

Functional limit theorems for the number of busy servers in a $G/G/\infty$ queue

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Abstract

We discuss weak convergence of the number of busy servers in a $G/G/\infty$ queue in the J_1 -topology on the Skorokhod space. We prove two functional limit theorems, with random and nonrandom centering, respectively, thereby solving two open problems stated in [16]. A new integral representation for the limit Gaussian process is given.

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1 Introduction

Let $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η) where both ξ and η are positive. No condition is imposed on the dependence structure between ξ and η .

Define

$$K(t) := \sum_{k \geq 0} \mathbb{1}_{\{S_k + \eta_{k+1} \leq t\}} \quad \text{and} \quad Z(t) := \sum_{k \geq 0} \mathbb{1}_{\{S_k \leq t < S_k + \eta_{k+1}\}}, \quad t \geq 0,$$

where¹ $(S_k)_{k \in \mathbb{N}_0}$ is the zero-delayed ordinary random walk with increments ξ_k for $k \in \mathbb{N}$, i.e., $S_0 = 0$ and $S_k = \xi_1 + \dots + \xi_k$, $k \in \mathbb{N}$. In a $G/G/\infty$ -queueing system, where customers arrive at times $S_0 = 0 < S_1 < S_2 < \dots$ and are immediately served by one of infinitely many idle servers, the service time of the k th customer being η_{k+1} , $K(t)$ gives the number of customers served up to and including time $t \geq 0$, whereas $Z(t)$ gives the number of busy servers at time t . Some other interpretations of $Z(t)$ can be found in [12]. The process $(Z(t))_{t \geq 0}$ was also used to model the

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¹ $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

number of active sources in a communication network (for instance, active sessions in a computer network) [13, 16, 17].

From a more theoretical viewpoint, $K(t)$ is the number of visits to the interval $[0, t]$ of a *perturbed random walk* $(S_k + \eta_{k+1})_{k \in \mathbb{N}_0}$ and $Z(t)$ is the difference between the number of visits to $[0, t]$ of the ordinary random walk $(S_k)_{k \in \mathbb{N}_0}$ and $(S_k + \eta_{k+1})_{k \in \mathbb{N}_0}$. To proceed, we need a definition. Denote by $X := (X(t))_{t \geq 0}$ a random process arbitrarily dependent on ξ . Let $(X_k, \xi_k)_{k \in \mathbb{N}}$ be i.i.d. copies of the pair (X, ξ) . Following [8] we call *random process with immigration* the random process $(Y(t))_{t \geq 0}$ defined by

$$Y(t) := \sum_{k \geq 0} X_{k+1}(t - S_k) \mathbb{1}_{\{S_k \leq t\}}, \quad t \geq 0.$$

If X is deterministic, the random process with immigration becomes a classical renewal shot noise process. Getting back to the mainstream we conclude that both $(K(t))_{t \geq 0}$ and $Z := (Z(t))_{t \geq 0}$ are particular instances of the random process with immigration which correspond to $X(t) = \mathbb{1}_{\{\eta \leq t\}}$ and $X(t) = \mathbb{1}_{\{\eta > t\}}$, respectively.

Let $D := D[0, \infty)$ be the Skorokhod² space of real-valued functions on $[0, \infty)$, which are right-continuous on $[0, \infty)$ with finite limits from the left at each positive point. We shall write $\xrightarrow{J_1}$ and $\xrightarrow{\mathbb{P}}$ to denote weak convergence in the J_1 -topology on D and convergence in probability, respectively. The classical references concerning the J_1 -topology are [2, 11, 15].

In this paper we shall prove weak convergence of $(Z(ut))_{u \geq 0}$, properly centered and normalized, in the J_1 -topology on D as $t \rightarrow \infty$. The same problem for $(K(ut))_{u \geq 0}$ which is much simpler was solved in [1]. We start with a functional limit theorem with a random centering.

Theorem 1.1. *Assume that $\mu := \mathbb{E}\xi \in (0, \infty)$ and that*

$$1 - F(t) = \mathbb{P}\{\eta > t\} \sim t^{-\beta} \ell(t), \quad t \rightarrow \infty \quad (1.1)$$

for some $\beta \in [0, 1)$ and some ℓ slowly varying at ∞ . Then

$$\frac{\sum_{k \geq 0} (\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}})}{\sqrt{\mu^{-1} \int_0^t (1 - F(y)) dy}} \xrightarrow{J_1} V_\beta(u), \quad t \rightarrow \infty, \quad (1.2)$$

where $V_\beta := (V_\beta(u))_{u \geq 0}$ is a centered Gaussian process with

$$\mathbb{E} V_\beta(u) V_\beta(s) = u^{1-\beta} - (u - s)^{1-\beta}, \quad 0 \leq s \leq u. \quad (1.3)$$

In the case where ξ and η are independent weak convergence of the finite-dimensional distributions in (1.2) was proved in Proposition 3.2 of [16]. In the general case treated here where ξ and η are arbitrarily dependent the aforementioned convergence outside zero (i.e., weak convergence of $(Z_t^*(u_1), \dots, Z_t^*(u_n))$ for any $n \in \mathbb{N}$ and any $0 < u_1 < \dots < u_n < \infty$, where $Z_t^*(u)$ denotes the left-hand side in (1.2)) follows from a specialization of Proposition 2.1 in [8]. In Section 5.2 of [16] the authors write: ‘We suspect that the’ finite-dimensional ‘convergence can be considerably strengthened’. Our Proposition 1.1 confirms their conjecture.

Also, the authors of [16] ask on p. 154: ‘When can the random centering’ in (1.2) ‘be replaced by a non-random centering?’ Our second result states that such

²The Skorokhod spaces $D(0, \infty)$ and $D[0, T]$ for $T > 0$ which appear below are defined similarly.

a replacement is possible whenever ξ possesses finite moments of sufficiently large positive orders. Our approach is essentially based on the decomposition³

$$\begin{aligned}
& \sum_{k \geq 0} \mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - \mu^{-1} \int_0^{ut} (1 - F(y)) dy \\
&= \left(\sum_{k \geq 0} \mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - \sum_{k \geq 0} \mathbb{E}(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} | S_k) \right) \\
&+ \left(\sum_{k \geq 0} \mathbb{E}(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} | S_k) - \mu^{-1} \int_0^{ut} (1 - F(y)) dy \right) \\
&= \sum_{k \geq 0} \left(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} \right) \\
&+ \left(\sum_{k \geq 0} (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} - \mu^{-1} \int_0^{ut} (1 - F(y)) dy \right). \quad (1.4)
\end{aligned}$$

Weak convergence on D of the first summand on the right-hand side, normalized by $\sqrt{\mu^{-1} \int_0^t (1 - F(y)) dy}$, was treated in Theorem 1.1. Thus, we are left with analyzing weak convergence of the second summand.

Theorem 1.2. *Suppose that condition (1.1) holds. If $\mathbb{E}\xi^r < \infty$ for some $r > 2(1 - \beta)^{-1}$, then*

$$\frac{\sum_{k \geq 0} \mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - \mu^{-1} \int_0^{ut} (1 - F(y)) dy}{\sqrt{\mu^{-1} \int_0^t (1 - F(y)) dy}} \xrightarrow{J_1} V_\beta(u), \quad t \rightarrow \infty, \quad (1.5)$$

where $\mu = \mathbb{E}\xi < \infty$ and V_β is a centered Gaussian process with covariance (1.3).

Under the assumption that ξ and η are independent weak convergence of the *one-dimensional* distributions in (1.5) was proved in Theorem 2 of [12]. Note that regular variation condition (1.1) is not needed for this convergence to hold. Weak convergence of the *finite-dimensional* distributions in (1.5) takes place under (1.1) and the weaker assumption $\mathbb{E}\xi^2 < \infty$. We do not know whether (1.1) and the second moment assumption are sufficient for weak convergence on D . More generally, weak convergence of the finite-dimensional distributions of $Z(ut)$, properly⁴ normalized and centered, holds whenever the distribution of ξ belongs to the domain of attraction of an α -stable distribution, $\alpha \in (0, 2] \setminus \{1\}$, see Theorem 3.3.21 in [7] which is a specialization of Theorems 2.4 and 2.5 in [8]. We do not state these results here because in this paper we are only interested in weak convergence on D .

The rest of the paper is structured as follows. Theorems 1.1 and 1.2 are proved in Sections 2 and 3, respectively. In Section 4 we discuss an integral representation of the limit process V_β which seems to be new. The appendix collects several auxiliary results.

³Investigating Z directly, i.e., not using (1.4), seems to be a formidable task unless ξ and η are independent, and the distribution of ξ is exponential (for the latter situation, see [17] and references therein). We note in passing that our Theorem 1.2 includes Theorem 1 in [17] as a particular case.

⁴The normalization is not necessarily of the form $\sqrt{\mu^{-1} \int_0^t (1 - F(y)) dy}$, and the limit process is not necessarily V_β .

2 Proof of Theorem 1.1

We start by observing that

$$a(t) := \sum_{k=0}^{[t]+1} (1 - F(k)) \sim \int_0^t (1 - F(y)) dy \sim (1 - \beta)^{-1} t^{1-\beta} \ell(t) \quad (2.1)$$

as $t \rightarrow \infty$, where the second equivalence follows from Karamata's theorem (Proposition 1.5.8 in [3]). In particular, the first equivalence enables us to replace the integral in the denominator of (1.2) with the sum. For each $t, u \geq 0$, denote by $\widehat{Z}(ut)$ the first summand in decomposition (1.4), i.e.,

$$\begin{aligned} \widehat{Z}(ut) &:= \sum_{k \geq 0} \left(\mathbb{1}_{\{S_k \leq ut < S_{k+\eta_{k+1}}\}} - (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} \right) \\ &= \sum_{k \geq 0} \left(\mathbb{1}_{\{S_{k+\eta_{k+1}} \leq ut\}} - F(ut - S_k) \mathbb{1}_{\{S_k \leq ut\}} \right) \end{aligned}$$

and then set

$$Z_t(u) := \frac{\sum_{k \geq 0} \left(\mathbb{1}_{\{S_k \leq ut < S_{k+\eta_{k+1}}\}} - (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} \right)}{\sqrt{a(t)}} = \frac{\widehat{Z}(ut)}{\sqrt{a(t)}}, \quad u \geq 0.$$

Our proof of Theorem 1.1 is similar to the proof of Theorem 1 in [17] which treats the case where ξ and η are independent, and the distribution of ξ is exponential (Poisson case). Lemma 2.1 given below is concerned with inevitable technical complications that appear outside the Poisson case. Put

$$\nu(t) := \inf\{k \in \mathbb{N}_0 : S_k > t\}, \quad t \in \mathbb{R}$$

and note that the random variable $\nu(1)$ has finite moments of all positive orders by Lemma 5.2.

Lemma 2.1. *Let $l \in \mathbb{N}$ and $0 \leq v < u$. For any chosen $A > 1$ and $\rho \in (0, 1 - \beta)$ there exist $t_1 > 1$ such that*

$$\mathbb{E}|Z_t(u) - Z_t(v)|^{2l} \leq c(l)(u - v)^{l(1-\beta-\rho)}$$

whenever $u - v < 1$ and $(u - v)t \geq t_1$, where $c(l) := 2C_l(4A)^l(u - v)^{l(1-\beta-\rho)}\mathbb{E}(\nu(1))^l$ and C_l is a finite positive constant.

Proof. With $u, v \geq 0$ fixed, $\widehat{Z}(ut) - \widehat{Z}(vt)$ equals the terminal value of the martingale $(R(k, t), \mathcal{F}_k)_{k \in \mathbb{N}_0}$, where $R(0, t) := 0$,

$$\begin{aligned} R(k, t) &:= \sum_{j=0}^{k-1} \left((\mathbb{1}_{\{S_j + \eta_{j+1} \leq ut\}} - F(ut - S_j) \mathbb{1}_{\{S_j \leq ut\}}) \right. \\ &\quad \left. - (\mathbb{1}_{\{S_j + \eta_{j+1} \leq vt\}} - F(vt - S_j) \mathbb{1}_{\{S_j \leq vt\}}) \right), \end{aligned}$$

$\mathcal{F}_0 := \{\Omega, \emptyset\}$ and $\mathcal{F}_k := \sigma((\xi_j, \eta_j) : 1 \leq j \leq k)$. We use the Burkholder-Davis-Gundy inequality (Theorem 11.3.2 in [4]) to obtain for any $l \in \mathbb{N}$

$$\begin{aligned} &\mathbb{E}(\widehat{Z}(ut) - \widehat{Z}(vt))^{2l} \\ &\leq C_l \left(\mathbb{E} \left(\sum_{k \geq 0} \mathbb{E}((R(k+1, t) - R(k, t))^2 | \mathcal{F}_k) \right)^l + \sum_{k \geq 0} \mathbb{E}(R(k+1, t) - R(k, t))^{2l} \right) \\ &=: C_l(I_1(t) + I_2(t)) \end{aligned} \quad (2.2)$$

for a positive constant C_l . We shall show that

$$I_1(t) \leq 2^l \mathbb{E}(\nu(1))^l (a((u-v)t))^l, \quad t \geq 0 \quad (2.3)$$

and that

$$I_2(t) \leq 2^{2l} \mathbb{E}\nu(1) a((u-v)t), \quad t \geq 0. \quad (2.4)$$

PROOF OF (2.3). We first observe that

$$\begin{aligned} & \sum_{k \geq 0} \mathbb{E} \left((R(k+1, t) - R(k, t))^2 | \mathcal{F}_k \right) \\ &= \int_{(vt, ut]} F(ut-y)(1-F(ut-y)) d\nu(y) \\ &+ \int_{[0, vt]} (F(ut-y) - F(vt-y))(1-F(ut-y) + F(vt-y)) d\nu(y) \\ &\leq \int_{(vt, ut]} (1-F(ut-y)) d\nu(y) + \int_{[0, vt]} (F(ut-y) - F(vt-y)) d\nu(y) \end{aligned}$$

whence

$$\begin{aligned} I_1(t) &\leq 2^{l-1} \left(\mathbb{E} \left(\int_{(vt, ut]} (1-F(ut-y)) d\nu(y) \right)^l \right. \\ &\quad \left. + \mathbb{E} \left(\int_{[0, vt]} (F(ut-y) - F(vt-y)) d\nu(y) \right)^l \right) \end{aligned}$$

having utilized $(x+y)^l \leq 2^{l-1}(x^l + y^l)$ for nonnegative x and y . Using Lemma 5.1 with $G(y) = (1-F(y)) \mathbb{1}_{[0, (u-v)t)}(y)$ and $G(y) = F((u-v)t+y) - F(y)$, respectively, we obtain

$$\begin{aligned} & \mathbb{E} \left(\int_{(vt, ut]} (1-F(ut-y)) d\nu(y) \right)^l \\ &= \mathbb{E} \left(\int_{[0, ut]} (1-F(ut-y)) \mathbb{1}_{[0, (u-v)t)}(ut-y) d\nu(y) \right)^l \\ &\leq \mathbb{E}(\nu(1))^l \left(\sum_{n=0}^{[ut]} \sup_{y \in [n, n+1)} ((1-F(y)) \mathbb{1}_{[0, (u-v)t)}(y)) \right)^l \\ &\leq \mathbb{E}(\nu(1))^l \left(\sum_{n=0}^{[(u-v)t]} (1-F(n)) \right)^l \leq \mathbb{E}(\nu(1))^l (a((u-v)t))^l. \quad (2.5) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left(\int_{[0, vt]} (F(ut-y) - F(vt-y)) d\nu(y) \right)^l \\ &\leq \mathbb{E}(\nu(1))^l \left(\sum_{n=0}^{[vt]} \sup_{y \in [n, n+1)} (F((u-v)t+y) - F(y)) \right)^l \\ &\leq \mathbb{E}(\nu(1))^l \left(\sum_{n=0}^{[vt]} (1-F(n)) - \sum_{n=0}^{[vt]} (1-F((u-v)t+n+1)) \right)^l \\ &\leq \mathbb{E}(\nu(1))^l \left(\sum_{n=0}^{[vt]} (1-F(n)) - \sum_{n=0}^{[ut]+2} (1-F(n)) + \sum_{n=0}^{[(u-v)t]+1} (1-F(n)) \right)^l \\ &\leq \mathbb{E}(\nu(1))^l (a((u-v)t))^l. \quad (2.6) \end{aligned}$$

Combining (2.5) and (2.6) yields (2.3).

PROOF OF (2.4). Let us calculate

$$\begin{aligned}
& \mathbb{E}((R(k+1, t) - R(k, t))^{2l} | \mathcal{F}_k) \\
& \leq 2^{2l-1} ((1 - F(ut - S_k))^{2l} F(ut - S_k) \\
& + (F(ut - S_k))^{2l} (1 - F(ut - S_k))) \mathbb{1}_{\{vt < S_k \leq ut\}} \\
& + ((1 - F(ut - S_k) + F(vt - S_k))^{2l} (F(ut - S_k) - F(vt - S_k)) \\
& + (F(ut - S_k) - F(vt - S_k))^{2l} (1 - F(ut - S_k) + F(vt - S_k))) \mathbb{1}_{\{S_k \leq vt\}} \\
& \leq 2^{2l-1} ((1 - F(ut - S_k)) \mathbb{1}_{\{vt < S_k \leq ut\}} + (F(ut - S_k) - F(vt - S_k)) \mathbb{1}_{\{S_k \leq vt\}}).
\end{aligned}$$

Therefore,

$$I_2(t) \leq 2^{2l-1} \left(\mathbb{E} \int_{(vt, ut]} (1 - F(ut - y)) d\nu(y) + \mathbb{E} \int_{[0, vt]} (F(ut - y) - F(vt - y)) d\nu(y) \right).$$

Using now formulae (2.5) and (2.6) with $l = 1$ yields (2.4).

In view of (2.1) we can invoke Potter's bound (Theorem 1.5.6(iii) in [3]) to conclude that for any chosen $A > 1$ and $\rho \in (0, 1 - \beta)$ there exists $t_1 > 1$ such that

$$\frac{a((u - v)t)}{a(t)} \leq A(u - v)^{1-\beta-\rho}$$

whenever $u - v < 1$ and $(u - v)t \geq t_1$. Note that $u - v < 1$ and $(u - v)t \geq t_1$ together imply $t \geq t_1$. Hence

$$\frac{I_1(t)}{(a(t))^l} \leq 2^l \mathbb{E}(\nu(1))^l \left(\frac{a((u - v)t)}{a(t)} \right)^l \leq (4A)^l \mathbb{E}(\nu(1))^l (u - v)^{l(1-\beta-\rho)}. \quad (2.7)$$

Increasing t_1 if needed we can assume that $t^{1-\beta-\rho}/a(t) \leq 1$ for $t \geq t_1$ whence

$$\begin{aligned}
\frac{1}{\sum_{n=0}^{[t]+1} (1 - F(n))} &= \frac{((u - v)t)^{1-\beta-\rho}}{((u - v)t)^{1-\beta-\rho} \sum_{n=0}^{[t]+1} (1 - F(n))} \\
&\leq \frac{(u - v)^{1-\beta-\rho}}{((u - v)t)^{1-\beta-\rho}} \leq (u - v)^{1-\beta-\rho}
\end{aligned}$$

because $((u - v)t)^{1-\beta-\rho} \geq t_1^{1-\beta-\rho} > 1$. This implies

$$\frac{I_2(t)}{(a(t))^l} \leq 2^{2l} \mathbb{E}\nu(1) \frac{a((u - v)t)}{a(t)} \frac{1}{(a(t))^{l-1}} \leq (4A)^l \mathbb{E}(\nu(1))^l (u - v)^{l(1-\beta-\rho)}, \quad (2.8)$$

where we have used $\mathbb{E}\nu(1) \leq \mathbb{E}(\nu(1))^l$ which is a consequence of $\nu(1) \geq 1$ a.s.

Now the claim follows from (2.2), (2.7) and (2.8). \square

We are ready to prove Theorem 1.1. As discussed in the paragraph following Theorem 1.1 weak convergence of $(Z_t(u_1), \dots, Z_t(u_n))$ for any $n \in \mathbb{N}$ and any $0 < u_1 < \dots, u_n < \infty$ was proved in earlier works. In view of $V_\beta(0) = 0$ a.s., this immediately extends to $0 \leq u_1 < \dots, u_n < \infty$. Thus, it remains to prove tightness on $D[0, T]$ for any $T > 0$. Since the normalization in (1.2) is regularly varying it is enough to investigate the case $T = 1$ only. Suppose we can prove that for any $\varepsilon > 0$ and $\gamma > 0$ there exist $t_0 > 0$ and $\delta > 0$ such that

$$\mathbb{P} \left\{ \sup_{0 \leq u, v \leq 1, |u-v| \leq \delta} |Z_t(u) - Z_t(v)| > \varepsilon \right\} \leq \gamma \quad (2.9)$$

for all $t \geq t_0$. Then, by Theorem 15.5 in [2] the desired tightness follows along with continuity of the paths of (some version of) the limit process.

On pp. 763-764 in [17] it is shown that (the specific form of Z_t plays no role here)

$$\begin{aligned} \sup_{0 \leq u, v \leq 1, |u-v| \leq 2^{-i}} |Z_t(u) - Z_t(v)| &\leq 2 \sum_{j=i}^I \max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| \\ &\quad + 2 \max_{0 \leq k \leq 2^I - 1} \sup_{0 \leq w \leq 2^{-I}} |Z_t(k2^{-I} + w) - Z_t(k2^{-I})| \end{aligned}$$

for any positive integers i and I , $i \leq I$. Hence (2.9) follows if we can check that for any $\varepsilon > 0$ and $\gamma > 0$ there exist $t_0 > 0$, $i \in \mathbb{N}$ and $I \in \mathbb{N}$, $i \leq I$ such that

$$\mathbb{P} \left\{ \sum_{j=i}^I \max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| > \varepsilon \right\} \leq \gamma, \quad t \geq t_0 \quad (2.10)$$

and that

$$\max_{0 \leq k \leq 2^I - 1} \sup_{0 \leq w \leq 2^{-I}} |Z_t(k2^{-I} + w) - Z_t(k2^{-I})| \xrightarrow{\mathbb{P}} 0, \quad t \rightarrow \infty. \quad (2.11)$$

PROOF OF (2.10). By Lemma 2.1, for any chosen $A > 1$ and $\rho \in (0, 1 - \beta)$ there exists $t_1 > 1$ such that

$$\mathbb{E}|Z_t(k2^{-j}) - Z_t((k-1)2^{-j})|^{2l} \leq c(l)2^{-jl(1-\beta-\rho)} \quad (2.12)$$

whenever $2^{-j}t \geq t_1$. Let $I = I(t)$ denote the integer number satisfying

$$2^{-I}t \geq t_1 > 2^{-I-1}t.$$

Then the inequalities (2.12) and

$$\begin{aligned} \mathbb{E} \left(\max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| \right)^{2l} &\leq \sum_{k=1}^{2^j} \mathbb{E}|Z_t(k2^{-j}) - Z_t((k-1)2^{-j})|^{2l} \\ &\leq c(l)2^{-j(l(1-\beta-\rho)-1)} \end{aligned}$$

hold whenever $j \leq I$. Pick now minimal $l \in \mathbb{N}$ such that $l(1 - \beta - \rho) > 1$. Given positive ε and γ choose minimal $i \in \mathbb{N}$ satisfying

$$2^{-i(l(1-\beta-\rho)-1)} \leq \varepsilon^{2l} (1 - 2^{-(l(1-\beta-\rho)-1)/(2l)})^{2l} \gamma / c(l).$$

Increase t if needed to ensure that $i \leq I$. Invoking Markov's inequality and then the triangle inequality for the L_{2l} -norm gives

$$\begin{aligned} &\mathbb{P} \left\{ \sum_{j=i}^I \max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| > \varepsilon \right\} \\ &\leq \varepsilon^{-2l} \mathbb{E} \left(\sum_{j=i}^I \max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| \right)^{2l} \\ &\leq \varepsilon^{-2l} \left(\sum_{j=i}^I \left(\mathbb{E} \left(\max_{1 \leq k \leq 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| \right)^{2l} \right)^{1/2l} \right)^{2l} \\ &\leq \varepsilon^{-2l} c(l) \left(\sum_{j \geq i} 2^{-j(l(1-\beta-\rho)-1)/(2l)} \right)^{2l} \\ &= \varepsilon^{-2l} c(l) \frac{2^{-i(l(1-\beta-\rho)-1)}}{(1 - 2^{-(l(1-\beta-\rho)-1)/(2l)})^{2l}} \leq \gamma \end{aligned}$$

for all t large enough.

PROOF OF (2.11). We shall use a decomposition

$$\begin{aligned}
& (a(t))^{1/2}(Z_t(k2^{-I} + w) - Z_t(k2^{-I})) \\
= & \sum_{j \geq 0} \left(\mathbb{1}_{\{S_j + \eta_{j+1} \leq (k2^{-I} + w)t\}} - F((k2^{-I} + w)t - S_j) \right) \mathbb{1}_{\{k2^{-I}t < S_j \leq (k2^{-I} + w)t\}} \\
+ & \sum_{j \geq 0} \left(\mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \leq (k2^{-I} + w)t\}} \right. \\
- & \left. (F((k2^{-I} + w)t - S_j) - F(k2^{-I}t - S_j)) \right) \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \\
= : & J_1(t, k, w) + J_2(t, k, w).
\end{aligned}$$

It suffices to prove that for $i = 1, 2$

$$(a(t))^{-1/2} \max_{0 \leq k \leq 2^{I-1}} \sup_{0 \leq w \leq 2^{-I}} |J_i(t, k, w)| \xrightarrow{\mathbb{P}} 0, \quad t \rightarrow \infty. \quad (2.13)$$

PROOF OF (2.13) FOR $i = 1$. Since $|J_1(t, k, w)| \leq \nu((k2^{-I} + w)t) - \nu(k2^{-I}t)$ and $\nu(t)$ is a.s. nondecreasing we infer $\sup_{0 \leq w \leq 2^{-I}} |J_1(t, k, w)| \leq \nu((k+1)2^{-I}t) - \nu(k2^{-I}t)$. By Boole's inequality and distributional subadditivity of $\nu(t)$ (see formula (5.7) on p. 58 in [6])

$$\begin{aligned}
& \mathbb{P} \left\{ \max_{0 \leq k \leq 2^{I-1}} (\nu((k+1)2^{-I}t) - \nu(k2^{-I}t)) > \delta(a(t))^{1/2} \right\} \\
\leq & \sum_{k=0}^{2^I-1} \mathbb{P} \left\{ \nu((k+1)2^{-I}t) - \nu(k2^{-I}t) > \delta(a(t))^{1/2} \right\} \\
\leq & 2^I \mathbb{P} \left\{ \nu(2^{-I}t) > \delta(a(t))^{1/2} \right\} \leq 2^I \mathbb{P} \left\{ \nu(2t_1) > \delta(a(t))^{1/2} \right\}
\end{aligned}$$

for any $\delta > 0$. The right-hand side converges to zero as $t \rightarrow \infty$ because $\nu(2t_1)$ has finite exponential moments of all positive orders (see Lemma 5.2).

PROOF OF (2.13) FOR $i = 2$. We have

$$\begin{aligned}
& \sup_{0 \leq w \leq 2^{-I}} |J_2(t, k, w)| \\
\leq & \sup_{0 \leq w \leq 2^{-I}} \left(\sum_{j \geq 0} \mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \leq (k2^{-I} + w)t\}} \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \right. \\
+ & \left. \sum_{j \geq 0} (F((k2^{-I} + w)t - S_j) - F(k2^{-I}t - S_j)) \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \right) \\
\leq & \sum_{j \geq 0} \mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \leq (k+1)2^{-I}t\}} \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \\
+ & \sum_{j \geq 0} (F(((k+1)2^{-I})t - S_j) - F(k2^{-I}t - S_j)) \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \\
\leq & \left| \sum_{j \geq 0} \left(\mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \leq (k+1)2^{-I}t\}} \right. \right. \\
- & \left. \left. (F(((k+1)2^{-I})t - S_j) - F(k2^{-I}t - S_j)) \right) \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \right| \\
+ & 2 \sum_{j \geq 0} (F(((k+1)2^{-I})t - S_j) - F(k2^{-I}t - S_j)) \mathbb{1}_{\{S_j \leq k2^{-I}t\}} \\
= : & J_{21}(t, k) + 2J_{22}(t, k).
\end{aligned}$$

Pick minimal $r \in \mathbb{N}$ satisfying $r(1 - \beta) > 1$ so that $\lim_{t \rightarrow \infty} t^{-1}(a(t))^r = \infty$. Using (2.6) with $u = (k + 1)2^{-I}$ and $v = k2^{-I}$ we obtain

$$\mathbb{E}(J_{22}(t, k))^{2r} \leq \mathbb{E}(\nu(1))^{2r} (a(2^{-I}t))^{2r} \leq \mathbb{E}(\nu(1))^{2r} (a(2t_1))^{2r}$$

which implies

$$\begin{aligned} (a(t))^{-r} \mathbb{E} \left(\max_{0 \leq k \leq 2^I - 1} J_{22}(t, k) \right)^{2r} &\leq (a(t))^{-r} 2^I \max_{0 \leq k \leq 2^I - 1} \mathbb{E}(J_{22}(t, k))^{2r} \\ &\leq (a(t))^{-r} 2^I \mathbb{E}(\nu(1))^{2r} (a(2t_1))^{2r}. \end{aligned}$$

The right-hand side converges to zero as $t \rightarrow \infty$ by our choice of r . Consequently, $(a(t))^{-1/2} \max_{0 \leq k \leq 2^I - 1} J_{22}(t, k) \xrightarrow{\mathbb{P}} 0$ as $t \rightarrow \infty$ by Markov's inequality.

Using a counterpart of the first inequality in (2.2) for the martingale $(R^*(l, t), \mathcal{F}_l)_{l \in \mathbb{N}_0}$, where $R^*(0, t) := 0$ and

$$R^*(l, t) := \sum_{j=0}^{l-1} \left(\mathbb{1}_{\{vt < S_j + \eta_{j+1} \leq ut\}} - (F(ut - S_j) - F(vt - S_j)) \right) \mathbb{1}_{\{S_j \leq vt\}}, \quad l \in \mathbb{N}$$

for $u = (k + 1)2^{-I}t$ and $v = k2^{-I}t$, one can check that

$$\begin{aligned} \mathbb{E}(J_{21}(t, k))^{2r} &\leq C_r \left(\mathbb{E} \left(\int_{[0, k2^{-I}t]} (F((k + 1)2^{-I}t - y) - F(k2^{-I}t - y)) d\nu(y) \right)^r \right. \\ &\quad \left. + \mathbb{E} \int_{[0, k2^{-I}t]} (F((k + 1)2^{-I}t - y) - F(k2^{-I}t - y)) d\nu(y) \right). \end{aligned}$$

In view of (2.6) the right-hand side does not exceed

$$C_r (\mathbb{E}(\nu(1))^r (a(2^{-I}t))^r + \mathbb{E}\nu(1)a(2^{-I}t)) \leq C_r (\mathbb{E}(\nu(1))^r (a(2t_1))^r + \mathbb{E}\nu(1)a(2t_1)).$$

Arguing as above we conclude that $(a(t))^{-1/2} \max_{0 \leq k \leq 2^I - 1} J_{21}(t, k) \xrightarrow{\mathbb{P}} 0$ as $t \rightarrow \infty$, and (2.13) for $i = 2$ follows. The proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

Set $f(t) := \sqrt{t(1 - F(t))}$ for $t > 0$. In view of (2.1)

$$\sqrt{\int_0^t (1 - F(y)) dy} \sim (1 - \beta)^{-1/2} t^{1/2 - \beta/2} (\ell(t))^{1/2} \sim (1 - \beta)^{-1/2} f(t) \quad (3.1)$$

as $t \rightarrow \infty$. Assuming that $\mathbb{E}\xi^r < \infty$ for some $r > 2(1 - \beta)^{-1}$ we intend to show that

$$\frac{\sup_{0 \leq u \leq T} \left| \sum_{k \geq 0} (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}} - \mu^{-1} \int_0^{ut} (1 - F(y)) dy \right|}{f(t)} \xrightarrow{\mathbb{P}} 0, \quad t \rightarrow \infty$$

for any $T > 0$. This in combination with (3.1) and Theorem 1.1 is sufficient for the proof of the J_1 -convergence.

We proceed by observing that

$$\sum_{k \geq 0} (1 - F(t - S_k)) \mathbb{1}_{\{S_k \leq t\}} - \mu^{-1} \int_0^t (1 - F(y)) dy = \int_{[0, t]} (1 - F(t - y)) d(\nu(y) - \mu^{-1}y).$$

Integration by parts yields

$$\begin{aligned}
& \int_{[0,t]} (1 - F(t-y)) d(\nu(y) - \mu^{-1}y) + \mathbb{P}\{\xi = t\} \\
&= \nu(t) - \mu^{-1}t - \int_{[0,t]} (\nu(t-y) - \mu^{-1}(t-y)) dF(y) = \left(\nu(t) - \mu^{-1}t - \sigma\mu^{-3/2}W(t) \right. \\
&- \left. \int_{[0,t]} (\nu(t-y) - \mu^{-1}(t-y) - \sigma\mu^{-3/2}W(t-y)) dF(y) \right) \\
&+ \sigma\mu^{-3/2} \left(W(t) - \int_{[0,t]} W(t-y) dF(y) \right) =: R_1(t) + \sigma\mu^{-3/2}R_2(t),
\end{aligned}$$

where $\sigma^2 = \text{Var } \xi < \infty$ and W is a standard Brownian motion as defined in Lemma 5.3. For any $T > 0$

$$\begin{aligned}
\sup_{0 \leq u \leq T} |R_1(ut)| &\leq \sup_{0 \leq u \leq T} |\nu(ut) - \mu^{-1}ut - \sigma\mu^{-3/2}W(ut)| \\
&+ \sup_{0 \leq u \leq T} \int_{[0,ut]} |\nu(ut-y) - \mu^{-1}(ut-y) - \sigma\mu^{-3/2}W(ut-y)| dF(y) \\
&\leq \sup_{0 \leq u \leq Tt} |\nu(u) - \mu^{-1}u - \sigma\mu^{-3/2}W(u)| \\
&+ \sup_{0 \leq u \leq T} \sup_{0 \leq y \leq ut} |\nu(y) - \mu^{-1}y - \sigma\mu^{-3/2}W(y)| \\
&\leq 2 \sup_{0 \leq u \leq Tt} |\nu(u) - \mu^{-1}u - \sigma\mu^{-3/2}W(u)|.
\end{aligned}$$

By Lemma 5.3 the right-hand side is $o(t^{1/r})$ a.s. as $t \rightarrow \infty$. Hence, our choice of r in combination with (3.1) ensure that

$$\lim_{t \rightarrow \infty} \frac{\sup_{0 \leq u \leq T} |R_1(ut)|}{f(t)} = 0 \quad \text{a.s.}$$

Further, we note that

$$R_2(t) = W(t)(1 - F(t)) + \int_{[0,t]} (W(t) - W(t-y)) dF(y) =: R_{21}(t) + R_{22}(t).$$

Pick now $\varepsilon \in (0, (1 - \beta)/2)$ if $\beta \in [1/2, 1)$ and $\varepsilon \in (0, 1/2 - \beta)$ if $\beta \in [0, 1/2)$. With this ε , we have for any $T > 0$

$$\begin{aligned}
\sup_{0 \leq u \leq T} |R_{22}(ut)| &\leq \sup_{0 \leq u \leq T} \int_{[0,ut]} \frac{|W(ut) - W(ut-y)|}{y^{1/2-\varepsilon}} y^{1/2-\varepsilon} dF(y) \\
&\leq \sup_{0 \leq u \leq T} \sup_{0 \leq x \leq ut} \frac{|W(ut) - W(ut-x)|}{x^{1/2-\varepsilon}} \int_{[0,ut]} y^{1/2-\varepsilon} dF(y) \\
&\leq \sup_{0 \leq v < u \leq Tt} \frac{|W(u) - W(v)|}{(u-v)^{1/2-\varepsilon}} \int_{[0,Tt]} y^{1/2-\varepsilon} dF(y) \\
&\stackrel{d}{=} \sup_{0 \leq v < u \leq T} \frac{|W(u) - W(v)|}{(u-v)^{1/2-\varepsilon}} t^\varepsilon \int_{[0,Tt]} y^{1/2-\varepsilon} dF(y).
\end{aligned}$$

Here,

$$\sup_{0 \leq v < u \leq T} \frac{|W(u) - W(v)|}{(u-v)^{1/2-\varepsilon}} < \infty \quad \text{a.s.}$$

because the Brownian motion W is locally Hölder continuous with exponent $1/2 - \varepsilon$ (for any $\varepsilon \in (0, 1/2)$), and the distributional equality denoted by $\stackrel{d}{=}$ is a consequence of self-similarity of W with index $1/2$. Now it is convenient to treat two cases separately.

CASE $\beta \in [1/2, 1)$ in which

$$\frac{t^\varepsilon \int_{[0, Tt]} y^{1/2-\varepsilon} dF(y)}{f(t)} \sim \frac{\mathbb{E}\eta^{1/2-\varepsilon}}{t^{1/2-\beta/2-\varepsilon}(\ell(t))^{1/2}} \rightarrow 0, \quad t \rightarrow \infty$$

by (3.1) and our choice of ε . This proves

$$\frac{\sup_{0 \leq u \leq T} |R_{22}(ut)|}{f(t)} \xrightarrow{\mathbb{P}} 0, \quad t \rightarrow \infty. \quad (3.2)$$

CASE $\beta \in [0, 1/2)$. Here, we conclude that

$$\frac{t^\varepsilon \int_{[0, Tt]} y^{1/2-\varepsilon} dF(y)}{f(t)} \sim \frac{T^{1/2-\beta-\varepsilon}(\ell(t))^{1/2}}{(1/2 - \beta - \varepsilon)t^{\beta/2}} \rightarrow 0, \quad t \rightarrow \infty$$

having utilized (3.1), Theorem 1.6.4 in [3] which is applicable by our choice of ε and the fact that $\lim_{t \rightarrow \infty} \ell(t) = 0$ when $\beta = 0$. Thus, (3.2) holds in this case, too.

It remains to check weak convergence⁵ on D of $R_{21}(\cdot)/f(t)$ to the zero function or equivalently

$$\frac{\sup_{0 \leq u \leq T} |R_{21}(ut)|}{f(t)} \xrightarrow{\mathbb{P}} 0, \quad t \rightarrow \infty \quad (3.3)$$

for each $T > 0$. We shall only consider the case where $T > 1$, the case $T \in (0, 1]$ being analogous and simpler. By Potter's bound (Theorem 1.5.6 (iii) in [3]), for any chosen $A > 1$ and $\delta > 0$ there exists $t_0 > 0$ such that $1 - F(ut)/(1 - F(t)) \leq Au^{-\beta-\delta}$ whenever $u \in (0, 1]$ and $ut \geq t_0$. With this t_0 , write

$$\sup_{0 \leq u \leq T} |R_{21}(ut)| \leq \sup_{0 \leq u \leq t_0/t} |R_{21}(ut)| \vee \sup_{t_0/t \leq u \leq 1} |R_{21}(ut)| \vee \sup_{1 \leq u \leq T} |R_{21}(ut)|.$$

For the first supremum on the right-hand side we have $\sup_{0 \leq u \leq t_0/t} |W(ut)|(1 - F(ut)) \leq \sup_{0 \leq u \leq t_0} |W(u)|$ which converges to zero a.s. when divided by $f(t)$.

For the third supremum,

$$\begin{aligned} \sup_{1 \leq u \leq T} |W(ut)|(1 - F(ut)) &\leq (1 - F(t)) \sup_{0 \leq u \leq T} |W(ut)| \\ &\stackrel{d}{=} t^{1/2}(1 - F(t)) \sup_{0 \leq u \leq T} |W(u)|, \end{aligned}$$

and the right hand-side divided by $f(t)$ converges to zero a.s. in view of (3.1).

Finally,

$$\frac{\sup_{t_0/t \leq u \leq 1} |W(ut)|(1 - F(ut))}{1 - F(t)} \leq A \sup_{t_0/t \leq u \leq 1} |W(ut)|u^{-\beta-\delta}. \quad (3.4)$$

As before we distinguish the two cases.

⁵Weak convergence on $D(0, \infty)$ follows immediately from the fact that $\lim_{t \rightarrow \infty} (1 - F(ut))/(1 - F(t)) = u^{-\beta}$ locally uniformly in u on $(0, \infty)$. A longer proof is needed to treat weak convergence on $D[0, \infty)$, i.e., with 0 included.

CASE $\beta \in [1/2, 1)$. Choose δ satisfying $\delta \in (0, (1 - \beta)/2)$. The law of the iterated logarithm for $|W|$ at large times guarantees that $\lim_{t \rightarrow \infty} |W(t)|t^{-\beta-\delta} = 0$ a.s. and thereupon $\sup_{u \geq t_0} |W(u)|u^{-\beta-\delta} < \infty$ a.s. With this at hand we continue (3.4) as follows:

$$\begin{aligned} \frac{\sup_{t_0/t \leq u \leq 1} |W(ut)|(1 - F(ut))}{f(t)} &\leq \frac{At^{\beta+\delta} \sqrt{1 - F(t)} \sup_{t_0 \leq u \leq t} (|W(u)|u^{-\beta-\delta})}{t^{1/2}} \\ &\sim \frac{A \sup_{u \geq t_0} (|W(u)|u^{-\beta-\delta})(\ell(t))^{1/2}}{t^{1/2-\beta/2-\delta}} \quad \text{a.s.} \end{aligned}$$

having utilized (3.1) for the last asymptotic equivalence. The right-hand side converges to zero a.s.

CASE $\beta \in [0, 1/2)$. Pick δ so small that $\beta + \delta < 1/2$. The law of the iterated logarithm for $|W|$ at small times entails $\lim_{t \rightarrow 0+} |W(t)|t^{-\beta-\delta} = 0$ a.s. whence $\sup_{0 \leq u \leq 1} |W(u)|u^{-\beta-\delta} < \infty$ a.s. Continuing (3.4) with the help of self-similarity of W we further infer

$$\frac{\sup_{t_0/t \leq u \leq 1} |W(ut)|(1 - F(ut))}{f(t)} \leq A \sup_{0 \leq u \leq 1} |W(u)|u^{-\beta-\delta} \sqrt{1 - F(t)}.$$

It remains to note that the right-hand side trivially converges to zero a.s.

Combining pieces together we conclude that (3.3) holds. The proof of Theorem 1.2 is complete.

4 Integral representation of the limit process V_β

First of all, we note that V_0 is a standard Brownian motion. Therefore, throughout the rest of the section we assume that $\beta \in (0, 1)$.

Denote by $B := (B(u, v))_{u, v \geq 0}$ a standard Brownian sheet, i.e., a two-parameter continuous centered Gaussian field with $\mathbb{E}B(u_1, v_1)B(u_2, v_2) = (u_1 \wedge u_2)(v_1 \wedge v_2)$. In particular, B is a Brownian motion in u (in v) for each fixed v (u). See Section 3 in [18] for more properties of B . It turns out that the limit process V_β can be represented as the integral of a deterministic function with respect to the Brownian sheet. Such integrals are constructed in [10]. Also, these can be thought of as particular instances of the integrals of the first kind with respect to the Brownian sheet, see Section 4 in [18]. Set

$$V_\beta^*(u) = \sqrt{1 - \beta} \int_{[0, u]} \int_{[0, \infty)} \mathbb{1}_{\{x+z^{-1/\beta} > u\}} dB(x, z), \quad u \geq 0. \quad (4.1)$$

Clearly, the process $V_\beta^* := (V_\beta^*(u))_{u \geq 0}$ is centered Gaussian. Since

$$\begin{aligned} &\mathbb{E}V_\beta^*(u)V_\beta^*(s) \\ &= (1 - \beta) \int_{[0, \infty)} \int_{[0, \infty)} \mathbb{1}_{\{x+z^{-1/\beta} > u\}} \mathbb{1}_{[0, u]}(x) \mathbb{1}_{\{x+z^{-1/\beta} > s\}} \mathbb{1}_{[0, s]}(x) dz dx \\ &= (1 - \beta) \int_0^s \int_0^\infty \mathbb{1}_{\{x+z^{-1/\beta} > u\}} dz dx = (1 - \beta) \int_0^s (u - x)^{-\beta} dx \\ &= u^{1-\beta} - (u - s)^{1-\beta} \end{aligned}$$

for $0 \leq s \leq u$, we conclude that V_β^* is a version of V_β .

The discussion above does not give a clue on where equality (4.1) comes from. Here is a non-rigorous argument based on the idea from [14] which allows one to guess (4.1). We start with an integral representation

$$\begin{aligned} & \frac{\sum_{k \geq 0} (\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - (1 - F(ut - S_k)) \mathbb{1}_{\{S_k \leq ut\}})}{\sqrt{\mu^{-1} \int_0^t (1 - F(y)) dy}} \\ &= \int_{[0, u]} \int_{[0, \infty)} \mathbb{1}_{\{x+z > u\}} d \left(\frac{\sum_{k=1}^{\nu(xt)} \mathbb{1}_{\{\eta_k \leq zt\}} - \nu(xt)F(zt)}{\sqrt{\mu^{-1} \int_0^t (1 - F(y)) dy}} \right) \end{aligned} \quad (4.2)$$

where $\nu(t) = \inf\{k \in \mathbb{N} : S_k > t\}$ for $t \geq 0$. It is likely that

$$\frac{\sum_{k=1}^{\lfloor xt \rfloor} \mathbb{1}_{\{\eta_k \leq zt\}} - \lfloor xt \rfloor F(zt)}{\sqrt{t(1 - F(t))}}$$

converges weakly as $t \rightarrow \infty$ to $B(x, z^{-\beta})$ on some appropriate space of functions $g : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ equipped with some topology which is strong enough to ensure continuity of composition. The latter together with (2.1) and the well-known relation $t^{-1}\nu(tx) \xrightarrow{J_1} \mu^{-1}x$ as $t \rightarrow \infty$ should entail that

$$\frac{\sum_{k=1}^{\nu(xt)} \mathbb{1}_{\{\eta_k \leq zt\}} - \nu(xt)F(zt)}{\sqrt{\mu^{-1} \int_0^t (1 - F(y)) dy}}$$

converges weakly to $\sqrt{1 - \beta}B(x, z^{-\beta})$. One may expect that the right-hand side of (4.2) converges weakly to the right-hand side of (4.1). On the other hand, the left-hand side of (4.2) converges weakly to V_β by Theorem 1.1.

5 Appendix

The following result can be found in the proof of Lemma 7.3 in [1].

Lemma 5.1. *Let $G : [0, \infty) \rightarrow [0, \infty)$ be a locally bounded function. Then, for any $l \in \mathbb{N}$*

$$\mathbb{E} \left(\sum_{k \geq 0} G(t - S_k) \mathbb{1}_{\{S_k \leq t\}} \right)^l \leq \left(\sum_{j=0}^{\lfloor t \rfloor} \sup_{y \in [j, j+1)} G(y) \right)^l \mathbb{E}(\nu(1))^l, \quad t \geq 0. \quad (5.1)$$

The second auxiliary result is well-known. See, for instance, Theorem 2.1 (b) in [9]. It is of principal importance here that ξ is a.s. positive rather than nonnegative.

Lemma 5.2. *For all $a > 0$ and all $t > 0$ $\mathbb{E}e^{a\nu(t)} < \infty$.*

Also, we need a classical strong approximation result, see Corollary 3.1 (ii) in [5].

Lemma 5.3. *Suppose that $\mathbb{E}\xi^r < \infty$ for some $r > 2$. Then there exists a standard Brownian motion W such that*

$$\lim_{t \rightarrow \infty} t^{-1/r} \sup_{0 \leq s \leq t} |\nu(s) - \mu^{-1}s - \sigma\mu^{-3/2}W(s)| = 0 \quad \text{a.s.},$$

where $\mu = \mathbb{E}\xi$ and $\sigma^2 = \text{Var } \xi$.

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