Applied Probability Trust (26 June 2021)

RENEWAL THEORY FOR ITERATED PERTURBED RANDOM WALKS ON A GENERAL BRANCHING PROCESS TREE: INTERMEDIATE GENERATIONS

VLADYSLAV BOHUN,* *Taras Shevchenko National University of Kyiv*

ALEXANDER IKSANOV,** *Taras Shevchenko National University of Kyiv*

ALEXANDER MARYNYCH,*** *Taras Shevchenko National University of Kyiv*

BOHDAN RASHYTOV,**** *Taras Shevchenko National University of Kyiv*

Abstract

An iterated perturbed random walk is a sequence of point processes defined by the birth times of individuals in subsequent generations of a general branching process provided that the birth times of the first generation individuals are given by a perturbed random walk. We prove counterparts of the classical renewal-theoretic results (the elementary renewal theorem, Blackwell's theorem and the key renewal theorem) for the number of the jth generation individuals with birth times $\leq t$, when $j, t \to \infty$ and $j(t) = o(t^{2/3})$. According to our terminology, such generations form a subset of the set of intermediate generations.

Keywords: Convolution, key renewal theorem, perturbed random walk, renewal theory

2020 Mathematics Subject Classification: Primary 60K05;60J80

Secondary 60G05

* Postal address: Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, Volodymyrska 64/13, Kyiv, Ukraine, 01601

* Email address: vladyslavbogun@gmail.com

** Email address: iksan@univ.kiev.ua

*** Email address: marynych@unicyb.kiev.ua

**** Email address: mr.rashytov@gmail.com

1. Introduction

Let $(\xi_i, \eta_i)_{i\in\mathbb{N}}$ be independent copies of a random vector (ξ, η) with arbitrarily dependent and almost surely (a.s.) strictly positive components. Denote by $S := (S_i)_{i\geq 0}$ the zero-delayed random walk with increments ξ_i for $i \in \mathbb{N}$, that is, $S_0 := 0$ and $S_i := \xi_1 + \cdots + \xi_i$ for $i \in \mathbb{N}$. Define

$$T_i := S_{i-1} + \eta_i, \quad i \in \mathbb{N}.$$

The sequence $T := (T_i)_{i\in\mathbb{N}}$ is called a *perturbed random walk* (PRW, in short).

Classical renewal theory is an area of applied probability dealing with nondecreasing random walks S and various derived processes like

- the *renewal process* $(R(t))_{t\geq 0}$ defined by

$$R(t) := \sum_{i\geq 1} \mathbb{1}_{\{S_i \leq t\}}, \quad t \geq 0;$$

- the *first passage time process* $(\nu(t))_{t\geq 0}$ defined by

$$\nu(t) := \inf\{i \in \mathbb{N} : S_i > t\}, \quad t \geq 0; \tag{1}$$

- the *undershoot* $t - S_{\nu(t)-1}$, the *overshoot* $S_{\nu(t)} - t$ and others.

A good overview of renewal theory can be found in [2, 12, 25] and more recent accounts [14] and [20]. A survey of various results for PRWs T (with not necessarily positive ξ and η) and particularly counterparts of some renewal-theoretic results can be found in the book [14]. An incomplete list of more recent papers addressing various aspects of the PRWs includes [1, 9, 17, 22, 23, 24].

We proceed by recalling the construction of a general branching process (a.k.a. Crump–Mode–Jagers branching process) in the special case in which it is generated by T. At time 0 there is one individual, the ancestor. The ancestor produces offspring (the first generation) with birth times given by the points of T. The first generation produces second-generation individuals. The shifts of birth times of the second generation individuals with respect to their mothers' birth times are distributed according to copies of T, and for different mothers these copies are independent. The second generation produces the third one, and so on. All individuals act independently of each other. See Figure 1 for an illustration.

Clearly, the random sequence $T^{(j)}$ defined by the birth times in the jth generation of the process ($j \geq 2$) is much more complicated than the perturbed random walk $T^{(1)} = T$ defining

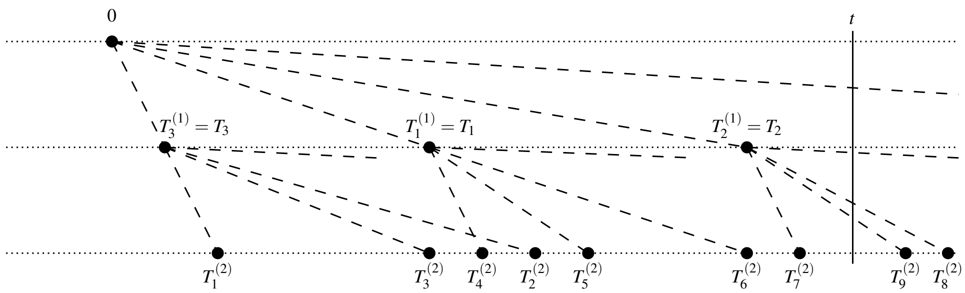

FIGURE 1: A general branching process generated by T. Superscripts indicate generation numbers. The shifts of birth times of the second generation individuals with respect to their mothers' birth times are distributed according to independent copies of T. For instance, $T_7^{(2)} - T_2^{(1)}$, $T_9^{(2)} - T_2^{(1)}$ and $T_8^{(2)} - T_2^{(1)}$ are distributed as the first three smallest elements of T. Note that, in general, $T = (T_k)_{k\in\mathbb{N}}$ is not monotone due to the perturbations. For example, $T_2 > T_3$ because $\eta_2 > \xi_2 + \eta_3$.

the birth times in the 1st generation. It is natural to call $(T^{(j)})_{j\geq 2}$ *iterated perturbed random walk on a general branching process tree*. If $\eta = \xi$ a.s., in which case $T = S$ a.s., the term *iterated random walk on a general branching process tree* may be used for the corresponding sequence $(S^{(j)})_{j\geq 2}$. This should not be confused with iterated renewal processes treated in [27]. In this paper we initiate a systematic study of $T^{(j)}$ for $j \geq 2$ and its derived processes, our primary purpose being obtaining counterparts of the classical renewal-theoretic results.

Now we introduce notation to be used throughout the paper. Put $N(t) := \sum_{i\geq 1} \mathbb{1}_{\{T_i \leq t\}}$ and $V(t) := \mathbb{E}N(t)$ for $t \in \mathbb{R}$. As usual, we shall write x_+ for $\max(x,0)$. Since η_i is independent of S_{i-1}, it is clear that

$$V(t) = \mathbb{E}U((t-\eta)_+) = (U * G)(t) = \int_{[0,t]} U(t-y)\mathrm{d}G(y), \quad t \geq 0. \tag{2}$$

where, for $t \in \mathbb{R}$, $U(t) := \mathbb{E}\nu(t) = \sum_{i\geq 0} \mathbb{P}\{S_i \leq t\}$ is the renewal function and $G(t) = \mathbb{P}\{\eta \leq t\}$. Note that $U(t) = V(t) = G(t) = 0$ for $t < 0$. Here and in what follows, we denote by $u * v$ the Lebesgue–Stieltjes convolution of two functions u and v of locally bounded variation. We also use the notation $u^{*(j)}$, $j \in \mathbb{N}$, for the jth convolution power of u.

For $t \geq 0$ and $j \in \mathbb{N}$, denote by $N_j(t)$ the number of the jth generation individuals with birth times $\leq t$ and put $V_j(t) := \mathbb{E}N_j(t)$. Then $N_1(t) = N(t)$, $V_1(t) = V(t)$ and

$$V_j(t) = (V_{j-1} * V)(t) = \int_{[0,t]} V_{j-1}(t-y)\mathrm{d}V(y), \quad j \geq 2, \quad t \geq 0.$$

For example, we have in Figure 1 $N_1(t) = N(t) = 3$ and $N_2(t) = 7$. For $r \in \mathbb{N}$, let $N_{j-1}^{(r)}(t)$ be the number of successors in the jth generation with birth times within $[T_r, t+T_r]$ of the first generation individual with birth time T_r. By the branching property, $(N_{j-1}^{(1)}(t))_{t\geq 0}$, $(N_{j-1}^{(2)}(t))_{t\geq 0}, \ldots$ are independent copies of N_{j-1} which are also independent of T. The basic decomposition that sheds light on the properties of $N_j := (N_j(t))_{t\geq 0}$ and also demonstrates its recursive structure is

$$N_j(t) = \sum_{r\geq 1} N_{j-1}^{(r)}(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}}, \quad j\geq 2, \quad t\geq 0. \tag{3}$$

Further, let $T^{(j-1)} := (T_r^{(j-1)})_{r\geq 1}$ be some enumeration of the birth times in the $(j-1)$th generation; $N_{1,j}^{(r)}(t)$ be the number of children in the jth generation with birth times within $[T_r^{(j-1)}, t+T_r^{(j-1)}]$ of the $(j-1)$th generation individual with birth time $T_r^{(j-1)}$. Again, by the branching property, $(N_{1,j}^{(1)}(t))_{t\geq 0}$, $(N_{1,j}^{(2)}(t))_{t\geq 0}, \ldots$ are independent copies of $(N(t))_{t\geq 0}$ which are also independent of $T^{(j-1)}$. With these ingredients we can write another recursive decomposition of N_j as follows

$$N_j(t) = \sum_{r\geq 1} N_{1,j}^{(r)}(t-T_r^{(j-1)})\,\mathbb{1}_{\{T_r^{(j-1)}\leq t\}}, \quad j\geq 2, \quad t\geq 0.$$

Note that, for $j\geq 2$, N_j is a particular instance of a random process with immigration at random times (the term was introduced in [8], see also [18]).

Our motivation behind introducing the iterated perturbed random walks is at least threefold.

1) For each integer $j\geq 2$, the sequence $T^{(j)}$ and the process N_j are a natural generalization of the perturbed random walk T and the counting process $(N(t))_{t\geq 0}$. It is interesting to investigate to which extent the renewal-theoretic properties of T and $(N(t))$ are inherited by $T^{(j)}$ and N_j. Thus, the activity undertaken in the present article can be thought of as the development of renewal theory for the iterated perturbed random walks.

2) The sequence $(T^{(j)})_{j\in\mathbb{N}}$ is a particular instance of a branching random walk in which the first generation point process is $(N(t))_{t\geq 0}$, the counting process of a perturbed random walk. Alternatively, and this is our preferred viewpoint, for $j\in\mathbb{N}$, $T^{(j)}$ can be interpreted as the sequence of birth times in the jth generation of a general branching process. Therefore, the results of the present article contribute towards better understanding of how the births occur within a particular generation. Being of intrinsic interest for the theory of general branching processes, this information also sheds light on the

organization of levels (the sets of vertices located at the same distance from the root) of some random trees (for instance, random recursive trees and binary search trees) that can be constructed as family trees of general branching processes stopped at suitable random times. We refer to [13] for more details and examples of the embeddable random trees.

3) Renewal theory for perturbed random walks is an inevitable ingredient for investigation of nested occupancy scheme in random environment generated by stick-breaking. Referring to [6, 15] for more details we only mention that the latter scheme is a generalization of the classical Karlin infinite balls-in-boxes occupancy scheme [11, 19]. Unlike the Karlin scheme in which the collection of boxes is unique, there is a nested hierarchy of boxes, and the hitting probabilities of boxes are defined in terms of iterated stick-breaking. Assuming that n balls have been thrown, denote by $K_n(j)$ the number of occupied boxes in the jth level which is the basic object of interest. It turns out that whenever $j = j_n = o((\log n)^{1/2})$ (the case of fixed j is included) the distributional behavior of $K_n(j)$ as $n \to \infty$ is the same as that of $N_j(\log n)$, when the underlying perturbed random walk T is appropriately chosen.

We call the jth generation *early*, *intermediate* or *late* depending on whether j is fixed, $j = j(t) \to \infty$ and $j(t) = o(t)$ as $t \to \infty$, or $j = j(t)$ is of order t. In view of Proposition 2.1 given in Section 2.1 there are no other regimes because $N_j(t) = 0$ a.s. for large enough t whenever $j = j(t)$ grows faster than t. Assume, for the time being, that j is a late generation and that T is a collection of random points, not necessarily the perturbed random walk. Nevertheless, we retain the notation N_j and V_j. In this case the asymptotic behavior of V_j and N_j is well-understood. For instance, a delicate counterpart of the key renewal theorem for V_j which includes both a version of the elementary renewal theorem and a version of Blackwell's theorem can be found in Theorem A of [4]. For the corresponding a.s. result for N_j, see Theorem B of the same paper and Theorem 4 in [5]. A strong law of large numbers for $N_j(bj)$ for appropriate $b > 0$ is given in formula (1.1) of [4]. From these and the other results of this flavor it follows that N_j forgets what was happening in the early history and particularly in the first generation. The behavior of N_j is universal for a wide class of input processes (responsible for the first generation). It is driven by limit theorems available for general branching processes like convergence of the Biggins martingales, large deviations etc.

While the present paper deals with some intermediate generations, the early generations which admit a much simpler analysis are treated in a separate paper [16]. The behavior of the iterated perturbed random walks in the early and intermediate generations is very different from that in the late generations. When j is a non-late generation, the process N_j inherits, for the most part, the properties of N, in a modified form. This statement is confirmed by counterparts of the elementary renewal theorem (Theorems 2.1 and 2.2), the key renewal theorem (Theorem 2.3) and Blackwell's theorem (Corollary 2.1) which are our main results. As far as early generations are concerned the claim is justified by results obtained in [16].

The remainder of the paper is structured as follows. Our main findings are formulated in Section 2 and then proved in Section 3. Also, Section 2 contains two previously known results concerning N_j and V_j. To our knowledge, all the results presented in this paper form the state-of-the-art as far as the intermediate generations of the iterated perturbed random walks are concerned. Finally, the Appendix collects the proofs of some auxiliary results.

2. Results

2.1. The height of a confined general branching process tree

For $t \geq 0$, put

$$H(t) := \inf\{j \in \mathbb{N} : N_j(t) = 0\}$$

and note that $N_j(t) = 0$ a.s. for all $j \geq H(t)$. We call the variable $H(t)$ the *height of a general branching process tree* generated by a perturbed random walk T and confined to the strip $[0,t]$.

Proposition 2.1. *For each $t \geq 0$, $H(t) < \infty$ a.s. Furthermore,*

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{\gamma} \in (0,\infty) \quad \text{a.s.,} \tag{4}$$

where $\gamma := \sup\{z > 0 : \mu(z) < 1\}$ and $\mu(z) := \inf_{s>0}(e^{zs}\frac{\mathbb{E}e^{-s\eta}}{1-\mathbb{E}e^{-s\xi}})$ for $z > 0$.

Proof. By assumption, $\mathbb{P}\{\eta = 0\} = 0$. This entails $\lim_{s\to\infty} \frac{\mathbb{E}e^{-s\eta}}{1-\mathbb{E}e^{-s\xi}} = 0$ and thereupon

$$\lim_{z\to 0+} \mu(z) = 0.$$

Also, $\lim_{z\to\infty} \mu(z) = \lim_{s\to 0+} \frac{\mathbb{E}e^{-s\eta}}{1-\mathbb{E}e^{-s\xi}} = \infty$. This shows that $\gamma \in (0,\infty)$.

Recall that, for $n \in \mathbb{N}$, $(T_r^{(n)})_{r\in\mathbb{N}}$ denotes some enumeration of the birth times in the nth generation of the general branching process. Put $B(n) := \inf_{r\geq 1} T_r^{(n)}$. By the famous Biggins result (Corollary on p. 635 in [3]),

$$\lim_{n\to\infty}\frac{B(n)}{n}=\gamma \quad \text{a.s.} \tag{5}$$

Since, for $n\in\mathbb{N}$ and $t>0$, $\{H(t)>n\}=\{B(n)\le t\}$ and, according to (5), $\lim_{n\to\infty}B(n)=+\infty$ a.s., we infer $H(t)<\infty$ a.s.

Finally, we have $B(H(t))>t\ge B(H(t)-1)$ a.s. The left-hand inequality ensures that $\lim_{t\to\infty}H(t)=+\infty$ a.s. which together with (5) proves (4) with the help of a standard sandwich argument. □

Proposition 2.1 implies, in particular, that if

$$\liminf_{t\to\infty}\frac{j(t)}{t}>\gamma^{-1},$$

then there exists an a.s. finite $t_0>0$ such that $N_{j(t)}(t)=0$ for all $t\ge t_0$. This observation justifies our classification of generations (early, intermediate, late). Furthermore, the analysis of N_j for the late generations can be restricted to the range in which $j=j(t)$ grows not faster than $\gamma^{-1}t$ as $t\to\infty$.

It is seldom possible to find the constant γ explicitly. Here is one happy exception. Let $(\xi,\eta)=(|\log W|,|\log(1-W)|)$, where W has a uniform distribution on [0, 1]. The distribution of the sequence $(e^{-T_i})_{i\in\mathbb{N}}$ is known as the *Griffiths–Engen–McCloskey distribution* with parameter 1. In this case, $\mu(z)=ez$ for $z>0$ which gives $\gamma=e^{-1}$.

2.2. Counterparts of the elementary renewal theorem for intermediate generations

The simplest result of renewal theory, called the elementary renewal theorem, tells us that

$$U(t)=\sum_{i\ge 0}\mathbb{P}\{S_i\le t\}\ \sim\ \frac{t}{\mathtt{m}},\quad t\to\infty,$$

where $\mathtt{m}:=\mathbb{E}\xi<\infty$, see, for instance, Theorem 3.3.3 on p. 191 in [25]. Here and hereafter, the notation $f(t)\sim g(t)$ means that the ratio $f(t)/g(t)$ tends to 1 as $t\to\infty$.

From (2) it follows that, without any assumptions on η,

$$V(t)\ \sim\ \frac{t}{\mathtt{m}},\quad t\to\infty. \tag{6}$$

This is a counterpart of the elementary renewal theorem for the perturbed random walks.

In this section we state two results on the first-order behavior of the convolutions powers V_j of V. Our first result, Theorem 2.1, deals with ‘early intermediate’ generations satisfying

$j = j(t) \to \infty$ and $j(t) = o(t^{1/2})$ as $t \to \infty$ as well as early generations. At this point we stress that even though both Theorem 2.1 and Theorem 2.2 hold true for early generations, the assumptions of these theorems are too restrictive as far as early generations are concerned. We refer to the companion article [16] for a proper version of the elementary renewal theorem in early generations. Recall the standard notation $x \wedge y = \min(x,y)$ for $x, y \in \mathbb{R}$.

Theorem 2.1. *Assume that either (i) $\mathbb{E}\xi^2 < \infty$ and $\mathbb{E}\eta < \infty$ or (ii) $\mathbb{E}\xi^2 = \infty$, $\mathbb{P}\{\xi > t\} = O(t^{-r})$ and $\mathbb{E}(\eta \wedge t) = O(t^{2-r})$ for some $r \in (1,2)$, as $t \to \infty$. Then, for any integer-valued function $j = j(t)$ satisfying $j(t) = o(t^{(r-1)/2})$ as $t \to \infty$, where we put $r = 2$ if conditions (i) prevail,*

$$V_j(t) \sim \frac{t^j}{\mathtt{m}^j j!}, \quad t \to \infty. \tag{7}$$

Here, $\mathtt{m} = \mathbb{E}\xi < \infty$.

Remark 2.1. The condition $\mathbb{E}\xi^r < \infty$ for some $r \in (1,2)$ is clearly sufficient for $\mathbb{P}\{\xi > t\} = O(t^{-r})$. Further, the condition $\mathbb{E}\eta^{r-1} < \infty$ is sufficient for $\mathbb{E}(\eta \wedge t) = O(t^{2-r})$, $t \to \infty$. This follows from

$$\begin{aligned}
\mathbb{E}(\eta \wedge t) &= \int_0^t \mathbb{P}\{\eta > y\}\mathrm{d}y \\
&\le \int_0^t \left(\frac{t}{y}\right)^{2-r} \mathbb{P}\{\eta > y\}\mathrm{d}y \\
&\le t^{2-r} \int_0^\infty y^{r-2}\mathbb{P}\{\eta > y\}\mathrm{d}y \\
&= (r-1)^{-1}\mathbb{E}\eta^{r-1}t^{2-r}.
\end{aligned}$$

Note that part (i) of Theorem 2.1 has already been obtained via a slightly different argument in formula (4.6) of [6].

Given next is a quite surprising result which shows that the convolution power V_j exhibits a phase transition in the generations j satisfying $j = j(t) \sim \mathrm{const} \cdot t^{1/2}$ as $t \to \infty$. Here, further moment and smoothness assumptions seem to be indispensable. In particular, we assume that the distribution of ξ is spread-out, which means that some convolution power of the distribution function $t \mapsto \mathbb{P}\{\xi \le t\}$ has an absolutely continuous component.

Theorem 2.2. *Assume that the distribution of ξ is spread-out, that $\mathbb{E}\xi^3 < \infty$ and $\mathbb{E}\eta^2 < \infty$. Then, for any integer-valued function $j = j(t)$ satisfying $j(t) = o(t^{2/3})$ as $t \to \infty$,*

$$V_j(t) \sim \frac{t^j}{\mathtt{m}^j j!} \exp\left(\frac{\gamma_0 \mathtt{m} j^2}{t}\right), \quad t \to \infty,$$

where

$$\gamma_0 := \int_{[0,\infty)} \mathrm{d}(V(y) - \mathtt{m}^{-1}y) = \lim_{t\to\infty}(V(t) - \mathtt{m}^{-1}t) = \frac{\mathbb{E}\xi^2}{2\mathtt{m}^2} - \frac{\mathbb{E}\eta}{\mathtt{m}} \tag{8}$$

may be positive, negative or zero.

Remark 2.2. Assume that $(\xi,\eta) = (|\log W|, |\log(1-W)|)$, where W is a random variable having the uniform distribution on $[0,1]$. Then, both ξ and η have the exponential distribution of unit mean. Therefore,

$$V(t) = \sum_{i\geq 1} \mathbb{P}\{S_{i-1} + \eta_i \leq t\} = \sum_{i\geq 1} \mathbb{P}\{S_i \leq t\} = U(t) - 1 = t, \quad t \geq 0,$$

where the last equality follows from $U(t) = t + 1$ for $t \geq 0$, see, for instance, the bottom of p. 211 in [25]. Thus, $V(t) = t$ for $t \geq 0$ and

$$V_j(t) = \int_0^t V_{j-1}(y)\mathrm{d}y = \frac{t^j}{j!}, \quad j \geq 2, \quad t \geq 0,$$

where the last equality follows by induction. This is in line with the asymptotics provided by Theorem 2.2, for, in this case, $\gamma_0 = 0$ and $\mathtt{m} = 1$.

2.3. Counterparts of the key renewal theorem and Blackwell's theorem for intermediate generations

We start by recalling a few standard notions of renewal theory.

Let $d > 0$. The distribution of ξ is *d-lattice* if it is concentrated on the set $(dn)_{n\in\mathbb{N}_0}$ and not concentrated on the set $(d_1 n)_{n\in\mathbb{N}_0}$ for any $d_1 > d$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The distribution of ξ is *nonlattice* if it is not d-lattice for any $d > 0$.

A function $f : [0,\infty) \to [0,\infty)$ is called *directly Riemann integrable* (dRi) on $[0,\infty)$, if
(a) $\overline{\sigma}(h) < \infty$ for each $h > 0$ and
(b) $\lim_{h\to 0+} \big(\overline{\sigma}(h) - \underline{\sigma}(h)\big) = 0$, where

$$\overline{\sigma}(h) := h \sum_{n\geq 1} \sup_{(n-1)h\leq y<nh} f(y) \quad \text{and} \quad \underline{\sigma}(h) := h \sum_{n\geq 1} \inf_{(n-1)h\leq y<nh} f(y).$$

Blackwell's theorem is the most important and complicated result of renewal theory. Here is its formulation (see, for instance, Theorem 1.10 on p. 18 in [20]). Recall that U denotes the renewal function.

Proposition 2.2. *Let $\mathtt{m} = \mathbb{E}\xi < \infty$. If the distribution of ξ is nonlattice, then, for any fixed $h > 0$,*

$$\lim_{t\to\infty}(U(t+h) - U(t)) = \frac{h}{\mathtt{m}}.$$

If the distribution of ξ is d-lattice, then, for any fixed positive integer n,

$$\lim_{t\to\infty}(U(t+dn)-U(t))=\frac{dn}{\mathtt{m}}.$$

If $\mathtt{m}=\infty$, then both limits are equal to 0*.*

Thus, Blackwell's theorem reads a bit differently for nonlattice and lattice distributions which justifies the necessity of distinguishing between these types of distributions. The same dichotomy is also needed for the *key renewal theorem*, see, for instance, Theorem 1.12 on p. 30 in [20].

In renewal theory the key renewal theorem is usually obtained as a corollary to Blackwell's theorem, see, for instance, p. 241-242 in [25]. We proceed differently by first proving a counterpart of the key renewal theorem (Theorem 2.3) and then obtain a counterpart of Blackwell's theorem (Corollary 2.1) as a corollary.

Theorem 2.3. *Let $f:[0,\infty)\to[0,\infty)$ be a directly Riemann integrable function on $[0,\infty)$. Assume that either (a) or (b) below holds true:*

(a) the distribution of ξ is nonlattice, the conditions of Theorem 2.1 hold and $j(t)=o(t^{(r-1)/2})$ as $t\to\infty$, with the same $r\in(1,2]$ as in Theorem 2.1;

(b) the conditions of Theorem 2.2 hold and $j(t)=o(t^{2/3})$ as $t\to\infty$.

Then

$$(f*V_j)(t)=\int_{[0,t]}f(t-y)\mathrm{d}V_j(y)\ \sim\ \Big(\frac{1}{\mathtt{m}}\int_0^\infty f(y)\mathrm{d}y\Big)V_{j-1}(t),\quad t\to\infty, \tag{9}$$

where $\mathtt{m}=\mathbb{E}\xi<\infty$, and $V_{j-1}(t)$ on the right-hand side can be replaced with $t^{j-1}/(m^{j-1}(j-1)!)$ in the case (a), or with $t^{j-1}/(m^{j-1}(j-1)!)\exp(\gamma_0\mathtt{m}j^2/t)$ in the case (b).

Remark 2.3. In part (b) of Theorem 2.3 one of the assumptions, coming from Theorem 2.2, is that the distribution of ξ is spread-out. We note that every spread-out distribution is nonlattice but not vice versa. To justify the second claim, observe that the distribution concentrated at points 1 and $\sqrt{2}$ is nonlattice but not spread-out.

Upon taking $f(y)=\mathbb{1}_{[0,h)}(y)$ in Theorem 2.3 we immediately obtain the following.

Corollary 2.1. *Let $h>0$ be fixed. Under the assumptions of Theorem 2.3*

$$V_j(t+h)-V_j(t)\ \sim\ \frac{h}{m}V_{j-1}(t),\quad t\to\infty. \tag{10}$$

2.4. Some previously known results

In this section we collect two previously known facts concerning the asymptotic behavior of N_j in the intermediate generations. They are borrowed from [15] and stated here for integrity and the reader's convenience. We write $\overset{\text{f.d.d.}}{\longrightarrow}$ to denote weak convergence of finite-dimensional distributions.

Theorem 2.4. (Multivariate central limit theorem for $(N_j(t))_{t\geq 0}$.) *Assume that* $\mathtt{s}^2 = \operatorname{Var}\xi \in (0,\infty)$ *and* $\mathbb{E}\eta < \infty$. *Let* $j = j(t)$ *be any positive integer-valued function satisfying* $j(t) \to \infty$ *and* $j(t) = o(t^{1/2})$ *as* $t \to \infty$. *Then, as* $t \to \infty$,

$$\left(\frac{\lfloor j(t)\rfloor^{1/2}(\lfloor j(t)u\rfloor - 1)!}{(\mathtt{s}^2\mathtt{m}^{-2\lfloor j(t)u\rfloor-1}t^{2\lfloor j(t)u\rfloor-1})^{1/2}}\left(N_{\lfloor j(t)u\rfloor}(t) - V_{\lfloor j(t)u\rfloor}(t)\right)\right)_{u>0} \overset{\text{f.d.d.}}{\longrightarrow} \left(\int_{[0,\infty)} e^{-uy}\mathrm{d}B(y)\right)_{u>0}, \tag{11}$$

where $(B(v))_{v\geq 0}$ *is a standard Brownian motion.*

According to Proposition 3.1, Theorems 3.2 and 3.3 in [6], the centering $V_{\lfloor j(t)u\rfloor}(t)$ in (11) can be replaced by its leading term

$$t^{\lfloor j(t)u\rfloor}/((\lfloor j(t)u\rfloor)!\mathtt{m}^{\lfloor j(t)u\rfloor}),$$

provided that $j(t) = o(t^{1/3})$. For functions $t \mapsto j(t)$ which grow faster, this is not always the case. Plainly, the possibility/impossibility of such a replacement is justified by a second-order behavior of V_j. It should come as no surprise that second-order results for V_j require more restrictive assumptions on the distributions of ξ and η than the corresponding first-order results. The following proposition which is concerned with the rate of convergence in the elementary renewal theorem for V_j was proved in Proposition 8.1 of [15].

Proposition 2.3. *Assume that the distribution of* ξ *has an absolutely continuous component, that* $\mathbb{E}e^{\beta_1\xi} < \infty$, $\mathbb{E}e^{\beta_2\eta} < \infty$ *for some* $\beta_1, \beta_2 > 0$ *and*

$$\gamma_0 = \frac{\mathbb{E}\xi^2}{2\mathtt{m}^2} - \frac{\mathbb{E}\eta}{\mathtt{m}} > 0.$$

Then

$$V_j(t) - \frac{t^j}{j!\mathtt{m}^j} \sim \frac{\gamma_0 j t^{j-1}}{(j-1)!\mathtt{m}^{j-1}}, \quad t \to \infty \tag{12}$$

whenever $j = j(t) = o(t^{1/2})$ *as* $t \to \infty$ *(*j *is allowed to be fixed).*

Formula (12) can be thought of as a generalization of formulae (8) and (14). These provide the second-order behavior of the functions V and U, respectively.

3. Proofs

3.1. Preparatory results

Recall that U denotes the renewal function for $(S_n)_{n\in\mathbb{N}_0}$. According to Lorden's inequality which holds whenever $\mathbb{E}\xi^2<\infty$,

$$U(t)-\mathtt{m}^{-1}t\leq c_0,\quad t\geq 0,\tag{13}$$

where $c_0:=\mathbb{E}\xi^2/\mathtt{m}^2$ and $\mathtt{m}=\mathbb{E}\xi<\infty$. See [7] for an elegant proof. We also note that

$$\lim_{t\to\infty}\big(U(t)-\mathtt{m}^{-1}t\big)=\frac{\mathbb{E}\xi^2}{2\mathtt{m}^2}\tag{14}$$

provided that the distribution of ξ is nonlattice and $\mathbb{E}\xi^2<\infty$, see, for instance, Example 3.10.3 on p. 242 in [25]. Further, by Wald's identity (see Proposition A10.2(a) in [2]) and the definition of $\nu(t)$ given in (1), $t\leq\mathbb{E}S_{\nu(t)}=\mathtt{m}\mathbb{E}\nu(t)=\mathtt{m}U(t)$. Thus,

$$U(t)\geq\mathtt{m}^{-1}t,\quad t\geq 0.\tag{15}$$

Since $V(t)\leq U(t)$ for $t\geq 0$ we infer

$$V(t)-\mathtt{m}^{-1}t\leq c_0,\quad t\geq 0.\tag{16}$$

On the other hand, assuming that $\mathbb{E}\eta<\infty$ (whereas the assumption $\mathbb{E}\xi^2<\infty$ is not needed here),

$$\begin{aligned}V(t)-\mathtt{m}^{-1}t&=\int_{[0,t]}(U(t-y)-\mathtt{m}^{-1}(t-y))\mathrm{d}G(y)-\mathtt{m}^{-1}\int_0^t(1-G(y))\mathrm{d}y\\&\geq-\mathtt{m}^{-1}\int_0^t(1-G(y))\mathrm{d}y\\&\geq-\mathtt{m}^{-1}\mathbb{E}\eta\end{aligned}\tag{17}$$

having utilized $U(t)\geq\mathtt{m}^{-1}t$. Thus, we have shown that, under the assumptions $\mathbb{E}\xi^2<\infty$ and $\mathbb{E}\eta<\infty$,

$$|V(t)-\mathtt{m}^{-1}t|\leq c_L,\quad t\geq 0\tag{18}$$

where $c_L=\max(c_0,\mathtt{m}^{-1}\mathbb{E}\eta)$.

Let $u,v,w:\mathbb{R}\mapsto\mathbb{R}$ be functions of locally bounded variation. Since the Lebesgue–Stieltjes convolution
$u*v(t)=\int_{\mathbb{R}}u(t-y)\mathrm{d}v(y)$ for $t\in\mathbb{R}$ will be frequently used in what follows, we recall its elementary properties which follow immediately from the definition.

Commutativity $u * v = v * u$.

Associativity $(u * v) * w = u * (v * w)$.

Distributivity $(u + v) * w = (u * w) + (v * w)$.

Existence of the identity If $z(t) = \mathbb{1}_{[0,\infty)}(t)$, then $u * z = z * u = u$. Thus, the function z is the identity with respect to the Lebesgue–Stieltjes convolution.

Further properties of the convolution operation can be found in Section 3.2 of [25] or Section 1.3.1 of [26].

3.2. Results on convolution powers of functions of a linear growth and proofs of Theorems 2.1 and 2.2

The results presented here are concerned with the following purely analytic problem. Assume that a nondecreasing function f exhibits a linear growth, that is, $f(t) \sim at$ as $t \to \infty$ for some $a > 0$. Then, for fixed $j \in \mathbb{N}$,

$$f^{*(j)}(t) \sim \frac{a^j t^j}{j!}, \quad t \to \infty.$$

Imposing various assumptions on the behavior of $f(t) - at$ we shall extend these asymptotics to the case when $j = j(t)$ diverges to infinity as $t \to \infty$.

Proposition 3.1. *Let $f : \mathbb{R} \to [0, \infty)$ be a nondecreasing right-continuous function vanishing on the negative half-line and satisfying*

$$f(t) = at + O(t^{\alpha}), \quad t \to \infty \tag{19}$$

for some $a > 0$ and $\alpha \in [0,1)$. Then, for any integer-valued function $j = j(t)$ such that $j(t) = o(t^{(1-\alpha)/2})$ as $t \to \infty$,

$$f_j(t) := f^{*(j)}(t) \sim \frac{a^j t^j}{j!}, \quad t \to \infty.$$

Proof. According to (19) there exists $C \geq 1$ such that

$$-C(t+1)^{\alpha} \leq f(t) - at \leq C(t+1)^{\alpha}, \quad t \geq 0. \tag{20}$$

For $j \in \mathbb{N}$ and $t \geq 0$, put

$$r_j(t) := \int_{[0,t]} f_j(t-y) \mathrm{d}(f(y) - ay) = \int_{[0,t]} (f(t-y) - a(t-y)) \,\mathrm{d} f_j(y)$$

and note that

$$f_j(t) = r_{j-1}(t) + a\int_0^t f_{j-1}(y)\mathrm{d}y, \quad j \ge 2,\ t \ge 0.$$

By virtue of (20), we conclude that

$$|r_j(t)| \le C(t+1)^{\alpha} f_j(t), \quad j \in \mathbb{N},\ t \ge 0.$$

Using this bound and mathematical induction we obtain

$$W_j^-(t) \le f_j(t) \le W_j^+(t), \quad j \in \mathbb{N},\ t \ge 0, \tag{21}$$

where $W_j^{\pm}$ is defined recursively by $W_0^{\pm}(t) := 1$ and

$$W_j^{\pm}(t) = \left(\pm C(t+1)^{\alpha} W_{j-1}^{\pm}(t) + a\int_0^t W_{j-1}^{\pm}(y)\mathrm{d}y \right)_+, \quad j \in \mathbb{N},\ t \ge 0.$$

Here, we recall that $x_+ = \max(x,0)$ and note that taking the nonnegative part is only relevant for W_j^- ensuring its nonnegativity, whereas it can be omitted for W_j^+.

It remains to show that

$$W_j^{\pm}(t) \ \sim\ \frac{a^j t^j}{j!}, \quad t \to \infty. \tag{22}$$

To this end, we first prove by induction that

$$W_j^+(t) \le \frac{a^j t^j}{j!} + \sum_{i=0}^{j-1} \binom{j}{i} \frac{a^i C^{j-i}(t+1)^{\alpha(j-i)+i}}{i!}, \quad j \in \mathbb{N},\ t \ge 0. \tag{23}$$

While for $j = 1$ this follows immediately because $W_1^+(t) = C(t+1)^{\alpha} + at$, the induction step works as follows

$$\begin{aligned}
W_{j+1}^+(t) &\le C(t+1)^{\alpha} \left(\frac{a^j t^j}{j!} + \sum_{i=0}^{j-1} \binom{j}{i} \frac{a^i C^{j-i}(t+1)^{\alpha(j-i)+i}}{i!} \right) \\
&\qquad + a\int_0^t \left(\frac{a^j y^j}{j!} + \sum_{i=0}^{j-1} \binom{j}{i} \frac{a^i C^{j-i}(y+1)^{\alpha(j-i)+i}}{i!} \right) \mathrm{d}y \\
&\le \sum_{i=0}^{j} \binom{j}{i} \frac{a^i C^{j+1-i}(t+1)^{\alpha(j+1-i)+i}}{i!} + \frac{a^{j+1}t^{j+1}}{(j+1)!} + \sum_{i=0}^{j-1} \binom{j}{i} \frac{a^{i+1}C^{j-i}}{i!} \frac{(t+1)^{\alpha(j-i)+i+1}}{\alpha(j-i)+i+1} \\
&\le \frac{a^{j+1}t^{j+1}}{(j+1)!} + \sum_{i=0}^{j} \binom{j}{i} \frac{a^i C^{j+1-i}(t+1)^{\alpha(j+1-i)+i}}{i!} + \sum_{i=0}^{j-1} \binom{j}{i} \frac{a^{i+1}C^{j-i}}{(i+1)!}(t+1)^{\alpha(j-i)+i+1} \\
&= \frac{a^{j+1}t^{j+1}}{(j+1)!} + \sum_{i=0}^{j} \binom{j}{i} \frac{a^i C^{j+1-i}(t+1)^{\alpha(j+1-i)+i}}{i!} + \sum_{i=1}^{j} \binom{j}{i-1} \frac{a^i C^{j+1-i}}{i!}(t+1)^{\alpha(j+1-i)+i} \\
&= \frac{a^{j+1}t^{j+1}}{(j+1)!} + \sum_{i=0}^{j} \binom{j+1}{i} \frac{a^i C^{j+1-i}(t+1)^{\alpha(j+1-i)+i}}{i!}
\end{aligned}$$

having utilized the binomial identity $\binom{j}{i}+\binom{j}{i-1}=\binom{j+1}{i}$ for the last step. Further, using $j(t)=o(t)$ we obtain

$$\begin{aligned}\frac{j!}{a^jt^j}\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^iC^{j-i}(t+1)^{\alpha(j-i)+i}}{i!} &\sim \frac{j!}{a^j(t+1)^j}\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^iC^{j-i}(t+1)^{\alpha(j-i)+i}}{i!}\\ &\le \sum_{i=0}^{j-1}\left(\frac{j!}{i!}\right)^2\left(\frac{C}{a}\right)^{j-i}(t+1)^{(1-\alpha)(i-j)}\\ &\le \sum_{i=0}^{j-1}(j^{j-i})^2(Ca^{-1})^{j-i}(t+1)^{(1-\alpha)(i-j)}\\ &\le \sum_{i\ge 1}\left(\frac{Ca^{-1}j^2}{(t+1)^{1-\alpha}}\right)^i\\ &= \frac{Ca^{-1}j^2}{(t+1)^{1-\alpha}}\left(1-\frac{Ca^{-1}j^2}{(t+1)^{1-\alpha}}\right)^{-1}.\end{aligned}$$

Thus, in view of the assumption $j(t)=o(t^{(1-\alpha)/2})$ we have

$$\limsup_{t\to\infty}\frac{j!}{a^jt^j}W_j^+(t)\le 1. \tag{24}$$

We use a similar reasoning to prove that

$$W_j^-(t)\ge\left(\frac{a^jt^j}{j!}-\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^iC^{j-i}(t+1)^{\alpha(j-i)+i}}{i!}\right)_+,\quad j\in\mathbb{N},\ t\ge 0. \tag{25}$$

While (25) is obviously true for $j=1$, we obtain with the help of induction, for $j\ge 2$,

$$\begin{aligned}W_{j+1}^-(t) &\ge -C(t+1)^\alpha W_j^-(t)+a\int_0^t W_j^-(y)\mathrm{d}y\\ &\ge -C(t+1)^\alpha W_j^+(t)+a\int_0^t W_j^-(y)\mathrm{d}y\\ &\ge -C(t+1)^\alpha W_j^+(t)+a\int_0^t\left(\frac{a^jy^j}{j!}-\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^iC^{j-i}(y+1)^{\alpha(j-i)+i}}{i!}\right)\mathrm{d}y\\ &\ge -C(t+1)^\alpha\left(\frac{a^jt^j}{j!}+\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^iC^{j-i}(t+1)^{\alpha(j-i)+i}}{i!}\right)\\ &\qquad +a\int_0^t\left(\frac{a^jy^j}{j!}-\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^iC^{j-i}(y+1)^{\alpha(j-i)+i}}{i!}\right)\mathrm{d}y\\ &\ge \frac{a^{j+1}t^{j+1}}{(j+1)!}-\left(\sum_{i=0}^{j}\binom{j}{i}\frac{a^iC^{j+1-i}(t+1)^{\alpha(j+1-i)+i}}{i!}\right.\\ &\qquad\left.+\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^{i+1}C^{j-i}}{i!}\int_0^t(y+1)^{\alpha(j-i)+i}\mathrm{d}y\right)\\ &\ge \frac{a^{j+1}t^{j+1}}{(j+1)!}-\sum_{i=0}^{j}\binom{j+1}{i}\frac{a^iC^{j+1-i}(t+1)^{\alpha(j+1-i)+i}}{i!}.\end{aligned}$$

We have used (21) and (23) for the second and the fourth inequality, respectively. Since W^{-}_{j+1} is nonnegative we arrive at (25). Thus,

$$\liminf_{t\to\infty} \frac{j!}{a^j t^j} W^{-}_{j}(t) \geq 1. \tag{26}$$

Combining (24) and (26) yields (22), thereby finishing the proof of Proposition 3.1. □

Proof of Theorem 2.1. It is enough to show that the function V satisfies all the assumptions of Proposition 3.1 with $a = \mathtt{m}^{-1}$ and $\alpha = 2-r$, where $r=2$ in case (i). We first prove that

$$U(t) = \mathtt{m}^{-1}t + O(t^{2-r}), \quad t\to\infty. \tag{27}$$

CASE (i). In this case (27) follows from (15) and Lorden's inequality (13).
CASE (ii). Let S_0^* be a random variable with distribution

$$\mathbb{P}\{S_0^*\in \mathrm{d}x\} = \mathtt{m}^{-1}\mathbb{P}\{\xi > x\}\,\mathbb{1}_{(0,\infty)}(x)\mathrm{d}x.$$

Note that, by formula (2) in [7],

$$\mathbb{E}U(t-S_0^*) = \mathtt{m}^{-1}t, \quad t\geq 0. \tag{28}$$

Then

$$U(t)-\mathtt{m}^{-1}t = \int_{[0,t]}\mathbb{P}\{S_0^* > t-y\}\mathrm{d}U(y) \ \sim\ \mathtt{m}^{-1}\int_0^t \mathbb{P}\{S_0^*>y\}\mathrm{d}y, \quad t\to\infty,$$

where the equality is nothing else but (28), and the asymptotic relation follows from Theorem 4 in [28]. The cited theorem applies because $\mathbb{E}\xi^2=\infty$ entails $\mathbb{E}S_0^*=\infty$. Observe that $\mathbb{P}\{S_0^*>t\} = \mathtt{m}^{-1}\int_t^\infty \mathbb{P}\{\xi>y\}\mathrm{d}y = O(t^{-(r-1)})$ as $t\to\infty$ as a consequence of $\mathbb{P}\{\xi>t\}=O(t^{-r})$. In view of this we infer

$$U(t)-\mathtt{m}^{-1}t \ \sim\ \mathtt{m}^{-1}\int_0^t \mathbb{P}\{S_0^*>y\}\mathrm{d}y = O(t^{2-r}), \quad t\to\infty$$

and thereupon (27).

It remains to check that under the assumption $\mathbb{E}(\eta\wedge t) = O(t^{2-r})$ as $t\to\infty$,

$$V(t) = \mathtt{m}^{-1}t + O(t^{2-r}), \quad t\to\infty. \tag{29}$$

Since $\int_0^t(1-G(y))\mathrm{d}y = \mathbb{E}(\eta\wedge t)$, an equivalent form of the equality in (17) is

$$V(t)-\mathtt{m}^{-1}t = \int_{[0,t]}(U(t-y)-\mathtt{m}^{-1}(t-y))\mathrm{d}G(y) - \mathtt{m}^{-1}\mathbb{E}(\eta\wedge t).$$

With this at hand relation (29) follows because each summand is $O(t^{2-r})$ by (27) and the assumption of the theorem, respectively.

$\square$

The next result provides asymptotics of convolution powers $f^{*(j)}$ for $j = j(t)$ which may grow faster than $t^{1/2}$ under the assumption that the function $|f(t) - at|$ has a finite total variation and satisfies an additional integrability assumption. We shall use a convention that, for a function $x : \mathbb{R} \to \mathbb{R}$, $x^{*(0)}(t) = \mathbb{1}_{[0,\infty)}(t)$, $t \in \mathbb{R}$. Also, we shall write $\mathscr{V}_I(x)$ for the total variation of x on the (possibly infinite) interval I. Finally, if x is a function of a finite total variation on $[a, b]$, $-\infty \le a < b \le \infty$ and y is a measurable function on I, we stipulate that

$$\int_{[a,b]} y(t)|\mathrm{d}x(t)| = \int_{[a,b]} y(t)\mathrm{d}\left(\mathscr{V}_{[a,t]}(x)\right),$$

where the integral on the right-hand side is understood in the Lebesgue–Stieltjes sense.

Proposition 3.2. *Let $f : \mathbb{R} \mapsto [0, \infty)$ be a nondecreasing right-continuous function vanishing on the negative half-line. Assume that the function ε defined by*

$$\varepsilon(t) := f(t) - at, \quad t \ge 0, \tag{30}$$

for some $a > 0$, satisfies

$$\int_{[0,\infty)} y|\mathrm{d}\varepsilon(y)| < \infty. \tag{31}$$

Then, for any integer-valued function $j = j(t)$ such that $j(t) = o(t^{2/3})$ as $t \to \infty$,

$$f_j(t) := f^{*(j)}(t) \sim \frac{a^j t^j}{j!} \exp\left(\frac{\gamma_0 j^2}{at}\right), \quad t \to \infty, \tag{32}$$

where $\gamma_0 := \int_{[0,\infty)} \mathrm{d}\varepsilon(y) = \lim_{t\to\infty}(f(t) - at)$.

Proof. The function ε, as the difference of two nondecreasing functions, has a finite total variation on every finite interval. In particular, (31) entails

$$\int_{[0,\infty)} |\mathrm{d}\varepsilon(y)| \le \int_{[0,1)} |\mathrm{d}\varepsilon(y)| + \int_{[1,\infty)} y|\mathrm{d}\varepsilon(y)| < \infty.$$

Thus, ε has a finite total variation on $[0,\infty)$. Write

$$\int_0^\infty |\varepsilon(y) - \gamma_0|\mathrm{d}y = \int_0^\infty \left|\int_{(y,\infty)} \mathrm{d}\varepsilon(z)\right| \mathrm{d}y \le \int_0^\infty \int_{(y,\infty)} |\mathrm{d}\varepsilon(z)|\,\mathrm{d}y = \int_{[0,\infty)} y\,|\mathrm{d}\varepsilon(y)| < \infty$$

having utilized integration by parts for the last equality. Hence, (31) implies that

$$\int_0^\infty |\varepsilon(y) - \gamma_0|\mathrm{d}y < \infty. \tag{33}$$

Now we modify (30) in a neighborhood of the origin, so that the essential properties of ε given by (31) and (33) are preserved. Put

$$f(t) = (at+\gamma_0)_+ + \widetilde{\varepsilon}(t) =: \ell(t) + \widetilde{\varepsilon}(t), \quad t \in \mathbb{R}. \tag{34}$$

Note that both summands can be nonzero in a bounded left neighborhood of the origin, yet

$$\int_{\mathbb{R}} |\widetilde{\varepsilon}(y)|\mathrm{d}y < \infty \quad \text{and} \quad \int_{\mathbb{R}} |y||\mathrm{d}\widetilde{\varepsilon}(y)| < \infty \tag{35}$$

because $t \mapsto \varepsilon(t) - \gamma_0 - \widetilde{\varepsilon}(t)$ has a bounded support. The advantage of (34) is justified by two facts:

(i) a simple formula for the convolution powers of ℓ, namely,

$$\ell^{*(j)}(t) = \frac{(at+\gamma_0 j)_+^j}{j!}, \quad j \in \mathbb{N}, \quad t \in \mathbb{R}, \tag{36}$$

(ii) the function $\widetilde{\varepsilon}$ decays sufficiently fast and, as such, is asymptotically negligible in the sense that $\ell^{*(j)}(t) \sim f^{*(j)}(t)$ as $t \to \infty$.

To check (36) we use mathematical induction. While, for $j=1$, the formula is trivial, the induction step works as follows: for $t \geq -a^{-1}\gamma_0(j+1)$,

$$\begin{aligned}
\ell^{*(j+1)}(t) &= \int_{\mathbb{R}} \frac{(a(t-y)+\gamma_0 j)_+^j}{j!}\mathrm{d}\ell(y) \\
&= a\int_{-\gamma_0 a^{-1}}^{t+j\gamma_0 a^{-1}} \frac{(a(t-y)+\gamma_0 j)^j}{j!}\mathrm{d}y \\
&= \int_0^{at+\gamma_0(j+1)} \frac{z^j}{j!}\mathrm{d}z \\
&= \frac{(at+\gamma_0(j+1))^{j+1}}{(j+1)!},
\end{aligned}$$

and $\ell^{*(j+1)}(t) = 0$ for $t < -a^{-1}\gamma_0(j+1)$.

As far as the point (ii) is concerned, using (34), commutativity and distributivity of the Lebesgue–Stieltjes convolution yields

$$f^{*(j)}(t) = \ell^{*(j)}(t) + \sum_{k=0}^{j-1} \binom{j}{k} \left(\ell^{*(k)} * \widetilde{\varepsilon}^{*(j-k)}\right)(t), \quad t \in \mathbb{R}.$$

We are going to show that the second summand is asymptotically negligible with respect to $\ell^{*(j)}(t)$ whenever $j(t) = o(t^{2/3})$. Assume this has already been done. Then (32) follows immediately because, for large enough t,

$$f^{*(j)}(t) \; = \; \ell^{*(j)}(t) \; = \; \frac{a^j t^j}{j!}\left(1+\frac{\gamma_0 j}{at}\right)^j \; = \; \frac{a^j t^j}{j!}\exp\left(j\log\left(1+\frac{\gamma_0 j}{at}\right)\right).$$

The right-hand side is asymptotically equivalent to $\frac{a^j t^j}{j!}\exp\left(\frac{\gamma_0 j^2}{at}\right)$ whenever $j=j(t)=o(t^{2/3})$ as $t\to\infty$.

Passing to the analysis of

$$R_j(t) := \sum_{k=0}^{j-1}\binom{j}{k}\left(\ell^{*(k)} * \widetilde{\varepsilon}^{*(j-k)}\right)(t), \quad t\geq 0$$

we first check that

$$\mathscr{V}_{\mathbb{R}}(\ell * \widetilde{\varepsilon}) \leq \widetilde{C} < \infty \tag{37}$$

for an absolute constant $C>0$. For $t\in\mathbb{R}$,

$$(\ell*\widetilde{\varepsilon})(t) = \int_{\mathbb{R}}\widetilde{\varepsilon}(t-y)\mathrm{d}\ell(y) = a\int_{-a^{-1}\gamma_0}^{\infty}\widetilde{\varepsilon}(t-y)\mathrm{d}y = a\int_{-\infty}^{t+a^{-1}\gamma_0}\widetilde{\varepsilon}(y)\mathrm{d}y.$$

Thus, (37) holds with $\widetilde{C} := a\int_{\mathbb{R}}|\widetilde{\varepsilon}(y)|\mathrm{d}y$. Put

$$g_{i,j}(t) := \mathscr{V}_{(-\infty,t]}(\ell^{*(i)} * \widetilde{\varepsilon}^{*(j)}), \quad i,j\in\mathbb{N}_0, \quad t\in\mathbb{R}.$$

Then, for $i,j\in\mathbb{N}$, taking into account commutativity of the Lebesgue–Stieltjes convolution we infer

$$\begin{aligned}
g_{i,j}(t) &= \mathscr{V}_{(-\infty,t]}\left((\ell^{*(i-1)} * \widetilde{\varepsilon}^{*(j-1)}) * (\ell*\widetilde{\varepsilon})\right)\\
&\leq \mathscr{V}_{(-\infty,t]}(\ell^{*(i-1)} * \widetilde{\varepsilon}^{*(j-1)})\mathscr{V}_{(-\infty,t]}(\ell*\widetilde{\varepsilon})\\
&\leq \mathscr{V}_{(-\infty,t]}(\ell^{*(i-1)} * \widetilde{\varepsilon}^{*(j-1)})\mathscr{V}_{\mathbb{R}}(\ell*\widetilde{\varepsilon})\\
&\leq \widetilde{C}g_{i-1,j-1}(t), \quad t\in\mathbb{R}.
\end{aligned}$$

Here, we have used the fact that the total variation of the convolution of two functions is bounded by the product of their total variations, see Theorem 1.3.2(c) in [26]. Iterating this inequality we conclude that

$$\begin{aligned}
|R_j(t)| &\leq \sum_{k=0}^{j-1}\binom{j}{k}g_{k,j-k}(t)\\
&\leq \sum_{k\leq j/2}\binom{j}{k}\widetilde{C}^k g_{0,j-2k}(t) + \sum_{j/2<k<j}\binom{j}{k}\widetilde{C}^{j-k}g_{2k-j,0}(t), \quad t\in\mathbb{R}.
\end{aligned} \tag{38}$$

Note that $g_{0,j-2k}(t)\leq \mathscr{V}_{\mathbb{R}}(\widetilde{\varepsilon}^{*(j-2k)}) \leq (\mathscr{V}_{\mathbb{R}}(\widetilde{\varepsilon}))^{j-2k} \leq \widetilde{C}_1^{j-2k}$ for $\widetilde{C}_1 := \int_{\mathbb{R}}|\mathrm{d}\widetilde{\varepsilon}(y)|<\infty$. Therefore, the first sum on the right-hand side of (38) satisfies

$$\begin{aligned}\sum_{k\le j/2}\binom{j}{k}\widetilde{C}^k g_{0,j-2k}(t) &\le \sum_{k\le j/2}\binom{j}{k}\widetilde{C}^k\widetilde{C}_1^{j-2k}\\ &\le (\widetilde{C}\widetilde{C}_1^{-1}+\widetilde{C}_1)^j\\ &= o\left(\frac{a^jt^j}{j!}\left(1+\frac{\gamma_0 j}{at}\right)^j\right),\quad t\to\infty,\end{aligned}$$

for $b^j=b^{j(t)}$ grows slower than $\frac{a^jt^j}{j!}\left(1+\frac{\gamma_0 j}{at}\right)^j$ as $t\to\infty$ for an arbitrary finite constant $b>0$. Now we analyze the second sum on the right-hand side of (38):

$$\begin{aligned}\sum_{j/2<k<j}\binom{j}{k}\widetilde{C}^{j-k}g_{2k-j,0}(t) &= \sum_{j/2<k<j}\binom{j}{k}\widetilde{C}^{j-k}\mathscr{V}_{(-\infty,t]}(\ell^{*(2k-j)})\\ &= \sum_{j/2<k<j}\binom{j}{k}\widetilde{C}^{j-k}\ell^{*(2k-j)}(t)\\ &= \sum_{j/2<k<j}\binom{j}{k}\widetilde{C}^{j-k}\frac{(at+\gamma_0(2k-j))^{2k-j}}{(2k-j)!}\\ &= \sum_{1\le k<j/2}\binom{j}{k}\widetilde{C}^{k}\frac{(at+\gamma_0(j-2k))^{j-2k}}{(j-2k)!}.\end{aligned}$$

Here, the second equality follows from monotonicity of ℓ and the third equality holds for t large enough. It is important for what follows that, for $k<j/2$ and $t>0$,

$$\frac{(at+\gamma_0(j-2k))^{j-2k}}{(j-2k)!}\le\frac{a^{j-2k}t^{j-2k}}{(j-2k)!}\exp\left(\frac{\gamma_0(j-2k)^2}{at}\right).$$

CASE $\gamma_0\ge 0$. We obtain, for $t>0$,

$$\begin{aligned}\sum_{1\le k<j/2}\binom{j}{k}\widetilde{C}^{k}\frac{(at+\gamma_0(j-2k))^{j-2k}}{(j-2k)!} &\le \exp\left(\frac{\gamma_0 j^2}{at}\right)\sum_{1\le k<j/2}\binom{j}{k}\widetilde{C}^k\frac{a^{j-2k}t^{j-2k}}{(j-2k)!}\\ &= \frac{a^jt^j}{j!}\exp\left(\frac{\gamma_0 j^2}{at}\right)\sum_{1\le k<j/2}\frac{(j!)^2}{(j-k)!(j-2k)!}\frac{1}{k!}\frac{\widetilde{C}^k}{a^{2k}t^{2k}}\\ &\le \frac{a^jt^j}{j!}\exp\left(\frac{\gamma_0 j^2}{at}\right)\sum_{k\ge1}j^{3k}\frac{1}{k!}\frac{\widetilde{C}^k}{a^{2k}t^{2k}}\\ &= \frac{a^jt^j}{j!}\exp\left(\frac{\gamma_0 j^2}{at}\right)\left(\exp\left(\frac{\widetilde{C}j^3}{a^2t^2}\right)-1\right).\end{aligned}$$

The last factor converges to zero whenever $j=j(t)=o(t^{2/3})$, whence the claim.

CASE $\gamma_0<0$. Arguing in the same vein it is enough to check that

$$\sum_{1\le k<j/2}\frac{1}{k!}\left(\frac{\widetilde{C}j^3}{a^2t^2}\right)^k\exp\left(\frac{\gamma_0(j-2k)^2}{at}\right)=o\left(\exp\left(\frac{\gamma_0 j^2}{at}\right)\right),\quad t\to\infty$$

which is equivalent to

$$I_t := \sum_{1\le k<j/2} \frac{1}{k!}\left(\frac{\widetilde{C} j^3}{a^2t^2}\right)^k \exp\left(\frac{4|\gamma_0|k(j-k)}{at}\right) = o(1), \quad t\to\infty.$$

Invoking the inequality $\exp\left(\frac{4|\gamma_0|k(j-k)}{at}\right) \le \exp(4|\gamma_0|a^{-1}k)$ for $1\le k<j$ and large enough t we infer

$$I_t \le \sum_{1\le k<j/2} \frac{1}{k!}\left(\frac{\widetilde{C} j^3 \exp(4|\gamma_0|a^{-1})}{a^2t^2}\right)^k \le \exp\left(\frac{\widetilde{C} j^3}{a^2t^2}\exp(4|\gamma_0|a^{-1})\right) - 1 \ \to\ 0, \quad t\to\infty.$$

The proof of Proposition 3.2 is complete. $\square$

Proof of Theorem 2.2. We intend to apply Proposition 3.2 with $f=V$, $a=\mathtt{m}^{-1}$. To this end, it is enough to check that, under the assumptions of Theorem 2.2,

$$\int_{[0,\infty)} y|\mathrm{d}(V(y)-\mathtt{m}^{-1}y)|<\infty.$$

Recall that $V=U*G$ and denote by Id the identity function on $[0,\infty)$, that is, $\mathrm{Id}(t):=t_+ = t\,\mathbb{1}_{[0,\infty)}(t)$ for $t\in\mathbb{R}$. Then

$$V-\mathtt{m}^{-1}\mathrm{Id} = (U-\mathtt{m}^{-1}\mathrm{Id})*G - \mathtt{m}^{-1}(\mathrm{Id}*(1-G)).$$

Using this and integration by parts yields

$$\begin{aligned}
\int_{[0,\infty)} y|\mathrm{d}(V(y)-\mathtt{m}^{-1}y)| &= -\int_{[0,\infty)} y\mathrm{d}\mathscr{V}_{[y,\infty)}(V-\mathtt{m}^{-1}\mathrm{Id})\\
&= \int_{[0,\infty)} \mathscr{V}_{[y,\infty)}(V-\mathtt{m}^{-1}\mathrm{Id})\mathrm{d}y\\
&\le \int_{[0,\infty)} \mathscr{V}_{[y,\infty)}(U-\mathtt{m}^{-1}\mathrm{Id})\mathrm{d}y + \mathtt{m}^{-1}\int_{[0,\infty)} \mathscr{V}_{[y,\infty)}(\mathrm{Id}*(1-G))\mathrm{d}y\\
&= \int_{[0,\infty)} y|\mathrm{d}(U(y)-\mathtt{m}^{-1}y)| + \mathtt{m}^{-1}\int_0^\infty\int_y^\infty (1-G(z))\mathrm{d}z\mathrm{d}y.
\end{aligned}$$

The first summand is finite by Remark 3.1.7(ii) on p. 121 in [10] and the second is finite in view of the assumption $\mathbb{E}\eta^2<\infty$.

The explicit form of γ_0 follows from the decomposition

$$V(t)-\mathtt{m}^{-1}t = \int_{[0,t]}(U(t-y)-\mathtt{m}^{-1}(t-y))\mathrm{d}G(y) - \mathtt{m}^{-1}\int_{[0,t]} y\mathrm{d}G(y) - \mathtt{m}^{-1}t(1-G(t)),$$

in which the first summand converges to $(2\mathtt{m}^2)^{-1}\mathbb{E}\xi^2$ by the dominated convergence theorem, (13) and (14); the second converges to $-\mathtt{m}^{-1}\mathbb{E}\eta$ and the third tends to zero as $t\to\infty$. $\square$

Finally, we give a general result on the behavior of $f^{*(j)}$ for arbitrary $j = j(t) = o(t)$. Unfortunately, this result can seldom be applied to the counting function V but is of independent interest and has at least two merits. On the one hand, it gives a probabilistic explanation of a rather mysterious appearance of the exponent in (32). On the other hand, it may be used for guessing the behavior of V_j for $j = j(t)$ growing at least as fast as $t^{2/3}$.

Proposition 3.3. *Let $(\widetilde{S}_j)_{j\in\mathbb{N}_0}$ be a nondecreasing zero-delayed random walk with $K(t) := \mathbb{P}\{\widetilde{S}_1 \le t\}$ for $t \in \mathbb{R}$. Assume that, for some $a > 0$,*

$$f(t) = at - \int_0^t (1-K(y))\mathrm{d}y, \quad t \ge 0.$$

Then

$$f^{*(j)}(t) = \frac{\mathbb{E}(at-\widetilde{S}_j)_+^j}{j!}, \quad j \in \mathbb{N}, \ t \ge 0.$$

In particular, if $\mathbb{E}\widetilde{S}_1^2 < \infty$ and $j = j(t) = o(t^{2/3})$ as $t \to \infty$, then (32) *holds with $\gamma_0 = -\mathbb{E}\widetilde{S}_1$.*

Remark 3.1. In the setting of Proposition 3.3, assume that $\mathbb{E}\widetilde{S}_1^3 < \infty$ and $j = j(t) = o(t^{3/4})$ as $t \to \infty$. We state, without going into details (which become rather technical), that

$$\mathbb{E}\big(at-\widetilde{S}_j\big)_+^j \ \sim \ a^j t^j \exp\big(\gamma_0 j^2/t + (\gamma_1/2-\gamma_0^2)j^3/t^2\big), \quad t \to \infty,$$

where $\gamma_0 = -\mathbb{E}\widetilde{S}_1$ and $\gamma_1 := \mathbb{E}\widetilde{S}_1^2$.

The proof of Proposition 3.3 will be given in the Appendix.

3.3. Proof of Theorem 2.3

Our proof of Theorem 2.3 relies on counterparts for perturbed random walks of some standard renewal-theoretic results. We start by recalling that the renewal function U is subadditive, which means that

$$U(x+y) \le U(x) + U(y), \quad x, y \in \mathbb{R}.$$

This follows, for instance, from formula (5.7) on p. 58 in [12]. The counterpart of this inequality for the function V defined by

$$V(x) = \mathbb{E}N(x) = \sum_{i\ge 1} \mathbb{P}\{S_{i-1} + \eta_i \le x\}, \quad x \in \mathbb{R}$$

is

$$V(x+y) - V(x) \le U(y), \quad x, y \in \mathbb{R}. \tag{39}$$

Indeed, for $x,y\geq 0$,

$$V(x+y)-V(x) = \mathbb{E}(U(x+y-\eta)-U(x-\eta))\,\mathbb{1}_{\{\eta\leq x\}}+\mathbb{E}U(x+y-\eta)\,\mathbb{1}_{\{x<\eta\leq x+y\}} \leq U(y)(\mathbb{P}\{\eta\leq x\}+\mathbb{P}\{x<\eta\leq x+y\})\leq U(y) \tag{40}$$

having utilized subadditivity and monotonicity of U for the penultimate inequality. If $x,y<0$, then both sides of (39) are zero. Finally, we use monotonicity of V to obtain: if $x<0$ and $y\geq 0$, then $V(x+y)-V(x)=V(x+y)\leq V(y)\leq U(y)$; and if $x\geq 0$ and $y<0$, then $V(x+y)-V(x)\leq 0=U(y)$.

Lemmas 3.1 and 3.2 below are counterparts for the function V of Blackwell's theorem and the key renewal theorem, respectively. Observe that the presence of the η_i plays no role, and the results are of the same form as for renewal function U.

Lemma 3.1. *Let $h>0$ be any fixed number.*
(a) Assume that the distribution of ξ is nonlattice and $\mathtt{m}=\mathbb{E}\xi<\infty$. Then

$$\lim_{t\to\infty}(V(t+h)-V(t))=\mathtt{m}^{-1}h.$$

(b) Assume that $\mathtt{m}=\infty$ (the assumption that the distribution of ξ is nonlattice is not needed). Then

$$\lim_{t\to\infty}(V(t+h)-V(t))=0. \tag{41}$$

Lemma 3.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a dRi function on $\mathbb{R}$.*
(a) Assume that $\mathtt{m}<\infty$ and that the distribution of ξ is nonlattice. Then

$$\lim_{t\to\infty}\int_{[0,\infty)}f(t-y)\mathrm{d}V(y)=\mathtt{m}^{-1}\int_{\mathbb{R}}f(y)\mathrm{d}y.$$

(b) Assume that $\mathtt{m}=\infty$ (the assumption that the distribution of ξ is nonlattice is not needed). Then

$$\lim_{t\to\infty}\int_{[0,\infty)}f(t-y)\mathrm{d}V(y)=0.$$

If f is dRi on $[0,\infty)$ or $(-\infty,0]$, then the ranges of integration $[0,\infty)$ and $\mathbb{R}$ should be replaced with $[0,t]$ and $[0,\infty)$ or $[t,\infty)$ and $(-\infty,0]$, respectively.

The proofs of Lemmas 3.1 and 3.2 are postponed to the Appendix.

In some cases the precision of Lemma 3.2 is not needed. In this situation the following 'light' version, borrowed from Lemma 9.1 in [15], may suffice.

Lemma 3.3. *Let $f:\mathbb{R}\to[0,\infty)$ be a dRi function on $\mathbb{R}$. Then for some $r>0$ and all $x\in\mathbb{R}$*

$$\int_{[0,\infty)} f(x-y)\mathrm{d}V(y)\leq r. \tag{42}$$

If f is dRi on $[0,\infty)$ or $(-\infty,0]$, then the range of integration $[0,\infty)$ should be replaced with $[0,x]$ or $[x,\infty)$ and then (42) holds for all $x\geq 0$ or all $x\leq 0$, respectively.

Having Lemmas 3.1, 3.2 and 3.3 at our disposal we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. For $t\geq 0$, put $g(t):=\int_{[0,t]} f(t-y)\mathrm{d}V(y)$ and $I:=\mathtt{m}^{-1}\int_0^\infty f(y)\mathrm{d}y$. By Lemma 3.2(a), given $\varepsilon>0$ there exists $t_0>0$ such that $|g(t)-I|\leq\varepsilon$ whenever $t\geq t_0$. Also, by Lemma 3.3, $g(t)\leq J$ for some $J>0$ and all $t\geq 0$. Hence, for $t\geq t_0$, by the associativity

$$\begin{aligned}(f*V_j)(t)=(f*(V*V_{j-1}))(t)&=(g*V_{j-1})(t)\\&=\int_{[0,t]} g(t-y)\mathrm{d}V_{j-1}(y)\\&=\int_{[0,t-t_0]} g(t-y)\mathrm{d}V_{j-1}(y)+\int_{(t-t_0,t]} g(t-y)\mathrm{d}V_{j-1}(y)\\&\leq (I+\varepsilon)V_{j-1}(t)+J(V_{j-1}(t)-V_{j-1}(t-t_0)).\end{aligned} \tag{43}$$

We claim that

$$\lim_{t\to\infty}\frac{V_{j(t)-1}(t)-V_{j(t)-1}(t-t_0)}{V_{j(t)-1}(t)}=0. \tag{44}$$

Note that (44) is not a direct consequence of the elementary renewal theorem, for the theorem provides the asymptotics of $V_{j(t-t_0)-1}(t-t_0)$ rather than $V_{j(t)-1}(t-t_0)$ which is actually needed for (44). To prove (44) we write with the help of (39)

$$0\leq V_{j(t)-1}(t)-V_{j(t)-1}(t-t_0)=\int_{[0,t]}(V(t-y)-V(t-t_0-y))\mathrm{d}V_{j(t)-2}(y)\leq U(t_0)V_{j(t)-2}(t),$$

for all $t\geq 0$. Thus, (44) follows from

$$\lim_{t\to\infty}\frac{V_{j(t)-2}(t)}{V_{j(t)-1}(t)}=0,$$

which is a consequence of Theorems 2.1 and 2.2 applied with $j=j(t)-1$ and $j=j(t)-2$.

Combining (43) and (44) we obtain

$$\limsup_{t\to\infty}\frac{(f*V_j)(t)}{V_{j-1}(t)}\leq I.$$

The converse inequality for the limit inferior follows analogously. The remaining statements of Theorem 2.3 are secured by Theorems 2.1 and 2.2. $\square$

Appendix A. Proof of Proposition 3.3

Proof. Replacing K with $t \mapsto K(at)$ we can and do assume that $a=1$, that is, $f(t) = \int_0^t K(y)\mathrm{d}y$ or, in short, $f = K * \mathrm{Id}$. Then, by the commutativity of the convolution,

$$f^{*(j)}(t) = \left((\mathrm{Id})^{*(j)} * K^{*(j)}\right)(t) = \int_{[0,t]} \frac{(t-y)^j}{j!} \mathrm{d}K^{*(j)}(y) = \frac{\mathbb{E}(t-\widetilde{S}_j)_+^j}{j!}, \quad t \geq 0.$$

If $\mathbb{E}\widetilde{S}_1^2 < \infty$, then $j = j(t) = o(t^{2/3})$ as $t \to \infty$ implies that

$$\mathbb{E}(t-\widetilde{S}_j)_+^j \sim t^j \exp\left(\frac{\gamma_0 j^2}{t}\right), \quad t \to \infty. \tag{45}$$

This can be justified as follows. We first note that $\gamma_0 < 0$. Further, in the decomposition

$$\mathbb{E}\left(1-\frac{\widetilde{S}_j}{t}\right)_+^j = \mathbb{E}(e^{j\log(1-\widetilde{S}_j/t)}\, \mathbb{1}_{\{\widetilde{S}_j \leq t/2\}}) + \mathbb{E}\left(1-\frac{\widetilde{S}_j}{t}\right)_+^j \mathbb{1}_{\{\widetilde{S}_j \in (t/2,t)\}} \tag{46}$$

the second summand is bounded by 2^{-j} and $2^{-j} = o\left(\exp\left(\frac{\gamma_0 j^2}{t}\right)\right)$ as $t \to \infty$, for $j^2/t = o(j)$. The first summand in (46) can be bounded with the help of the inequalities

$$-x-x^2 \leq \log(1-x) \leq -x, \quad x \in [0,\, 1/2] \quad \text{and} \quad 1-x \leq e^{-x}, \quad x \in \mathbb{R}.$$

Indeed, we obtain, for $j \geq 4$,

$$\begin{aligned}\mathbb{E}e^{-j\widetilde{S}_j/t}\left(1-\frac{j\widetilde{S}_j^2}{t^2}\right) &\leq \mathbb{E}e^{-j\widetilde{S}_j/t}\left(1-\frac{j\widetilde{S}_j^2}{t^2}\right)\mathbb{1}_{\{\widetilde{S}_j \leq t/2\}} \leq \mathbb{E}e^{-j\widetilde{S}_j/t}e^{-j\widetilde{S}_j^2/t^2}\,\mathbb{1}_{\{\widetilde{S}_j \leq t/2\}} \\ &\leq \mathbb{E}(e^{j\log(1-\widetilde{S}_j/t)}\,\mathbb{1}_{\{\widetilde{S}_j \leq t/2\}}) \leq \mathbb{E}e^{-j\widetilde{S}_j/t}\,\mathbb{1}_{\{\widetilde{S}_j \leq t/2\}} \leq \mathbb{E}e^{-j\widetilde{S}_j/t}.\end{aligned}$$

For $\lambda \geq 0$, put $\phi(\lambda) := \mathbb{E}e^{-\lambda \widetilde{S}_1}$. In view of $\mathbb{E}\widetilde{S}_1^2 < \infty$ we infer

$$\mathbb{E}e^{-j\widetilde{S}_j/t} = \phi^j(j/t) = \left(1 + \frac{\gamma_0 j}{t} + O\left(\frac{j^2}{t^2}\right)\right)^j.$$

The right-hand side is asymptotically equivalent to $\exp(\gamma_0 j^2/t)$ as $t \to \infty$ under the assumption $j = j(t) = o(t^{2/3})$. Finally, the relation

$$\mathbb{E}e^{-j\widetilde{S}_j/t}\left(\frac{j\widetilde{S}_j^2}{t^2}\right) = o\left(\mathbb{E}e^{-j\widetilde{S}_j/t}\right), \quad t \to \infty$$

can be checked using the equality $\mathbb{E}e^{-j\widetilde{S}_j/t}\widetilde{S}_j^2 = \frac{\partial^2}{\partial\lambda^2}(\phi^j(\lambda))\Big|_{\lambda=j/t}$ in conjunction with the assumptions $j = j(t) = o(t^{2/3})$ and $\mathbb{E}\widetilde{S}_1^2 < \infty$. $\square$

Appendix B. Proof of Lemma 3.1

PROOF OF PART (a). According to Blackwell's theorem (Proposition 2.2),

$$\lim_{t\to\infty}(U(t+h)-U(t)) = \mathtt{m}^{-1}h. \tag{47}$$

In view of (47), $\lim_{t\to\infty}(U(t+h-\eta)-U(t-\eta))\,\mathbb{1}_{\{\eta\le t-t^{1/2}\}} = \mathtt{m}^{-1}h$ a.s. Recalling (39) we infer

$$\lim_{t\to\infty}\mathbb{E}(U(t+h-\eta)-U(t-\eta))\,\mathbb{1}_{\{\eta\le t-t^{1/2}\}} = \mathtt{m}^{-1}h$$

by Lebesgue's dominated convergence theorem. Another appeal to (39) yields

$$\mathbb{E}(U(t+h-\eta)-U(t-\eta))\,\mathbb{1}_{\{t-t^{1/2}<\eta\le t\}} \le U(h)\mathbb{P}\{t-t^{1/2}<\eta\le t\},$$

and the right-hand side converges to 0 as $t\to\infty$. Finally, by monotonicity,

$$\mathbb{E}U(t+h-\eta)\,\mathbb{1}_{\{t<\eta\le t+h\}} \le U(h)\mathbb{P}\{t<\eta\le t+h\},$$

and the right-hand side converges to 0 as $t\to\infty$. Invoking the first equality in (40) with $x=t$ and $y=h$ completes the proof of part (a).

PROOF OF PART (b). If the distribution of ξ is nonlattice, then, by Blackwell's theorem,

$$\lim_{t\to\infty}(U(t+h)-U(t)) = 0. \tag{48}$$

If the distribution of ξ is d-lattice, then, by Blackwell's theorem (Proposition 2.2), (48) holds for $h=nd$, $n\in\mathbb{N}$. However, using monotonicity of U we can ensure that (48) holds for any fixed $h>0$ in both nonlattice and lattice cases. With this at hand, repeating verbatim the proof of part (a) we arrive at (41).

Appendix C. Proof of Lemma 3.2

PROOF OF PART (a). We only prove the claim under the assumption that f is dRi on $\mathbb{R}$ which is equivalent to the fact that f_+ and f_- (nonnegative and nonpositive parts of f) are dRi on $\mathbb{R}$. Thus, we can and do assume that $f\ge 0$ on $\mathbb{R}$. Obviously, it is enough to show that

$$\lim_{t\to\infty}\int_{[0,t]} f(t-y)\mathrm{d}V(y) = \mathtt{m}^{-1}\int_0^\infty f(y)\mathrm{d}y$$

and that

$$\lim_{t\to\infty}\int_{(t,\infty)} f(t-y)\mathrm{d}V(y) = \mathtt{m}^{-1}\int_{-\infty}^0 f(y)\mathrm{d}y.$$

The proof of the first relation with U replacing V can be found on p. 241–242 in [25]. We only check the second limit relation by following closely aforementioned Resnick's proof.

We proceed via three steps complicating successively the structure of f.

STEP 1. Suppose first that

$$f(t) = \mathbb{1}_{[(n-1)h,\,nh)}(t), \quad t<0$$

for fixed nonpositive integer n and $h>0$. Then $f(t-y)=1$ if, and only if, $y \in (t-nh, t-(n-1)h]$ which entails

$$\int_{(t,\infty)} f(t-y)\mathrm{d}V(y) = V(t-(n-1)h) - V(t-nh).$$

By Lemma 3.1(a), the last difference tends to $\mathtt{m}^{-1}h$ as $t\to\infty$, thereby proving that

$$\lim_{t\to\infty}\int_{(t,\infty)} f(t-y)\mathrm{d}V(y) = \mathtt{m}^{-1}h = \mathtt{m}^{-1}\int_{-\infty}^{0} f(y)\mathrm{d}y.$$

STEP 2. Suppose now that

$$f(t) = \sum_{n\le 0} c_n\, \mathbb{1}_{[(n-1)h,\,nh)}(t), \quad t<0,$$

where $\big(c_n\big)_{n\le 0}$ is a sequence of nonnegative numbers satisfying $\sum_{n\le 0} c_n<\infty$. An argument similar to that used in the previous step enables us to assert that

$$\int_{(t,\infty)} f(t-y)\mathrm{d}V(y) = \sum_{n\le 0} c_n\big(V(t-(n-1)h) - V(t-nh)\big).$$

Using Lemma 3.1(a) in combination with (39) we infer with the help of Lebesgue's dominated convergence theorem,

$$\lim_{t\to\infty}\int_{(t,\infty)} f(t-y)\mathrm{d}V(y) = \mathtt{m}^{-1}h\sum_{n\le 0} c_n = \mathtt{m}^{-1}\int_{-\infty}^{0} f(y)\mathrm{d}y.$$

STEP 3. Let now f be an arbitrary nonnegative dRi function on $\mathbb{R}$ (actually, for the present proof it is enough it is dRi on $(-\infty,0)$). For each $h>0$, put

$$\overline{f}_h(t) := \sum_{n\le 0}\sup_{(n-1)h\le y<nh} f(y)\,\mathbb{1}_{[(n-1)h,\,nh)}(t), \quad t<0$$

and

$$\underline{f}_h(t) := \sum_{n\le 0}\inf_{(n-1)h\le y<nh} f(y)\,\mathbb{1}_{[(n-1)h,\,nh)}(t), \;\; t<0.$$

By the definition of direct Riemann integrability,

$$\sum_{n\le 0}\sup_{(n-1)h\le y<nh} f(y)<\infty \quad \text{and} \quad \sum_{n\le 0}\inf_{(n-1)h\le y<nh} f(y)<\infty$$

for each $h>0$. Thus, the functions $\overline{f}_h$ and $\underline{f}_h$ have the same structure as the functions discussed in Step 2. According to the result of Step 2,

$$\lim_{t\to\infty}\int_{(t,\infty)}\overline{f}_h(t-y)\mathrm{d}V(y)=\mathtt{m}^{-1}h\sum_{n\leq 0}\sup_{(n-1)h\leq y<nh}f(y)=:\mathtt{m}^{-1}\overline{\sigma}(h)$$

and

$$\lim_{t\to\infty}\int_{(t,\infty)}\underline{f}_h(t-y)\mathrm{d}V(y)=\mathtt{m}^{-1}h\sum_{n\leq 0}\inf_{(n-1)h\leq y<nh}f(y)=:\mathtt{m}^{-1}\underline{\sigma}(h)$$

for all $h>0$. Since, for each $h>0$,

$$\underline{f}_h(t)\leq f(t)\leq\overline{f}_h(t),\quad t<0,$$

it follows that

$$\begin{aligned}\mathtt{m}^{-1}\underline{\sigma}(h)&=\liminf_{t\to\infty}\int_{(t,\infty)}\underline{f}_h(t-y)\mathrm{d}V(y)\\&\leq\liminf_{t\to\infty}\int_{(t,\infty)}f(t-y)\mathrm{d}V(y)\\&\leq\limsup_{t\to\infty}\int_{(t,\infty)}f(t-y)\mathrm{d}V(y)\\&\leq\limsup_{t\to\infty}\int_{(t,\infty)}\overline{f}_h(t-y)\mathrm{d}V(y)\\&=\mathtt{m}^{-1}\overline{\sigma}(h).\end{aligned}$$

We have $\lim_{h\to 0+}\big(\overline{\sigma}(h)-\underline{\sigma}(h)\big)=0$ by the definition of direct Riemann integrability. Also, it is known that $\lim_{h\to 0+}\overline{\sigma}(h)=\int_{-\infty}^0 f(y)\mathrm{d}y$. Letting $h\to 0+$ in the last chain of inequalities completes the proof of part (a).

PROOF OF PART (b). Use part (b) of Lemma 3.1 in place of part (a) and proceed as above. This finishes the proof of Lemma 3.2.

Acknowledgements

We thank the anonymous referee for many useful suggestions which greatly improved the presentation of our results.

Funding Information

The present work was supported by the National Research Foundation of Ukraine (project 2020.02/0014 “Asymptotic regimes of perturbed random walks: on the edge of modern and classical probability”).

Competing Interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] G. Alsmeyer, A. Iksanov and A. Marynych, *Functional limit theorems for the number of occupied boxes in the Bernoulli sieve*. Stoch. Proc. Appl. **127** (2017), 995–1017.

[2] S. Asmussen, *Applied probability and queues*. 2nd Edition, Springer-Verlag, 2003.

[3] J. D. Biggins, *Chernoff's theorem in the branching random walk*. J. Appl. Probab. **14** (1977), 630–636.

[4] J. D. Biggins, *Growth rates in the branching random walk*. Z. Wahrscheinlichkeitstheorie Verw. Geb. **48** (1979), 17–34.

[5] J. D. Biggins, *Uniform convergence of martingales in the branching random walk*. Ann. Probab. **20** (1992), 137–151.

[6] D. Buraczewski, B. Dovgay and A. Iksanov, *On intermediate levels of nested occupancy scheme in random environment generated by stick-breaking I*. Electron. J. Probab. **25**, paper no. 123, 24 pp.

[7] H. Carlsson and O. Nerman, *An alternative proof of Lorden's renewal inequality*. Adv. Appl. Probab. **18** (1986), 1015–1016.

[8] C. Dong and A. Iksanov, *Weak convergence of random processes with immigration at random times*. J. Appl. Probab. **57** (2020), 250–265.

[9] J.-J. Duchamps, J. Pitman and W. Tang, *Renewal sequences and record chains related to multiple zeta sums*. Trans. Amer. Math. Soc. **371** (2019), 5731–5755.

[10] J. B. G. Frenk, *On Banach algebras, renewal measures and regenerative processes*. Stichting Mathematisch Centrum, 1987.

[11] A. Gnedin, A. Hansen and J. Pitman, *Notes on the occupancy problem with infinitely many boxes: general asymptotics and power laws*. Probab. Surv. **4** (2007), 146–171.

[12] A. Gut, *Stopped random walks. Limit theorems and applications*. 2nd Edition, Springer, 2009.

[13] C. Holmgren and S. Janson, *Fringe trees, Crump–Mode–Jagers branching processes and m-ary search trees*. Probab. Surv. **14** (2017), 53–154.

[14] A. Iksanov, *Renewal theory for perturbed random walks and similar processes*. Birkhäuser, 2016.

[15] A. Iksanov, A. Marynych and I. Samoilenko, *On intermediate levels of nested occupancy scheme in random environment generated by stick-breaking II*. Preprint (2020) available at https://arxiv.org/abs/2011.12231

[16] A. Iksanov, B. Rashytov and I. Samoilenko, *Renewal theory for iterated perturbed random walks on a general branching process tree: early levels*. Preprint (2021) available at https://arxiv.org/abs/2105.02846

[17] A. Iksanov, A. Pilipenko and I. Samoilenko, *Functional limit theorems for the maxima of perturbed random walks and divergent perpetuities in the M_1-topology*. Extremes. **20** (2017), 567–583.

[18] A. Iksanov and B. Rashytov, *A functional limit theorem for general shot noise processes*. J. Appl. Probab. **57** (2020), 280–294.

[19] S. Karlin, *Central limit theorems for certain infinite urn schemes*. J. Math. Mech. **17** (1967), 373–401.

[20] K. V. Mitov and E. Omey, *Renewal processes*. Springer, 2014.

[21] N. R. Mohan, *Teugels' renewal theorem and stable laws*. Ann. Probab., **4(5)** (1976), 863–868.

[22] J. Pitman and W. Tang, *Regenerative random permutations of integers*. Ann. Probab. **47** (2019), 1378–1416.

[23] J. Pitman and Yu. Yakubovich, *Gaps and interleaving of point processes in sampling from a residual allocation model*. Bernoulli. **25** (2019), 3623–3651.

[24] B. Rashytov, *Power moments of first passage times for some oscillating perturbed random walks*. Theory Stoch. Proc. **23(39)** (2018), 93–97.

[25] S. I. Resnick, *Adventures in stochastic processes*. 3rd printing, Birkhäuser, 2002.

[26] W. Rudin, *Fourier analysis on groups*. John Wiley & Sons, 1962.

[27] P. K. Sen, *Weak convergence of an iterated renewal process*. J. Appl. Probab. **18** (1981), 291–296.

[28] M. S. Sgibnev, *Renewal theorem in the case of an infinite variance*. Sib. Math. J. **22** (1982), 787–796.