2. *If $(X_n)_{n\geq 0}$ is locally contractive and recurrent, it possesses a unique (up to a multiplicative constant) invariant Radon measure ν.*

In view of this result, $(X_n)_{n\geq 0}$ is called *positive recurrent* if $\nu(L) < \infty$ and *null-recurrent*, otherwise. Equivalent conditions for the transience and recurrence of $(X_n)_{n\geq 0}$ are listed in the next proposition which may easily be proved with the help of Lemma 2.1 and Lemma 2.3 in [1].

PROPOSITION 2.3. *A locally contractive Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{R}$ is transient iff it satisfies one of the following equivalent assertions:*

(a) $\lim_{n\to\infty} |X_n^x| = \infty$ *a.s. for all $x \in \mathbb{R}$.*
(b) $\mathbb{P}(X_n^x \in U \text{ i.o.}) < 1$ *for any bounded open $U \subset \mathbb{R}$ and some/all $x \in \mathbb{R}$.*
(c) $\sum_{n\geq 0} \mathbb{P}(X_n^x \in K) < \infty$ *for any compact $K \subset \mathbb{R}$ and some/all $x \in \mathbb{R}$.*

On the other hand, each of the following is equivalent to the recurrence of the chain:

(a) $\liminf_{n\to\infty} |X_n^x - x| < \infty$ *a.s. for all $x \in \mathbb{R}$.*

(b) $\liminf_{n\to\infty}|X_n|<\infty$ *a.s.*

(c) $\sum_{n\ge 0}\mathbb{P}\{X_n^x\in K\}=\infty$ *for a nonempty compact set K and some/all $x\in\mathbb{R}$.*

It was shown in [2, Thm. 3.1] that any RDE-chain associated with an $\mathbb{R}^2$-valued random vector (M,Q) satisfying (4) and (6) is locally contractive. Their proof hinges on a number of nontrivial potential-theoretic arguments, but simplifies considerably if M and Q are nonnegative as also mentioned by them, see [2, Rem. 1 on p. 486]. In fact, under this restriction, the result is easily extended to any critical RDE-chain satisfying our basic assumptions.

Proposition 2.4. *A critical RDE-chain $(X_n)_{n\ge 0}$ with associated random vector (M,Q) in $\mathbb{R}_+^2$ satisfying (2) and (3) is locally contractive.*

Proof. Let $(X_n^x)_{n\ge 0}$ be defined by (1) with $X_0=x\ge 0$ and K an arbitrary compact subset of $\mathbb{R}_+$. Denote by τ_n, $n\in\mathbb{N}$, the successive epochs when the chain visits K, with the usual convention that $\tau_n:=\infty$ if the number of visits is less than n. We must verify (24) for the given K only on $E:=\{\tau_n<\infty \text{ for all } n\in\mathbb{N}\}$ because it trivially holds on the complement of this event. Use (23) and the boundedness of K to infer that

$$\sup_{n\ge 1}\Pi_{\tau_n}(x+{}^{\#}\widehat{X}_{\tau_n}^0)\ =\ \sup_{n\ge 1}X_{\tau_n}^x\ <\ \infty\quad\text{on } E. \tag{27}$$

Since $({}^{\#}X_n)_{n\ge 0}$ is also a critical RDE-chain satisfying (2) and (3) and hence *not* positive recurrent by the Goldie-Maller theorem [17, Thm. 2.1], it follows that ${}^{\#}\widehat{X}_n^0\uparrow\infty$ a.s. But in combination with (27), this further entails $\Pi_{\tau_n}\to 0$ a.s. on E and thereupon

$$X_{\tau_n}^x-X_{\tau_n}^y\ =\ \Pi_{\tau_n}(x-y)\ \xrightarrow{n\to\infty}\ 0\quad\text{a.s. on } E$$

for all $x,y\ge 0$ as required. □

3. The embedded ladder RDE-chain. Recalling from (7) the definition of the ladder epochs $\sigma_n^<$, put $X_0^<:=X_0$ and

$$X_n^<\ :=\ X_{\sigma_n^<}\ =\ \Psi_n^<\cdots\Psi_1^<(X_0)$$

for $n\in\mathbb{N}$, where

$$\Psi_n^<(x)\ :=\ \Psi_{\sigma_n^<}\cdots\Psi_{\sigma_{n-1}^<+1}(x)\ =\ M_n^<x\,+\,Q_n^<, \tag{28}$$

and

$$(M_n^<, Q_n^<) := \frac{\Pi_{\sigma_n^<}}{\Pi_{\sigma_{n-1}^<}} \cdot \left(1, \sum_{k=\sigma_{n-1}^<+1}^{\sigma_n^<} \frac{\Pi_{\sigma_{n-1}^<}}{\Pi_k} Q_k\right) \tag{29}$$

The $(\Psi_n^<, M_n^<, Q_n^<)$ being again i.i.d., we infer that $(X_n^<)_{n\ge 0}$ is again a RDE-chain, with associated nonnegative random vector $(M^<, Q^<) = (M_1^<, Q_1^<)$, i.e.

$$(M^<, Q^<) := \Pi_{\sigma^<} \cdot \left(1, \sum_{k=1}^{\sigma^<} \Pi_k^{-1} Q_k\right) = e^{S_{\sigma^<}} \left(1, \sum_{k=1}^{\sigma^<} e^{-S_k} Q_k\right). \tag{30}$$

It is called *embedded ladder RDE-chain* hereafter. Since $M^< < 1$ by definition of $\sigma^<$, it is trivially strongly contractive, and under Condition (6), it further satisfies

$$\mathbb{E} \log_+ Q^< < \infty \tag{31}$$

as was shown by Elie [15, Lemma 5.49]. This implies the positive recurrence of the chain and the existence of a unique stationary distribution, a fact that formed an essential ingredient in [2]. A somewhat different approach is used here, which embarks on the strong contractivity of the ladder RDE-chain, combines it with appropriate tail estimates for $Q^<$ instead of (31) and then draws on results recently obtained in [1]. It is furnished by the subsequent lemma.

LEMMA 3.1. *Given a critical RDE-generated Markov chain $(X_n)_{n\ge 0}$ with associated random vector (M, Q) in $\mathbb{R}_+^2$ satisfying (2) and (3) and embedded ladder RDE-chain $(X_n^<)_{n\ge 0}$, the following equivalence holds true:*

$$(X_n)_{n\ge 0} \text{ recurrent} \iff (X_n^<)_{n\ge 0} \text{ recurrent.}$$

PROOF. We must only show that the transience of $(X_n^<)_{n\ge 0}$ implies the transience of $(X_n)_{n\ge 0}$. Observing that

$$\Psi_{\sigma_n^<+k} \cdots \Psi_{\sigma_n^<+1}(x) \ge \frac{\Pi_{\sigma_n^<+k}}{\Pi_{\sigma_n^<}} x \ge x$$

for all $x \in \mathbb{R}_+$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $1 \le k < \sigma_{n+1}^< - \sigma_n^<$, this follows from

$$X_{\sigma_n^<+k} = \Psi_{\sigma_n^<+k} \cdots \Psi_{\sigma_n^<+1}(X_n^<) \ge X_n^<$$

in combination with $\lim_{n\to\infty} X_n^< = \infty$ a.s. □

All previous considerations including the lemma remain true when replacing the $\sigma_n^<$ by the *level* $\log\gamma$ *ladder epochs* $\sigma_n^<(\gamma)$ for an arbitrary $\gamma \in (0,1)$, defined by $\sigma_0^<(\gamma) := 0$ and, recursively,

$$\sigma_n^<(\gamma) \ := \ \inf\{k > \sigma_{n-1}^<(\gamma) : S_k - S_{\sigma_{n-1}^<(\gamma)} < \log\gamma\}, \quad n \in \mathbb{N}.$$

The sequence $(X_n^<(\gamma))_{n\ge 0} := (X_{\sigma_n^<(\gamma)})_{n\ge 0}$, which then replaces $(X_n^<)_{n\ge 0}$, is a RDE-chain with associated random vector

$$(M^<(\gamma), Q^<(\gamma)) \ := \ \Pi_{\sigma^<(\gamma)} \cdot \left(1\,, \sum_{k=1}^{\sigma^<(\gamma)} \Pi_k^{-1} Q_k\right), \tag{32}$$

where $\sigma^<(\gamma) := \sigma_1^<(\gamma)$. Naturally, the $(M_n^<(\gamma), Q_n^<(\gamma))$ are defined accordingly, that is (compare (29))

$$(M_n^<(\gamma), Q_n^<(\gamma)) \ := \ \frac{\Pi_{\sigma_n^<(\gamma)}}{\Pi_{\sigma_{n-1}^<(\gamma)}} \cdot \left(1, \sum_{k=\sigma_{n-1}^<(\gamma)+1}^{\sigma_n^<(\gamma)} \frac{\Pi_{\sigma_{n-1}^<(\gamma)}}{\Pi_k} Q_k\right) \tag{33}$$

for all $n \in \mathbb{N}$. Note that $M^<(\gamma) = e^{S^<(\gamma)} < \gamma$.

4. A threshold result. The subsequent proposition, needed particularly for the proof of Theorem 1.3, shows that, as intuitively predictable, the family of RDE-chains $(X_{p,n})_{n\ge 0}$ defined below for $p \ge 0$ exhibits a phase transition from recurrence to transience at a critical value p_0 which, however, may be zero or infinite.

PROPOSITION 4.1. *Given a sequence of i.i.d. random vectors $(M_n, Q_n)_{n\ge 1}$ in $\mathbb{R}_+^2$ with generic copy (M,Q) satisfying (2), (3) and (11), let $(X_{p,n})_{n\ge 0}$ for $p > 0$ denote the RDE-chain defined by $X_{p,0} := 0$ and*

$$X_{p,n} \ := \ M_n X_{p,n-1} \,+\, Q_n^p, \quad \textit{for } n \in \mathbb{N}.$$

Then there exists $p_0 \in [0,\infty]$ such that $(X_{p,n})_{n\ge 0}$ is transient for $p > p_0$ (thus never if $p_0 = \infty$) and recurrent for $p < p_0$ (thus never if $p_0 = 0$).

The proof is based on the subsequent lemma.

LEMMA 4.2. *In the situation of Proposition 4.1, let $(M_n^<(\gamma), Q_n^<(\gamma))$ for any fixed $\gamma \in (0,1)$ be given by (33). Further define $X_0^* = Y_{p,0} := 0$ and*

$$X_n^* \ := \ M_n X_{n-1}^* \,+\, 1,$$

$$Y_{p,n} \;:=\; M_n Y_{p,n-1} \,+\, (Q_n \vee 1)^p$$

for $n \in \mathbb{N}$. *Then*

$$X_n^* \vee X_{p,n} \;\le\; Y_{p,n} \;\le\; X_n^* + X_{p,n}, \tag{34}$$

$$0 \;\le\; Y_{p,\sigma_n^<(\gamma)} - X_{p,\sigma_n^<(\gamma)} \;\le\; \frac{1}{1-\gamma} \tag{35}$$

for each $n \in \mathbb{N}_0$, *and the recurrence of* $(X_{p,n})_{n\ge 0}$ *and* $(Y_{p,n})_{n\ge 0}$ *are equivalent.*

Proof. Since (34) follows by a straightforward induction, we turn directly to (35) and prove inductively that

$$0 \;\le\; Y_{p,\sigma_n^<(\gamma)} - X_{p,\sigma_n^<(\gamma)} \;\le\; \sum_{k=0}^{n-1} \gamma^k$$

for all $n \in \mathbb{N}$. For $n = 1$, this follows from

$$Y_{p,\sigma^<(\gamma)} - X_{p,\sigma^<(\gamma)} \;=\; (Q^<(\gamma) \vee 1)^p - Q^<(\gamma)^p \;=\; (1 - Q^<(\gamma)^p)^+ \;\in [0,1].$$

Assuming it be true for arbitrary n, we obtain

$$\begin{aligned} 0 \;&\le\; Y_{p,\sigma_{n+1}^<(\gamma)} - X_{p,\sigma_{n+1}^<(\gamma)} \\ &=\; M_n^<(\gamma)\big(Y_{p,\sigma_n^<(\gamma)} - X_{p,\sigma_n^<(\gamma)}\big) \,+\, (1 - Q_{n+1}^<(\gamma)^p)^+ \;\le\; \gamma \sum_{k=0}^{n-1} \gamma^k \,+\, 1 \end{aligned}$$

and thus the desired result.

It remains to prove the final equivalence statement. By (35), the recurrence of the two level γ ladder RDE-chain $(X_{p,\sigma_n^<(\gamma)})_{n\ge 0}$ and $(Y_{p,\sigma_n^<(\gamma)})_{n\ge 0}$ are obviously equivalent. Hence we arrive at the desired conclusion because, by Lemma 3.1, the joint recurrence of $(X_{p,n})_{n\ge 0}$ and $(Y_{p,n})_{n\ge 0}$ is equivalent to the joint recurrence of their aforementioned respective ladder RDE-chains. □

Proof of Proposition 4.1. For the $Y_{p,n}$, $(p,n) \in (0,\infty) \times \mathbb{N}_0$, considered in the previous lemma, we obviously have $Y_{p,n} \le Y_{q,n}$ whenever $p < q$. Consequently, if $(Y_{q,n})_{n\ge 0}$ is recurrent, then the same holds true for $(Y_{p,n})_{n\ge 0}$. The set

$$\{p > 0 : (Y_{p,n})_{n\ge 0} \text{ recurrent}\}$$

must therefore be an interval which may be empty. But the previous lemma further ensures that this set remains the same when replacing $(Y_{p,n})_{n\ge 0}$ with $(X_{p,n})_{n\ge 0}$. □

As one can readily check, Proposition 4.1 remains valid if the criticality condition (3) is replaced with $\lim_{n\to\infty}\Pi_n = 0$ a.s. and (5). Then it covers also the positive recurrent case when

$$\lim_{n\to\infty}\Pi_n \;=\; 0 \;\text{ a.s. and }\; I_Q \;<\; \infty \tag{36}$$

hold true, see [17, Thm. 2.1], and the *divergent contractive case*, thus called and studied in [1], when

$$\lim_{n\to\infty}\Pi_n \;=\; 0 \;\text{ a.s. and }\; I_Q \;=\; \infty. \tag{37}$$

Here $I_Q := \mathbb{E}J_-(\log_+ Q)$ with $J_-(x) := x/\mathbb{E}(x\wedge\log_- M)$ for $x > 0$ and $J_-(0) := 0$. Having stated this, the next two propositions are easily obtained by combining Proposition 4.1 with [17, Thm. 2.1] and the main results in [1], respectively. They should be viewed as the counterparts of Theorem 1.3(c) for these cases.

PROPOSITION 4.3. *Given a sequence of i.i.d. random vectors $(M_n,Q_n)_{n\ge1}$ in $\mathbb{R}_+^2$ with generic copy (M,Q) satisfying (2), (5) and (36), the sequence $(X_{p,n})_{n\ge0}$ is positive recurrent for all $p\ge0$, thus $p_0=\infty$.*

PROOF. The result is immediate by [17, Thm. 2.1] when observing that $I_Q<\infty$ is equivalent to $I_{Q^p}<\infty$ for all $p>0$. □

For the corresponding result in the divergent contractive case, we define

$$r_*(\overline{F}) \;:=\; \liminf_{t\to\infty} t\,\overline{F}(t) \quad\text{and}\quad r^*(\overline{F}) \;:=\; \limsup_{t\to\infty} t\,\overline{F}(t)$$

for any survival function $\overline{F}$.

PROPOSITION 4.4. *Given a sequence of i.i.d. random vectors $(M_n,Q_n)_{n\ge1}$ in $\mathbb{R}_+^2$ with generic copy (M,Q) satisfying (2), (5), (37), and*

$$r^*(\overline{F}) \;<\; \infty$$

for $\overline{F}(t) := \mathbb{P}(\log Q > t)$, the following assertions hold true:

(a) If $\mathfrak{m} := \mathbb{E}\log M \in (-\infty,0)$, then there exists a critical exponent

$$p_0 \;\in\; \left[\frac{|\mathfrak{m}|}{r^*(\overline{F})},\frac{|\mathfrak{m}|}{r_*(\overline{F})}\right]$$

such that $(X_{p,n})_{n\ge0}$ is null-recurrent for all $p<p_0$ and transient for $p>p_0$.

(b) If $\mathfrak{m} = -\infty$ or does not exist, then $(X_{p,n})_{n\geq 0}$ is null-recurrent for all $p \geq 0$.

PROOF. Noting that $\overline{F}_p(t) := \mathbb{P}(\log Q^p > t) = \overline{F}(t/p)$ for all $t \in \mathbb{R}$, we see that $r_*(\overline{F}_p) = pr_*(\overline{F})$ and $r^*(\overline{F}_p) = pr^*(\overline{F})$.

(a) Suppose that $\mathfrak{m} \in (-\infty, 0)$. By [1, Thm. 3.1], we then infer the null-recurrence of $(X_{p,n})_{n\geq 0}$ if $pr^*(\overline{F}) < |\mathfrak{m}|$, and the transience if $pr_*(\overline{F}) > |\mathfrak{m}|$. The assertion about p_0 follows.

(b) If $\mathfrak{m} = -\infty$ or does not exist, then we obtain the null-recurrence for all p, in the first case by another appeal to [1, Thm. 3.1] and in the second case by [1, Thm. 3.2]. □

5. A tail lemma. In order to prove our results by a look at the embedded ladder RDE-chain, we need information on the tail behavior of

$$\log Q^< \;=\; \log\left(\sum_{k=1}^{\sigma^<} e^{S_{\sigma^<}-S_k} Q_k\right)$$

which satisfies the two inequalities

$$\log Q^< \;\leq\; \log \sigma^< \,+\, \max_{1\leq k\leq \sigma^<} \log Q_k, \tag{38}$$

and

$$\log Q^< \;\geq\; \max_{1\leq k\leq \sigma^<} \big((\log Q_k) + (S_{\sigma^<} - S_k)\big), \tag{39}$$

as one can readily see.

LEMMA 5.1. *Given an RDE-chain with associated random vector (M, Q) in $\mathbb{R}_+^2$ satisfying (2), (3), and (10) for some $\rho \in (0,1)$, the following assertions hold:*

(a) Condition (15) for all sufficiently large t and a survival function $\overline{F}$ entails

$$\limsup_{t\to\infty} t\,\mathbb{P}\,(Q^< > t) \;\leq\; s^*(\overline{F}) \;\in\; [0,\infty].$$

(b) If (9) is additionally assumed, then Condition (19) for all sufficiently large t and a survival function $\overline{G}$ entails

$$\limsup_{t\to\infty} t\,\mathbb{P}\,(Q^< > t) \;\geq\; s_*(\overline{G}) \;\in\; [0,\infty].$$

Here $s^*(\overline{F})$ and $s_*(\overline{G})$ are as in (18) and (20), respectively.

PROOF. (a) By (38), we have for any $\varepsilon \in (0,1)$ and $t > 0$,

$$\mathbb{P}(\log Q^< > t) \;\le\; \mathbb{P}(\log \sigma^< > \varepsilon t) \,+\, \mathbb{P}\left(\max_{1\le k\le \sigma^<} \log Q_k > (1-\varepsilon)t\right).$$

Since $\mathbb{E}\log\sigma^< < \infty$ (by (11)) entails $\mathbb{P}(\log\sigma^< > \varepsilon t) = o(t)$ as $t\to\infty$, it suffices to show that

$$\limsup_{t\to\infty} t\,\mathbb{P}\left(\max_{1\le k\le\sigma^<} \log Q_k > t\right) \;\le\; s^*(\overline{F}),$$

which in turn follows from

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\le k\le\sigma^<} \log Q_k > t\right)\\
&\quad= \sum_{n\ge1}\int_{\{\sigma^<=n\}} \mathbb{P}\left(\max_{1\le k\le n} \log Q_k > t \,\middle|\, M_1,\ldots,M_n\right)\, d\mathbb{P}\\
&\quad= \sum_{n\ge1}\int_{\{\sigma^<=n\}} 1-\prod_{k=1}^{n}\mathbb{P}(\log Q_k \le t|M_k)\; d\mathbb{P}\\
&\quad\le \sum_{n\ge1}\int_{\{\sigma^<=n\}} 1-(1-\overline{F}(t))^n\; d\mathbb{P}\\
&\quad= 1\,-\,\mathbb{E}(1-\overline{F}(t))^{\sigma^<}\\
&\quad\sim \overline{F}(t)^{\rho}\,\ell_\rho^<(1/\overline{F}(t)) \quad \text{as } t\to\infty,
\end{aligned}$$

where we have used (15) for the fourth line and [5, Cor. 8.1.7 on p. 334] for the last one.

(b) Without loss of generality $\overline{G}(t)$ can be assumed to be regularly varying of index $-1/\rho$ at infinity and, for sufficiently large t, also smooth and convex with negative and concave derivative $\overline{G}'(t)$ which is regularly varying of index $-(1+\rho)/\rho$. To see this, observe that (20) provides $\liminf_{t\to\infty} t h_\rho(\overline{G}(t)) > \kappa'$ for some $\kappa' > \kappa$, where $h_\rho(t) = t^\rho \ell_\rho(1/t)$ is regularly varying of index ρ and thus ultimately increasing with regularly varying inverse h_ρ^{-1} of index $1/\rho$. Hence $\overline{G}(t) > h_\rho^{-1}(\kappa'/t)$ for all sufficiently large t which in turn ensures that (19) and (20) remain valid for $\overline{G}(t) := h_\rho^{-1}(\kappa'/t)$. But this function has the asserted form, including the additional smoothness properties when referring to [5, Prop. 1.8.1]. With $\overline{G}$ thus chosen, we infer

$$\mathbb{P}\left(\max_{1\le k\le\sigma^<} \big((\log Q_k) + (S_{\sigma^<} - S_k)\big) > t\right)$$

$$
\begin{aligned}
&= \sum_{n\ge 1}\int_{\{\sigma^<=n\}} \mathbb{P}\left(\max_{1\le k\le n}\big((\log Q_k)+(S_n-S_k)\big)>t\,\middle|\,M_1,\ldots,M_n\right)\,d\mathbb{P}\\
&\ge \sum_{n\ge 1}\int_{\{\sigma^<=n\}} 1-\prod_{k=1}^{n}(1-\overline{G}(t-S_n+S_k))\,d\mathbb{P}\\
&= 1-\mathbb{E}\exp\left(\sum_{k=1}^{\sigma^<}\log(1-\overline{G}(\zeta_k(t))\right),\quad \zeta_k(t)\,:=\,t-S_{\sigma^<}+S_k\\
&\ge 1-\mathbb{E}\exp\left(-\sum_{k=1}^{\sigma^<}\overline{G}(\zeta_k(t))\right)\\
&= 1-\mathbb{E}\exp\left(-\overline{G}(t)\,\sigma^<\left[1-\frac{1}{\sigma^<}\sum_{k=1}^{\sigma^<}\left(1-\frac{\overline{G}(\zeta_k(t))}{\overline{G}(t)}\right)\right]\right)
\end{aligned}
$$

With the additional properties of $\overline{G}$ and $\overline{G}'(t)$, we further obtain for all sufficiently large t that

$$
\begin{aligned}
\frac{1}{\sigma^<}\sum_{k=1}^{\sigma^<}\left(1-\frac{\overline{G}(\zeta_k(t))}{\overline{G}(t)}\right) &\le \frac{2}{\sigma^<}\sum_{k=1}^{\sigma^<}\left(1-\left(\frac{t}{\zeta_k(t)}\right)^{1/\rho}\right)\\
&= \frac{2}{\sigma^<}\sum_{k=1}^{\sigma^<}\left(\frac{\zeta_k(t)^{1/\rho}-t^{1/\rho}}{t^{1/\rho}\,\zeta_k(t)^{1/\rho}}\right) \le \frac{2}{\sigma^<}\sum_{k=1}^{\sigma^<}\left(\frac{\zeta_k(t)-t}{\rho\,t^{1/\rho}\,\zeta_k(t)}\right)\\
&\le \frac{2}{\rho\,t^{1/\rho}} =:\ \varepsilon(t)
\end{aligned}
$$

and then, by another appeal to [5, Cor. 8.1.7 on p. 334],

$$
\begin{aligned}
&\mathbb{P}\left(\max_{1\le k\le\sigma^<}\big((\log Q_k)+(S_{\sigma^<}-S_k)\big)>t\right)\\
&\qquad\ge 1-\mathbb{E}\exp\big(-\overline{G}(t)\,(1-\varepsilon(t))\sigma^<\big)\\
&\qquad\sim \overline{G}(t)^\rho(1-\varepsilon(t))^\rho\,\ell\left(\frac{1}{\overline{G}(t)\,(1-\varepsilon(t))}\right)\\
&\qquad\sim \overline{G}(t)^\rho\,\ell(1/\overline{G}(t))
\end{aligned}
$$

as $t\to\infty$. This completes the proof of the lemma. □

6. Proofs of main results.

6.1. *Proof of Theorem 1.1.* By Lemma 3.1, it is enough to show recurrence of the ladder RDE-chain $(X_n^<)_{n\ge 0}$ with associated random vector

$(M^<, Q^<)$. But since the latter chain is contractive ($\Pi_{\sigma_n^<} \to 0$ a.s.), this follows directly from either [17, Thm. 2.1] (positive recurrence) or [1, Theorem 3.1(i)] in combination with Lemma 5.1 (null-recurrence). □

6.2. *Proof of Theorem 1.3.* (a) Again, the result follows from Theorem 3.1(i) in [1] applied to the ladder RDE-chain $(X_n^<)_{n\ge 0}$ after noting that the latter is mean contractive, i.e. $\mathbb{E}\log M^< = \mathbb{E}S_{\sigma^<} \in (-\infty, 0)$, and $Q^<$ satisfies

$$\limsup_{t\to\infty} t\,\mathbb{P}(\log Q^< > t) \;\le\; s^*(\overline{F}) \;<\; \mathbb{E}S_{\sigma^<}$$

by Lemma 5.1(a) and (18).

(b) Here the mean contractivity combines with

$$\liminf_{t\to\infty} t\,\mathbb{P}(\log Q^< > t) \;\ge\; s_*(\overline{G}) \;>\; \mathbb{E}S_{\sigma^<}$$

by Lemma 5.1(b) and (20). Hence, the ladder RDE-chain is transient by Theorem 3.1(ii) in [1].

(c) With $(M_1, Q_1), (M_2, Q_2), \ldots$ denoting i.i.d. copies of (M, Q), let the RDE-chain $(X_{p,n})_{n\ge 0}$ be as defined in Proposition 4.1 for $p > 0$. By another use of Lemma 5.1, here applied to $(X_{p,n})_{n\ge 0}$ with associated random vector (M, Q^p), we obtain

$$ps_*(\overline{G}) \;\le\; \liminf_{t\to\infty} t\,\mathbb{P}(\log Q_{p,\sigma^<} > t) \;\le\; \limsup_{t\to\infty} t\,\mathbb{P}(\log Q_{p,\sigma^<} > t) \;\le\; ps^*(\overline{F})$$

where $Q_{p,\sigma^<}$ takes the role of $Q^<$ for the ladder RDE-chain $(X_{p,\sigma_n^<})_{n\ge 0}$. To see this, note that Q^p for any p still satisfies (15) and (19), but with $\overline{F}(\cdot/p)$ and $\overline{G}(\cdot/p)$ instead of $\overline{F}$ and $\overline{G}$, respectively. From the already shown parts (a) and (b), we finally infer the recurrence of $(X_{p,\sigma_n^<})_{n\ge 0}$ and thus $(X_{p,n})_{n\ge 0}$ whenever $ps^*(\overline{F}) < \kappa$, and the transience whenever $ps_*(\overline{G}) > \kappa$. And so the critical exponent p_0 must lie between the asserted bounds $\kappa/s^*(\overline{F})$ and $\kappa/s_*(\overline{G})$. □

6.3. *Proof of Theorem 1.5.* In view of Lemma 3.1, it suffices to argue that the embedded ladder RDE-chain $(X_n^<)_{n\ge 0}$ is positive recurrent. By (30), its associated random vector has here the form

$$(M^<, Q^<) \;:=\; \Pi_{\sigma^<}\cdot\left(1, \sum_{k=1}^{\sigma^<}\Pi_k^{-1}(aM_k + b)\right),$$

and since

$$Q^< \;=\; \Pi_{\sigma^<}\left(a\sum_{k=0}^{\sigma^<-1}\Pi_k^{-1} + b\sum_{k=1}^{\sigma^<}\Pi_k^{-1}\right) \;\le\; (a+b)\sigma^<$$

we infer $\mathbb{E}\log_+ Q^< < \infty$ with the help of (11). Since $(X_n^<)_{n\ge 0}$ is clearly contractive, positive recurrence follows from [17, Thm. 2.1] (or more general results like [16, Thm. 3] or [10, Thm. 1.1]). □

6.4. *Proof of Theorem 1.6.* In view of Theorem 1.5, we must only prove the transience of the chain when M and Q are independent. As mentioned earlier, $\mathbb{E}\log_-^2 M < \infty$ in combination with (4) ensures $\kappa = \mathbb{E}|S_{\sigma^<}| < \infty$. Furthermore, the slow variation of $L(t) = t^{1/\rho}\,\overline{F}(t)$ for $\overline{F}(t) := \mathbb{P}(\log M > t) = \mathbb{P}(\log Q > t)$ and $\rho \in (\frac{1}{2}, 1)$ is then equivalent to the validity of (10), by [11, Thm. 1], which in turn entails (11). Finally, we arrive at the desired conclusion by invoking Theorem 1.3(b) if we still show that $L(t) \to \infty$ implies

$$s_*(\overline{F}) \;=\; \liminf_{t\to\infty} t\overline{F}(t)^\rho \ell_\rho^<(1/\overline{F}(t)) \;=\; \infty.$$

To this end, we will actually prove that $\ell_\rho^<(s) \to \infty$ as $s \to \infty$ and point out first that, similar to (12), we have

$$\mathbb{P}(\sigma^> > n) \;\sim\; \frac{\ell_{1-\rho}^>(n)}{\Gamma(\rho)\, n^{1-\rho}} \quad \text{as } n \to \infty \tag{40}$$

for the first strictly ascending ladder epoch $\sigma^> := \inf\{n \ge 1 : S_n > 0\}$, where

$$\ell_{1-\rho}^>(s) \;:=\; \exp\left(\sum_{n\ge 1} \frac{(1-s^{-1})^n}{n}\big(1-\rho-\mathbb{P}(S_n > 0)\big)\right), \quad s \in (1,\infty),$$

is slowly varying and obviously related to $\ell_\rho^<$ by the identity

$$\ell_{1-\rho}^>(s) \;=\; \exp\left(\sum_{n\ge 1} \frac{(1-s^{-1})^n}{n}\,\mathbb{P}(S_n = 0)\right) \frac{1}{\ell_\rho^<(s)},$$

and thus

$$\ell_{1-\rho}^>(s) \;\sim\; \frac{\theta}{\ell_\rho^<(s)} \quad \text{as } s \to \infty.$$

Here

$$\theta \;:=\; \exp\left(\sum_{n\ge 1} \frac{1}{n}\,\mathbb{P}(S_n = 0)\right)$$

is well-known to be always finite, see e.g. [20, Cor. 3.3]. So it remains to verify that $\ell_{1-\rho}^>(s) \to 0$ as $s \to \infty$. Now use another result by Doney [13, Thm. 2] to infer that, under the assumptions of the theorem, the relation $\overline{F}(t) \sim t^{-1/\rho}L(t)$ is actually equivalent to the relation

$$\mathbb{P}(\sigma^{>} > n) \ \sim \ \frac{c}{L_{1/\rho}^{*}(n)\, n^{1-\rho}} \quad \text{as } n \to \infty, \tag{41}$$

for some $c > 0$ and a slowly varying function $L_{1/\rho}^{*}$ which is related to L by

$$L(s)^{-\rho} L_{1/\rho}^{*}(s^{1/\rho}/L(s)) \ \to \ 1 \quad \text{as } s \to \infty$$

and unique up to asymptotic equivalence. Since $L(s) \to \infty$, also $L_{1/\rho}^{*}(s) \to \infty$ holds, and we finally infer $\ell_{1-\rho}^{>}(s) \to 0$ as $s \to \infty$ when combining (40) with (41). □

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