On nested infinite occupancy scheme in random environment

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Abstract

We consider an infinite balls-in-boxes occupancy scheme with boxes organised in nested hierarchy, and random probabilities of boxes defined in terms of iterated fragmentation of a unit mass. We obtain a multivariate functional limit theorem for the cumulative occupancy counts as the number of balls approaches infinity. In the case of fragmentation driven by a homogeneous residual allocation model our result generalises the functional central limit theorem for the block counts in Ewens' and more general regenerative partitions.

Key words: Bernoulli sieve; Ewens' partition; functional limit theorem; infinite occupancy; nested hierarchy
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1 Introduction

In the infinite multinomial occupancy scheme balls are thrown independently in a series of boxes, so that each ball hits box $k = 1, 2, \dots$ with probability p_k, where $p_k > 0$ and $\sum_{k\in\mathbb{N}} p_k = 1$. This classical model is sometimes named after Karlin due to his seminal contribution [31]. Features of the occupancy pattern emerging after the first n balls are thrown have been intensely studied, see [6, 20, 27] for survey and references and [8, 13, 14, 15] for recent advances. Statistics in focus of most of the previous work, and also relevant to the subject of this paper, are not sensitive to the labelling of boxes but rather only depend on the integer partition of n comprised of nonzero occupancy numbers.

In the infinite occupancy scheme in a random environment the (hitting) probabilities of boxes are positive random variables $(P_k)_{k\in\mathbb{N}}$ with an arbitrary joint distribution satisfying $\sum_{k\in\mathbb{N}} P_k = 1$ almost surely. Conditionally on $(P_k)_{k\in\mathbb{N}}$, balls are thrown independently, with probability P_k of hitting box k. Instances of this general setup have received considerable attention within the circle of questions around exchangeable partitions, discrete random measures and their applications to population genetics, Bayesian statistics and computer science. In the most studied and analytically best tractable case the probabilities of boxes are representable as the residual allocation (or stick-breaking) model

$$P_k = U_1U_2\cdots U_{k-1}(1-U_k), \quad k\in\mathbb{N}, \tag{1}$$

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where the U_i's are independent with beta$(\theta, 1)$ distribution[1] on $(0,1)$ and $\theta > 0$. In this case the distribution of the sequence $(P_k)_{k\in\mathbb{N}}$ is known as the Griffiths-Engen-McCloskey (GEM) distribution with parameter θ. The sequence of the P_k's arranged in decreasing order has the Poisson-Dirichlet (PD) distribution with parameter θ, and the induced exchangeable partition on the set of n balls follows the celebrated Ewens sampling formula [3, 33, 35, 36]. Generalisations have been proposed in various directions. The two-parameter extension due to Pitman and Yor [33] involves probabilities of form (1) with independent but not identically distributed U_i's, where the distribution of U_i is beta$(\theta + \alpha i, 1 - \alpha)$ (with $0 < \alpha < 1$ and $\theta > -\alpha$). Residual allocation models with other choices of parameters for the U_i's with different beta distributions are found in [29, 37]. Much effort has been devoted to the occupancy scheme, known as the Bernoulli sieve, which is based on a *homogeneous* residual allocation model (1), that is, with independent and identically distributed (iid) factors U_i having arbitrary distribution on $(0,1)$, see [2, 16, 22, 27, 28, 34]. The homogeneous model has a multiplicative regenerative property, also inherited by the partition of the set of balls.

In more sophisticated constructions of random environments probabilities $(P_k)_{k\in\mathbb{N}}$ are identified with some arrangement in sequence of masses of a purely atomic random probability measure. A widely explored possibility is to define a random cumulative distribution function F by transforming the path of an increasing drift-free Lévy process (subordinator) $(X(t))_{t\geq 0}$. In particular, in the Poisson-Kingman model $F(t) = X(t)/X(1)$ for a measure supported by $[0,1]$, see [18, 33]. In the regenerative model $F(t) = 1 - e^{-X(t)},\ t \geq 0$, called in the statistical literature neutral-to-the right prior [18], see [5, 21, 23, 24].

Following [9, 12, 30] we shall study a nested infinite occupancy scheme in random environment. In this context we regard $(P_k)_{k\in\mathbb{N}}$ as a random *fragmentation law* (with $P_k > 0$ and $\sum_{k\in\mathbb{N}} P_k = 1$ a.s.). To introduce hierarchy of boxes, for each $j \in \mathbb{N}_0$ let $\mathbb{V}_j$ be the set of words of length j over $\mathbb{N}$, where $\mathbb{V}_0 := \{\varnothing\}$. The set $\mathbb{V} = \bigcup_{j\in\mathbb{N}_0} \mathbb{V}_j$ of all finite words has the natural structure of a ∞-storey tree with root $\varnothing$ and ∞-ary branching at every node, where $v1, v2, \dots \in \mathbb{V}_{j+1}$ are the immediate followers of $v \in \mathbb{V}_j$. Let $\{(P_k^{(v)})_{k\in\mathbb{N}},\ v \in \mathbb{V}\}$ be a family of independent copies of $(P_k)_{k\in\mathbb{N}}$. With each $v \in \mathbb{V}$ we associate a box divided in sub-boxes $v1, v2, \dots$ of the next level. The probabilities of boxes are defined recursively by

$$P(\varnothing) = 1, \quad P(vk) = P(v)P_k^{(v)} \quad \text{for } v \in \mathbb{V}, k \in \mathbb{N} \tag{2}$$

(note that the factors $P(v)$ and $P_k^{(v)}$ are independent). Given $(P(v))_{v\in\mathbb{V}}$, balls are thrown independently, with probability $P(v)$ of hitting box v. Since $\sum_{v\in\mathbb{V}_j} P(v) = 1$ the allocation of balls in boxes of level j occurs according to the ordinary Karlin's occupancy scheme.

Recursion (2) defines a discrete-time mass-fragmentation process, where the generic mass splits in proportions according to the same fragmentation law, independently of the history and masses of the co-existing fragments. The nested occupancy scheme can be seen as a combinatorial version of this fragmentation process. Initially all balls are placed in box $\varnothing$, and at each consecutive step $j+1$ each ball in box $v \in \mathbb{V}_j$ is placed in sub-box vk with probability $P_k^{(v)}$. The inclusion relation on the hierarchy of boxes induces a combinatorial structure on the (labelled) set of balls called total partition, that is a sequence of refinements from the trivial one-block partition down to the partition in singletons. The paper [17] highlights the role of exchangeability and gives the general de Finetti-style connection between mass-fragmentations and total partitions.

[1] Recall that a random variable X has a beta distribution with parameters $\alpha > 0$ and $\beta > 0$ if $\mathbb{P}\{X \in \mathrm{d}x\} = (1/\mathrm{B}(\alpha,\beta))x^{\alpha-1}(1-x)^{\beta-1}\,\mathbb{1}_{(0,1)}(x)\mathrm{d}x$. Here, $\mathrm{B}(\cdot,\cdot)$ is the beta function.

We consider the random probabilities of the hierarchy of boxes and the outcome of throwing infinitely many balls all defined on the same underlying probability space. For $j, r \in \mathbb{N}$, denote by $K_{n,j,r}$ the number of boxes $w \in \mathbb{W}_j$ of the jth level that contain exactly r out of n first balls, and let

$$K_{n,j}(s) := \sum_{r=\lceil n^{1-s} \rceil}^{n} K_{n,j,r}, \quad s \in [0,1], \tag{3}$$

be a cumulative count of occupied boxes, where $\lceil \cdot \rceil$ is the integer ceiling function. With probability one the random function $s \mapsto K_{n,j}(s)$ is nondecreasing and right-continuous, hence belongs to the Skorokhod space $D[0,1]$. Also observe that $K_{n,j}(0) = K_{n,j,n}$ is zero unless all balls fall in the same box and that $K_{n,j}(1)$ is the number of occupied boxes in the jth level. In [9] a central limit theorem with random centering was proved for $K_{n,j}(1)$ for j growing with n at certain rate. Our focus is different. We are interested in the joint weak convergence of $((K_{n,j_1}(s), \ldots, K_{n,j_m}(s)))_{s\in[0,1]}$, properly normalised and centered, for any finite collection of occupancy levels $1 \leq j_1 < \ldots < j_m$ as the number of balls n tends to ∞. As far as we know, this question has not been addressed so far. We prove a multivariate functional limit theorem (Theorem 2.1) applicable to the fragmentation laws representable by homogeneous residual allocations models (including the GEM/PD distribution) and some other models where the sequence of P_k's arranged in decreasing order approaches zero sufficiently fast. A univariate functional limit for $(K_{n,1}(s))_{s\in[0,1]}$ in the case of Bernoulli sieve was previously obtained in [2].

2 Main result

For given fragmentation law $(P_k)_{k\in\mathbb{N}}$, let $\rho(s) := \#\{k \in \mathbb{N} : P_k \geq 1/s\}$ for $s > 0$, and $N(t) := \rho(e^t), V(t) := \mathbb{E}N(t)$ for $t \in \mathbb{R}$. The joint distribution of $K_{n,j,r}$'s is completely determined by the probability law of the random function $\rho(\cdot)$, which captures the fragmentation law up to re-arrangement of P_k's. For our purposes therefore we can make no difference between fragmentation laws with the same $\rho(\cdot)$.

Similarly, using probabilities of boxes in level $j \in \mathbb{N}$ define $\rho_j(s) := \#\{v \in \mathbb{V}_j : P(v) \geq 1/s\}$ for $s > 0$, and $N_j(t) := \rho_j(e^t), V_j(t) := \mathbb{E}N_j(t)$ for $t \in \mathbb{R}$. Note that $N_j(t) = 0$ for $t \leq 0$. Since $\sum_{v\in\mathbb{V}_j} P(v) = 1$ a.s. we have $\rho_j(s) \leq s$, whence $N_j(t) \leq e^t$ a.s. and $V_j(t) < e^t$.

Let $T_k := -\log P_k$ for $k \in \mathbb{N}$. Here is a basic decomposition of principal importance for what follows:

$$N_j(t) = \sum_{k\in\mathbb{N}} N_{j-1}^{(k)}(t - T_k), \quad t \in \mathbb{R}, \tag{4}$$

where $(N_{j-1}^{(k)}(t))_{t\geq0}$ for $k \in \mathbb{N}$ are independent copies of $(N_{j-1}(t))_{t\geq0}$ which are also independent of $T_1, T_2, \ldots$ An immediate consequence of (4) is a recursion for the expectations

$$V_j(t) = \int_{[0,\,t]} V_{j-1}(t - y)\mathrm{d}V(y), \quad t \geq 0, \ j \geq 2, \tag{5}$$

which shows that $V_j(\cdot)$ is the jth convolution power of $V(\cdot)$.

The assumptions on fragmentation law and the functional limit will involve a centered Gaussian process $W := (W(s))_{s\geq0}$ which is a.s. locally Hölder continuous with exponent $\beta > 0$ and satisfy $W(0) = 0$. In particular, for any $T > 0$

$$|W(x) - W(y)| \leq M_T|x - y|^\beta, \quad 0 \leq x, y \leq T \tag{6}$$

for some a.s. finite random variable M_T. For each $u>0$, we set further

$$R_1^{(u)}(s) := W(s), \quad R_j^{(u)}(s) := \int_{[0,\,s]} (s-y)^{u(j-1)} \mathrm{d}W(y), \quad s \geq 0,\ j \geq 2.$$

For $j \geq 2$, the process $R_j^{(u)}$ is understood as the result of integration by parts

$$R_j^{(u)}(s) = u(j-1) \int_0^s (s-y)^{u(j-1)-1} W(y) \mathrm{d}y, \quad s \geq 0.$$

In particular, when $u(j-1)$ is a positive integer,

$$R_j^{(u)}(s) = (u(j-1))! \int_0^{s_1} \int_0^{s_2} \ldots \int_0^{s_{u(j-1)}} W(y) \mathrm{d}y \mathrm{d}s_{u(j-1)} \ldots \mathrm{d}s_2, \quad s \geq 0,\ j \geq 2,$$

where $s_1 = s$, which can be seen with the help of repeated integration by parts.

Throughout the paper $D := D[0,\infty)$ denotes the standard Skorokhod space. Here is our main result.

Theorem 2.1. *Assume the following conditions hold:*

(i)

$$b_0 + b_1 t^{\omega-\varepsilon_1} \leq V(t) - ct^{\omega} \leq a_0 + a_1 t^{\omega-\varepsilon_2} \tag{7}$$

for all $t \geq 0$ and some constants $c, \omega, a_0, a_1 > 0$, $0 < \varepsilon_1, \varepsilon_2 \leq \omega$ and $b_0, b_1 \in \mathbb{R}$,

(ii)

$$\mathbb{E} \sup_{s\in[0,\,t]} (N(s) - V(s))^2 = O(t^{2\gamma}), \quad t \to \infty \tag{8}$$

for some $\gamma \in (\omega - \min(1, \varepsilon_1, \varepsilon_2), \omega)$.

(iii)

$$\frac{N(t\,\cdot) - c(t\,\cdot)^{\omega}}{at^{\gamma}} \ \Rightarrow\ W(\cdot), \quad t \to \infty \tag{9}$$

in the J_1-topology on D for some $a > 0$.

Then

$$\left(\frac{K_{n,j}(\cdot) - c_j(\log n(\cdot))^{\omega j}}{ac_{j-1}(\log n)^{\gamma+\omega(j-1)}}\right)_{j\in\mathbb{N}} \ \Rightarrow\ (R_j^{(\omega)}(\cdot))_{j\in\mathbb{N}}, \quad n \to \infty \tag{10}$$

in the J_1-topology on $D[0,1]^{\mathbb{N}}$, where $\Gamma(\cdot)$ is the gamma function and

$$c_j := \frac{(c\Gamma(\omega+1))^j}{\Gamma(\omega j + 1)}, \quad j \geq 0. \tag{11}$$

Remark 2.2. The assumption $0 < \varepsilon_1, \varepsilon_2 \leq \omega$ ensures that $\gamma > 0$. Furthermore, in view of (7) and the choice of γ relation (9) is equivalent to

$$\frac{N(t\,\cdot) - V(t\,\cdot)}{at^{\gamma}} \ \Rightarrow\ W(\cdot), \quad t \to \infty \tag{12}$$

in the J_1-topology on D. Similarly, in view of (13) given below relation (10) is equivalent to

$$\left(\frac{K_{n,j}(\cdot) - V_j(\log n(\cdot))}{ac_{j-1}(\log n)^{\gamma+\omega(j-1)}}\right)_{j\in\mathbb{N}} \ \Rightarrow\ (R_j^{(\omega)}(\cdot))_{j\in\mathbb{N}}, \quad n \to \infty$$

in the J_1-topology on $D[0,1]^{\mathbb{N}}$.

3 Proof of Theorem 2.1

3.1 Auxiliary results

Lemma 3.1. (a) *Condition* (7) *ensures that, for* $j \in \mathbb{N}$ *and* $t \geq 0$,

$$b_{0,j} + b_{1,j} t^{\omega j - \varepsilon_1} \leq V_j(t) - c_j t^{\omega j} \leq a_{0,j} + a_{1,j} t^{\omega j - \varepsilon_2}, \tag{13}$$

where c_j *is given by* (11), $a_{0,j}, a_{1,j} > 0$ *and* $b_{0,j}, b_{1,j} \in \mathbb{R}$ *are constants with* $a_{0,1} := a_0$, $a_{1,1} := a_1$, $b_{0,1} := b_0$ *and* $b_{1,1} := b_1$. *In particular, for* $j \in \mathbb{N}$,

$$V_j(t) \ \sim \ c_j t^{\omega j}, \quad t \to \infty \tag{14}$$

and, for $j \in \mathbb{N}$ *and* $u, v \geq 0$,

$$\begin{aligned} V_j(u+v) - V_j(v) &\leq c_j (\mathbb{1}_{\{\omega j \in (0,1]\}} u^{\omega j} + \mathbb{1}_{\{\omega j > 1\}} \omega j (u+v)^{\omega j - 1} u) \\ &+ a_{0,j} + a_{1,j}(u+v)^{\omega j - \varepsilon_2} - b_{0,j} - b_{1,j} v^{\omega j - \varepsilon_1}. \end{aligned} \tag{15}$$

(b) *Suppose* (7) *and* (8). *Then*

$$\lim_{t \to \infty} \frac{N(t)}{V(t)} = 1 \quad \text{a.s.} \tag{16}$$

(c) *Suppose* (7) *and* (8). *Then, for* $j \in \mathbb{N}$,

$$\mathbb{E} \sup_{s \in [0,\, t]} (N_j(t) - V_j(t))^2 = O(t^{2\gamma + 2\omega(j-1)}), \quad t \to \infty \tag{17}$$

Proof. (a) We only prove the second inequality in (13). To this end, we first check that for any $b > 0$

$$\int_{[0,\, t]} (t-y)^b \mathrm{d}V(y) \leq a_0 t^b + b a_1 \mathrm{B}(b, 1+\omega-\varepsilon) t^{\omega - \varepsilon + b} + bc \mathrm{B}(b, 1+\omega) t^{\omega + b},$$

where $\mathrm{B}(\cdot, \cdot)$ is the beta function, and we write ε for ε_2 to ease notation. Indeed, using (7) we obtain

$$\begin{aligned} \int_{[0,\, t]} (t-y)^b \mathrm{d}V(y) &= b \int_0^t (V(t-y) - c(t-y)^\omega) y^{b-1} \mathrm{d}y + bc \int_0^t (t-y)^\omega y^{b-1} \mathrm{d}y \\ &\leq b a_0 \int_0^t y^{b-1} \mathrm{d}y + b a_1 \int_0^t (t-y)^{\omega - \varepsilon} y^{b-1} \mathrm{d}y + bc \int_0^t (t-y)^\omega y^{b-1} \mathrm{d}y \\ &= a_0 t^b + b a_1 \mathrm{B}(b, 1+\omega-\varepsilon) t^{\omega - \varepsilon + b} + bc \mathrm{B}(b, 1+\omega) t^{\omega + b}. \end{aligned}$$

To prove the second inequality in (13) we use induction. The case $j = 1$ is covered by (7). Assume the inequality holds for $j = k - 1$. Then, for $t \geq 0$ recalling (5) we obtain

$$\begin{aligned} V_k(t) &= \int_{[0,\, t]} (V_{k-1}(t-y) - c_{k-1}(t-y)^{\omega(k-1)}) \mathrm{d}V(y) + c_{k-1} \int_{[0,\, t]} (t-y)^{\omega(k-1)} \mathrm{d}V(y) \\ &\leq a_{0,k-1} V(t) + a_{1,k-1} \int_{[0,\, t]} (t-y)^{\omega(k-1) - \varepsilon} \mathrm{d}V(y) + c_{k-1} \int_{[0,\, t]} (t-y)^{\omega(k-1)} \mathrm{d}V(y) \\ &\leq a_{0,k-1} V(t) + a_{1,k-1} \big(a_0 t^{\omega(k-1) - \varepsilon} + (\omega(k-1) - \varepsilon) a_1 \mathrm{B}(\omega(k-1) - \varepsilon, 1 + \omega - \varepsilon) t^{\omega k - 2\varepsilon} \\ &+ (\omega(k-1) - \varepsilon) c \mathrm{B}(\omega(k-1) - \varepsilon, 1+\omega) t^{\omega k - \varepsilon} \big) \\ &+ c_{k-1} \big(a_0 t^{\omega(k-1)} + \omega(k-1) a_1 \mathrm{B}(\omega(k-1), 1 + \omega - \varepsilon) t^{\omega k - \varepsilon} \\ &+ \omega(k-1) c \mathrm{B}(\omega(k-1), 1+\omega) t^{\omega k} \big) \leq c_k t^{\omega k} + a_{0,k} + a_{1,k} t^{\omega k - \varepsilon} \end{aligned}$$

for appropriate positive $a_{0,k}$ and $a_{1,k}$, where we used

$$c_k = c_{k-1}\omega(k-1)c\mathrm{B}(\omega(k-1), 1+\omega). \tag{18}$$

Further, (14) is an immediate consequence of (13). To prove (15), we use (13) to obtain, for $j \in \mathbb{N}$ and $u, v \geq 0$,

$$V_j(u+v) - V_j(v) \leq c_j((u+v)^{\omega j} - v^{\omega j}) + a_{0,j} + a_{1,j}(u+v)^{\omega j - \varepsilon_2} - b_{0,j} - b_{1,j} v^{\omega j - \varepsilon_1}.$$

If $\omega j \in (0,1]$, we have $(u+v)^{\omega j} - v^{\omega j} \leq u^{\omega j}$ by subadditivity. If $\omega j > 1$, we have $(u+v)^{\omega j} - v^{\omega j} \leq \omega j (u+v)^{\omega j - 1} u$ by the mean value theorem and monotonicity. This completes the proof of (15).

(b) Condition (8) ensures that $\mathrm{Var}\, N(t) = O(t^{2\gamma})$ as $t \to \infty$. Pick any $\delta > 0$ such that $\delta(\omega - \gamma) > 1/2$. An application of Markov's inequality yields, for any $\varepsilon > 0$ and positive integer ℓ,

$$\mathbb{P}\{|N(\ell^\delta) - V(\ell^\delta)| > \varepsilon V(\ell^\delta)\} \leq \frac{\mathrm{Var}\, N(\ell^\delta)}{\varepsilon^2 V(\ell^\delta)^2} = O(\ell^{-2\delta(\omega-\gamma)}), \quad \ell \to \infty.$$

This entails $\lim_{\ell\to\infty}(N(\ell^\delta)/V(\ell^\delta)) = 1$ a.s. by the Borel-Cantelli lemma. For any $t > 1$ there exists an integer $\ell \geq 2$ such that $(\ell-1)^\delta < t \leq \ell^\delta$, whence, by monotonicity,

$$\frac{N((\ell-1)^\delta)}{V((\ell-1)^\delta)} \frac{V((\ell-1)^\delta)}{V(\ell^\delta)} \leq \frac{N(t)}{V(t)} \leq \frac{N(\ell^\delta)}{V(\ell^\delta)} \frac{V(\ell^\delta)}{V((\ell-1)^\delta)}.$$

Since $\lim_{\ell\to\infty}(V(\ell^\delta)/V((\ell-1)^\delta)) = 1$ we infer (16).

(c) We use the induction on j. When $j = 1$, relation (17) holds according to (8). Assuming that (17) holds for $j = i-1$ we intend to show that it also holds for $j = i$.

Recalling (4), write, for $i \geq 2$ and $t \geq 0$,

$$\begin{aligned} N_i(t) - V_i(t) &= \sum_{k\in\mathbb{N}} \big(N_{i-1}^{(k)}(t-T_k) - V_{i-1}(t-T_k)\big) \\ &+ \Big(\sum_{k\in\mathbb{N}} V_{i-1}(t-T_k) - V_i(t)\Big) =: X_i(t) + Y_i(t). \end{aligned} \tag{19}$$

An integration by parts yields, for $s \geq 0$,

$$\begin{aligned} |Y_i(s)| &= \Big|\int_{[0,\,s]} V_{i-1}(s-x)\mathrm{d}(N_1(x) - V_1(x))\Big| \leq \int_{[0,\,s]} |N_1(s-x) - V_1(s-x)|\mathrm{d}V_{i-1}(x) \\ &\leq \sup_{y\in[0,\,s]} |N_1(y) - V_1(y)| V_{i-1}(s). \end{aligned}$$

Hence,

$$\mathbb{E}[\sup_{s\in[0,\,t]} Y_i(s)]^2 \leq \mathbb{E}[\sup_{y\in[0,\,t]} (N(y) - V(y))]^2 V_{i-1}(t)^2 = O(t^{2\gamma+2\omega(i-1)}), \quad t \to \infty$$

by (8) and (14).

Passing to the analysis of X_i we obtain, for $s \geq 0$

$$\begin{aligned} [\sup_{s\in[0,\,t]} X_i(s)]^2 &\leq \sup_{s\in[0,\,t]} \Big(N_1(s) \sum_{k\in\mathbb{N}} \big(N_{i-1}^{(k)}(s-T_k) - V_{i-1}(s-T_k)\big)^2 \mathbb{1}_{\{T_k \leq s\}}\Big) \\ &\leq N_1(t) \sum_{k\in\mathbb{N}} \sup_{s\in[0,\,t]} \big(N_{i-1}^{(k)}(s) - V_{i-1}(s)\big)^2 \mathbb{1}_{\{T_k \leq t\}}. \end{aligned}$$

Therefore, $\mathbb{E}[\sup_{s\in[0,\,t]} X_i(s)]^2 \le \mathbb{E}N(t)^2\mathbb{E}[\sup_{s\in[0,\,t]}(N_{i-1}(s)-V_{i-1}(s))]^2 = O(t^{2\gamma+2\omega(i-1)})$ as $t\to\infty$ by the induction assumption and the asymptotics $\mathbb{E}[N(t)]^2 = \operatorname{Var} N(t) + V(t)^2 \sim V(t)^2$ as $t\to\infty$. It remains to note that

$$\mathbb{E}[\sup_{s\in[0,\,t]}(N_i(s)-V_i(s))]^2 \le 2\big(\mathbb{E}[\sup_{s\in[0,\,t]} X_i(s)]^2 + \mathbb{E}[\sup_{s\in[0,\,t]} Y_i(s)]^2\big) = O(t^{2\gamma+2\omega(i-1)}),\quad t\to\infty.$$

□

Our main result, Theorem 2.1, is an immediate consequence of Proposition 3.7 given in Section 3.2, Theorem 3.2 given next and its corollary.

Theorem 3.2. *Suppose* (7), (8) *and* (9). *Then*

$$\Big(\frac{N_j(t\cdot)-V_j(t\cdot)}{ac_{j-1}t^{\gamma+\omega(j-1)}}\Big)_{j\in\mathbb{N}} \;\Rightarrow\; (R_j^{(\omega)}(\cdot))_{j\in\mathbb{N}} \tag{20}$$

in the J_1-topology on $D^{\mathbb{N}}$.

Corollary 3.3. *Relation* (20) *entails that, for $j\in\mathbb{N}$ and $h>0$,*

$$t^{-\gamma-\omega(j-1)}\sup_{y\in[0,\,1]}(N_j(yt+h)-N_j(yt)) \;\overset{\mathrm{P}}{\to}\; 0,\quad t\to\infty. \tag{21}$$

It is convenient to prove Corollary 3.3 at this early stage.

Proof. Fix any $j\in\mathbb{N}$. Since $R_j^{(\omega)}$ is a.s. continuous, relation (20) in combination with (13) ensures that, for any $h>0$,

$$\Big(\frac{N_j(t\cdot)-c_j(t\cdot)^{\omega j}}{ac_{j-1}t^{\gamma+\omega(j-1)}}, \frac{N_j(t\cdot+h)-c_j(t\cdot+h)^{\omega j}}{ac_{j-1}t^{\gamma+\omega(j-1)}}\Big) \;\Rightarrow\; (R_j^{(\omega)}(\cdot), R_j^{(\omega)}(\cdot)),\quad t\to\infty$$

in the J_1-topology on $D\times D$, whence

$$t^{-\gamma-\omega(j-1)}\sup_{y\in[0,\,1]}(N_j(yt+h)-N_j(yt)-c_j((yt+h)^{\omega j}-(yt)^{\omega j})) \;\overset{\mathrm{P}}{\to}\; 0,\quad t\to\infty.$$

Using

$$\sup_{y\in[0,\,1]}((yt+h)^{\omega j}-(yt)^{\omega j}) \;\le\; \mathbb{1}_{\{\omega j\in(0,1]\}}\,h^{\omega j} + \mathbb{1}_{\{\omega j>1\}}\,\omega j(t+h)^{\omega j-1}h$$

we conclude that the right-hand side is $o(t^{\gamma+\omega(j-1)})$ as $t\to\infty$ because $\gamma>\omega-1$ by assumption. The proof is complete. □

Theorem 3.2 follows, in its turn, from Propositions 3.4 and 3.5. Below we use the processes X_j and Y_j as defined in (19).

Proposition 3.4. *Suppose* (7) *and* (9). *Then*

$$\Big(\frac{N_1(t\cdot)-V_1(t\cdot)}{at^{\gamma}}, \Big(\frac{Y_j(t\cdot)}{ac_{j-1}t^{\gamma+\omega(j-1)}}\Big)_{j\ge 2}\Big) \;\Rightarrow\; (R_j^{(\omega)}(\cdot))_{j\in\mathbb{N}},\quad t\to\infty, \tag{22}$$

in the J_1-topology on $D^{\mathbb{N}}$.

Proposition 3.5. *Suppose* (7), (8) *and* (9). *Then, for each integer $j\ge 2$ and each $T>0$,*

$$t^{-(\gamma+\omega(j-1))}\sup_{y\in[0,\,T]} X_j(ty) \;\overset{\mathrm{P}}{\to}\; 0,\quad t\to\infty. \tag{23}$$

3.2 Connecting two ways of box-counting

We retreat for a while from our main theme to focus on Karlin's occupancy scheme with deterministic probabilities $(p_k)_{k\in\mathbb{N}}$. By the law of large numbers a box of probability p gets occupied by about np balls, provided np is big enough. This suggests to relate counting the boxes occupied by at least n^{1-s} balls to the number of boxes with probability at least n^{-s}. Let $\bar\rho(t) := \#\{k\in\mathbb{N} : p_k \ge 1/t\}$ for $t>0$, and let $\bar K_{n,r}$ be the number of boxes containing exactly r out of n balls. We shall estimate uniformly the difference between

$$\bar K_n(s) := \sum_{r=\lceil n^{1-s}\rceil}^{n} \bar K_{n,r}\,, \quad s\in[0,1],$$

and $(\bar\rho(n^s))_{s\in[0,1]}$. Proposition 4.1 in [2]. However, we did not succeed to apply the cited proposition directly and will combine the estimates obtained in its proof.

Proposition 3.6. *The following universal estimate holds for each $n\in\mathbb{N}$*

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,1]}\big|\bar K_n(s)-\bar\rho(n^s)\big| &\le 4\big(\bar\rho(n)-\bar\rho(y_0 n(\log n)^{-2})\big)+2\bar\rho(n)(\log n)^{-1}\\ &+\int_1^\infty x^{-2}(\bar\rho(nx)-\bar\rho(n))\mathrm{d}x+2\sup_{s\in[0,1]}(\bar\rho(en^s)-\bar\rho(e^{-1}n^s)),\end{aligned}\qquad(24)$$

where $y_0\in(0,1)$ is a constant which does not depend on n, nor on $(p_k)_{k\in\mathbb{N}}$.

Proof. For $k\in\mathbb{N}$, denote by $\bar Z_{n,k}$ the number of balls falling in the kth box, so that

$$\bar K_n(s) = \sum_{k\in\mathbb{N}} \mathbb{1}_{\{n^{1-s}\le \bar Z_{n,k}\le n\}},\ \ s\in[0,1].$$

Then, for $n\in\mathbb{N}$ and $s\in[0,1]$,

$$\begin{aligned}|\bar K_n(1-s)-\bar\rho(n^{1-s})| &\le \sum_{k\in\mathbb{N}}\mathbb{1}_{\{\bar Z_{n,k}\ge n^s,\,1\le np_k\le n^s\}}+\sum_{k\in\mathbb{N}}\mathbb{1}_{\{\bar Z_{n,k}\ge n^s,\,np_k<1\}}+\sum_{k\in\mathbb{N}}\mathbb{1}_{\{\bar Z_{n,k}\le n^s,\,np_k\ge n^s\}}\\ &:= A_n(s)+B_n(s)+C_n(s).\end{aligned}$$

In [2] it was shown that, for $n\in\mathbb{N}$,

$$\mathbb{E}\sup_{s\in[0,1]}A_n(s)\le 2(\bar\rho(n)-\bar\rho(y_0n(\log n)^{-2}))+\frac{\bar\rho(n)}{\log n}+\sup_{s\in[0,1]}(\bar\rho(en^s)-\bar\rho(n^s))$$

(see [2], pp. 1004–1005) and

$$\mathbb{E}\sup_{s\in[0,1]}C_n(s)\le 2(\bar\rho(n)-\bar\rho(y_0n(\log n)^{-2}))+\frac{\bar\rho(n)}{\log n}+\sup_{s\in[0,1]}(\bar\rho(n^s)-\bar\rho(e^{-1}n^s))$$

(see [2], p. 1006). Finally, for $n\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,1]}B_n(s) &= \mathbb{E}\sum_{k\in\mathbb{N}}\mathbb{1}_{\{\bar Z_{n,k}\ge 1,\,np_k<1\}}=\sum_{k\in\mathbb{N}}(1-(1-p_k)^n)\,\mathbb{1}_{\{np_k<1\}}\le\sum_{k\in\mathbb{N}}np_k\,\mathbb{1}_{\{np_k<1\}}\\ &= \int_{(1,\infty)}\frac{1}{x}\mathrm{d}(\bar\rho(nx)-\bar\rho(n))=\int_1^\infty\frac{\bar\rho(nx)-\bar\rho(n)}{x^2}\mathrm{d}x.\end{aligned}$$

Combining the estimates we arrive at (24) because

$$\sup_{s\in[0,1]}\big|\bar K_n(s)-\bar\rho(n^s)\big|=\sup_{s\in[0,1]}\big|\bar K_n(1-s)-\bar\rho(n^{1-s})\big|.$$

□

We apply next Proposition 3.6 to the setting of Theorem 2.1. This result shows that (10) is equivalent to the analogous limit relation with $\rho_j(n^t) = N_j(t\log n)$ replacing $K_{n,j}(t)$.

Proposition 3.7. *Suppose* (7) *and* (9). *Then, for each* $j \in \mathbb{N}$,

$$\frac{\sup_{s\in[0,1]}\big|K_{n,j}(s) - \rho_j(n^s)\big|}{(\log n)^{\gamma+\omega(j-1)}} \overset{\mathrm{P}}{\to} 0, \quad n\to\infty. \tag{25}$$

Proof. Fix any $j\in\mathbb{N}$. By Proposition 3.6, for $n\in\mathbb{N}$,

$$\begin{aligned} & \mathbb{E}\Big(\sup_{s\in[0,1]}\big|K_{n,j}(s) - \rho_j(n^s)\big|\Big|(P_k)_{k\in\mathbb{N}}\Big) \qquad (26)\\ \leq\ & 4\big(\rho_j(n) - \rho_j(y_0 n(\log n)^{-2})\big) + 2\rho_j(n)(\log n)^{-1} \\ +\ & \int_1^\infty x^{-2}(\rho_j(nx)-\rho_j(n))\mathrm{d}x + 2\sup_{s\in[0,1]}(\rho_j(en^s) - \rho_j(e^{-1}n^s)). \end{aligned}$$

Recall the notation

$$c_j = \frac{(c\Gamma(\omega+1))^j}{\Gamma(\omega j+1)}, \quad j\in\mathbb{N}$$

and our choice of $\gamma > \omega - \min(1,\varepsilon_1,\varepsilon_2)$. In view of (14),

$$\frac{\mathbb{E}\rho_j(n)}{\log n} = \frac{V_j(\log n)}{\log n} \sim c_j(\log n)^{\omega j-1} = o((\log n)^{\gamma+\omega(j-1)}), \quad n\to\infty. \tag{27}$$

The next step is to show that

$$\mathbb{E}\int_1^\infty x^{-2}(\rho_j(nx)-\rho_j(n))\mathrm{d}x = o((\log n)^{\gamma+\omega(j-1)}), \quad n\to\infty. \tag{28}$$

As a preparation for the proof of (28) we first note that according to (15)

$$\begin{aligned} \mathbb{E}(\rho_j(nx)-\rho_j(n)) &= V_j(\log n + \log x) - V_j(\log n) \\ &\leq c_j(\mathbb{1}_{\{\omega j\in(0,1]\}}(\log x)^{\omega j} + \mathbb{1}_{\{\omega j>1\}}\,\omega j(\log n+\log x)^{\omega j-1}\log x) \\ &+ a_{0,j} + a_{1,j}(\log n+\log x)^{\omega j-\varepsilon_2} - b_{0,j} + |b_{1,j}|(\log n)^{\omega j-\varepsilon_1} \end{aligned}$$

for $n\in\mathbb{N}$ and $x\geq 1$. Further, using the inequality $(u+v)^\alpha \leq (2^{\alpha-1}\wedge 1)(u^\alpha+v^\alpha)$ which holds for $\alpha>0$ and $u,v\geq 0$ yields

$$\int_1^\infty x^{-2}(\log n+\log x)^{\omega j-\varepsilon_2}\mathrm{d}x = O((\log n)^{\omega j-\varepsilon_2}), \quad n\to\infty$$

and

$$\int_1^\infty x^{-2}(\log n+\log x)^{\omega j-1}\mathrm{d}x = O((\log n)^{\omega j-1}), \quad n\to\infty,$$

and (28) follows.

An appeal to (13) enables us to conclude that for large enough n

$$\begin{aligned} & \mathbb{E}(\rho_j(n) - \mathbb{E}\rho_j(y_0 n(\log n)^{-2})) \\ =\ & V_j(\log n) - V_j(\log n + \log y_0 - 2\log\log n) \\ \leq\ & c_j(\log n)^{\omega j}\Big(1-\Big(1-\frac{2\log\log n - \log y_0}{\log n}\Big)^{\omega j}\Big) \\ +\ & a_{0,j} + a_{1,j}(\log n)^{\omega j-\varepsilon_2} - b_{0,j} - b_{1,j}(\log n+\log y_0 - 2\log\log n)^{\omega j-\varepsilon_1} \\ \leq\ & 4\omega j c_j(\log n)^{\omega j-1}\log\log n + a_{0,j} + a_{1,j}(\log n)^{\omega j-\varepsilon_2} \\ -\ & b_{0,j} + |b_{1,j}|(\log n+\log y_0 - 2\log\log n)^{\omega j-\varepsilon_1}. \end{aligned}$$

Hence,

$$\mathbb{E}(\rho_j(n) - \rho_j(y_0 n(\log n)^{-2})) = o((\log n)^{\gamma+\omega(j-1)}), \quad n \to \infty \tag{29}$$

by the same reasoning as above. Finally,

$$\frac{\sup_{s\in[0,1]}(\rho_j(en^s) - \rho_j(e^{-1}n^s))}{(\log n)^{\gamma+\omega(j-1)}} = \frac{\sup_{s\in[0,1]}(N_j(s\log n+1) - N_j(s\log n - 1))}{(\log n)^{\gamma+\omega(j-1)}} \overset{\mathrm{P}}{\to} 0, \quad n\to\infty \tag{30}$$

by Corollary 3.3. Using (27), (28), (29) and (30) in combination with Markov's inequality (applied to the first three terms on the right-hand side of (26)) shows that the left-hand side of (26) converges to zero in probability as $n \to \infty$. Now (25) follows by another application of Markov's inequality and the dominated convergence theorem. □

3.3 Proof of Proposition 3.4

We shall use an integral representation which has already appeared in the proof of Lemma 3.1 (c):

$$Y_j(t) = \sum_{k\in\mathbb{N}} V_{j-1}(t-T_k) - V_j(t) = \int_{[0,\,t]} V_{j-1}(t-y)\mathrm{d}(N_1(y) - V_1(y)) \tag{31}$$

for $j \geq 2$ and $t \geq 0$.

In view of (12) Skorokhod's representation theorem ensures that there exist versions $\widehat{N}_1$ and $\widehat{W}$ such that

$$\lim_{t\to\infty} \sup_{y\in[0,\,T]} \left| \frac{\widehat{N}_1(ty) - V_1(ty)}{at^\gamma} - \widehat{W}(y)\right| = 0 \quad \text{a.s.} \tag{32}$$

for all $T > 0$. This implies that (22) is equivalent to

$$\left(\widehat{W}(\cdot), \left(\frac{\widehat{Z}_j(t,\cdot)}{c_{j-1}t^{\omega(j-1)}}\right)_{j\geq 2}\right) \;\Rightarrow\; (R_j^{(\omega)}(\cdot))_{j\in\mathbb{N}}, \quad t\to\infty, \tag{33}$$

where we set $\widehat{Z}_j(t,x) := \int_{(0,\,x]} \widehat{W}(y)\mathrm{d}_y(-V_{j-1}(t(x-y))$ for $j \geq 2$ and $t, x \geq 0$. As far as the first coordinate is concerned the equivalence is an immediate consequence of (32). As for the other coordinates, integration by parts yields, for $s > 0$ fixed and $j \geq 2$

$$\begin{aligned}\int_{[0,\,st]} \frac{V_{j-1}(st-y)}{c_{j-1}t^{\omega(j-1)}}\mathrm{d}_y \frac{\widehat{N}_1(y) - V_1(y)}{at^\gamma} &= \int_{(0,\,s]}\left(\frac{\widehat{N}_1(ty) - V_1(ty)}{at^\gamma} - \widehat{W}(y)\right)\mathrm{d}_y \frac{-V_{j-1}(t(s-y))}{c_{j-1}t^{\omega(j-1)}} \\ &+ \int_{(0,\,s]} \widehat{W}(y)\mathrm{d}_y \frac{-V_{j-1}(t(s-y))}{c_{j-1}t^{\omega(j-1)}}.\end{aligned}$$

Observe that the first term is a counterpart of (31) in which $\widehat{N}_1$ replaces N_1. Denoting by $L(t)$ the first term on the right-hand side, we infer

$$|L(t)| \leq \sup_{0\leq y\leq s}\left|\frac{\widehat{N}_1(ty) - V_1(ty)}{at^\gamma} - \widehat{W}(y)\right|\left(\left(c_{j-1}t^{\omega(j-1)}\right)^{-1} V_{j-1}(st)\right) \to 0 \;\; \text{a.s. for } t\to\infty$$

in view of (14) which implies that

$$\lim_{t\to\infty}\left(c_{j-1}t^{\omega(j-1)}\right)^{-1} V_{j-1}(st) = s^{\omega(j-1)} \tag{34}$$

and (32).

For $j\geq 2$ and $t,x\geq 0$, set $Z_j(t,x):=\int_{(0,\,x]}W(y){\rm d}_y(-V_{j-1}(t(x-y))$ and note that (33) is equivalent to

$$\Big(W(\cdot),\Big(\frac{Z_j(t,\cdot)}{c_{j-1}t^{\omega(j-1)}}\Big)_{j\geq 2}\Big)\ \Rightarrow\ (R_j^{(\omega)}(\cdot))_{j\in\mathbb{N}},\quad t\to\infty \tag{35}$$

because the left-hand sides of (33) and (35) have the same distribution.

It remains to check two properties:

(a) weak convergence of finite-dimensional distributions, i.e. that for all $n\in\mathbb{N}$, all $0\leq s_1<s_2<\ldots<s_n<\infty$ and all integer $\ell\geq 2$

$$\Big(W(s_i),\Big(\frac{Z_j(t,s_i)}{c_{j-1}t^{\omega(j-1)}}\Big)_{2\leq j\leq \ell}\Big)_{1\leq i\leq n}\ \stackrel{\rm d}{\to}\ (R_j^{(\omega)}(s_i))_{1\leq j\leq \ell,\,1\leq i\leq n} \tag{36}$$

as $t\to\infty$;

(b) tightness of the distributions of coordinates in (35), excluding the first one.

Proof of (36). If $s_1=0$, we have $W(s_1)=Z_j(t,s_1)=R_k^{(\omega)}(s_1)=0$ a.s. for $j\geq 2$ and $k\in\mathbb{N}$. Hence, in what follows we consider the case $s_1>0$. Both the limit and the converging vectors in (36) are Gaussian. In view of this it suffices to prove that

$$\lim_{t\to\infty}t^{-\omega(k+j-2)}\mathbb{E}[Z_k(t,s)Z_j(t,u)]=c_{k-1}c_{j-1}\mathbb{E}[R_k^{(\omega)}(s)R_j^{(\omega)}(u)] \tag{37}$$

$$=\begin{cases}c_{k-1}c_{j-1}\int_0^s\int_0^u r(s-y,u-z){\rm d}y^{\omega(k-1)}{\rm d}z^{\omega(j-1)}, & \text{if } k,j\geq 2,\\ c_{j-1}\int_0^u r(s,u-z){\rm d}z^{\omega(j-1)}, & \text{if } k=1, j\geq 2\end{cases}$$

for $k,j\in\mathbb{N}$, $k+j\geq 3$ and $s,u>0$, where we set $Z_1(t,\cdot)=W(\cdot)$ and $r(x,y):=\mathbb{E}[W(x)W(y)]$ for $x,y\geq 0$. We only consider the case where $k,j\geq 2$, the complementary case being similar and simpler.

To prove (37) we need some preparation. For each $t>0$ denote by $\theta_{k,t}$ and $\theta_{j,t}$ independent random variables with the distribution functions $\mathbb{P}\{\theta_{k,t}\leq y\}=V_{k-1}(ty)/V_{k-1}(ts)$ on $[0,s]$ and $\mathbb{P}\{\theta_{j,t}\leq y\}=V_{j-1}(ty)/V_{j-1}(tu)$ on $[0,u]$, respectively. Further, let θ_k and θ_j denote independent random variables with the distribution functions $\mathbb{P}\{\theta_k\leq y\}=(y/s)^{\omega(k-1)}$ on $[0,s]$ and $\mathbb{P}\{\theta_j\leq y\}=(y/u)^{\omega(j-1)}$ on $[0,u]$, respectively. According to (14), $(\theta_{k,t},\theta_{j,t})\stackrel{\rm d}{\to}(\theta_k,\theta_j)$ as $t\to\infty$. Now observe that the function $r(x,y)=\mathbb{E}[W(x)W(y)]$ is continuous, hence bounded, on $[0,T]\times[0,T]$ for every $T>0$. This follows from the assumed a.s. continuity of W, the dominated convergence theorem in combination with $\mathbb{E}[\sup_{z\in[0,T]}W(z)]^2<\infty$ for every $T>0$ (for the latter, see Theorem 3.2 on p. 63 in [1]). As a result, $r(s-\theta_{k,t},u-\theta_{j,t})\stackrel{\rm d}{\to}r(s-\theta_k,u-\theta_j)$ as $t\to\infty$ and thereupon

$$\lim_{t\to\infty}\mathbb{E}r(s-\theta_{k,t},u-\theta_{j,t})=\mathbb{E}r(s-\theta_k,u-\theta_j)$$

by the dominated convergence theorem.

This together with (34) leads to formula (37):

$$\begin{aligned}&\mathbb{E}[t^{-\omega(k+j-2)}Z_k(t,s)Z_j(t,u)]\\ =\ &\frac{V_{k-1}(ts)}{t^{\omega(k-1)}}\frac{V_{j-1}(tu)}{t^{\omega(j-1)}}\int_0^s\int_0^u r(s-y,u-z){\rm d}_y\Big(\frac{V_{k-1}(ty)}{V_{k-1}(ts)}\Big){\rm d}_z\Big(\frac{V_{j-1}(tz)}{V_{j-1}(tu)}\Big)\\ =\ &\frac{V_{k-1}(ts)}{t^{\omega(k-1)}}\frac{V_{j-1}(tu)}{t^{\omega(j-1)}}\mathbb{E}r(s-\theta_{k,t},u-\theta_{j,t})\ \to\ c_{k-1}s^{k-1}c_{j-1}s^{j-1}\mathbb{E}r(s-\theta_k,u-\theta_j)\\ =\ &c_{k-1}c_{j-1}\int_0^s\int_0^u r(s-y,u-z){\rm d}y^{\omega(k-1)}{\rm d}z^{\omega(j-1)}\end{aligned}$$

as $t \to \infty$.

Proof of tightness. Choose $j \geq 2$. We intend to prove tightness of $(t^{-\omega(j-1)} Z_j(t,u))_{u \geq 0}$ on $D[0,T]$ for all $T > 0$. Since the function $t \mapsto t^{-\omega(j-1)}$ is regularly varying at ∞ it is enough to investigate the case $T = 1$ only. By Theorem 15.5 in [10] it suffices to show that for any $\kappa_1 > 0$ and $\kappa_2 > 0$ there exist $t_0 > 0$ and $\delta > 0$ such that

$$\mathbb{P}\Big\{ \sup_{0 \leq u, v \leq 1, |u-v| \leq \delta} |Z_j(t,u) - Z_j(t,v)| > \kappa_1 t^{-\omega(j-1)} \Big\} \leq \kappa_2 \tag{38}$$

for all $t \geq t_0$. We only analyze the case where $0 \leq v < u \leq 1$, the complementary case being analogous.

Set $W(x) = 0$ for $x < 0$. The basic observation for the subsequent proof is that (6) extends to

$$|W(x) - W(y)| \leq M_T |x - y|^\beta \tag{39}$$

whenever $-\infty < x, y \leq T$ for the same positive random variable M_T as in (6). This is trivial when $x \vee y \leq 0$ and a consequence of (6) when $x \wedge y \geq 0$. Assume that $x \wedge y \leq 0 < x \vee y$. Then $|W(x) - W(y)| = |W(x \vee y)| \leq M_T (x \vee y)^\beta \leq M_T |x-y|^\beta$, where the first inequality follows from (6) with $y = 0$.

Let $0 \leq v < u \leq 1$ and $u - v \leq \delta$ for some $\delta \in (0,1]$. Using (39) and (14) we obtain

$$\begin{aligned} t^{-\omega(j-1)} |Z_j(t,u) - Z_j(t,v)| &= t^{-\omega(j-1)} \Big| \int_{[0,\,u)} \big(W(u-y) - W(v-y)\big) \mathrm{d}V_{j-1}(ty) \Big| \\ &\leq M_1 (u-v)^\beta (t^{-\omega(j-1)} V_{j-1}(t)) \leq M_1 \delta^\beta \lambda \end{aligned}$$

for large enough t and a positive constant λ. This proves (38).

3.4 Proof of Proposition 3.5

Relation (23) will be proved by induction in two steps.

Step 1. Assume that (23) holds for $j = 2, \ldots, k$. We claim that then

$$\left(\frac{N_j(t\cdot) - V_j(t\cdot)}{a c_{j-1} t^{\gamma + \omega(j-1)}} \right)_{j=1,\ldots,k} \Rightarrow (R_j^{(\omega)}(\cdot))_{j=1,\ldots k} \tag{40}$$

in the J_1-topology on D^k. Indeed, in view of (19) and the induction hypothesis relation (40) is equivalent to

$$\left(\frac{N_1(t\cdot) - V_1(t\cdot)}{a t^\gamma}, \left(\frac{Y_j(t\cdot)}{a c_{j-1} t^{\gamma + \omega(j-1)}} \right)_{j=2,\ldots,k} \right) \Rightarrow (R_j^{(\omega)}(\cdot))_{j=1,\ldots,k}, \quad t \to \infty. \tag{41}$$

The latter holds by Proposition 3.4.

Step 2. Using

$$\frac{N_k(t\cdot) - V_k(t\cdot)}{a c_{k-1} t^{\gamma + \omega(k-1)}} \Rightarrow R_k^{(\omega)}(\cdot), \quad t \to \infty \tag{42}$$

in the J_1-topology on D which is a consequence of (40) we shall prove that (23) holds with $j = k+1$.

In view of (42) and the fact that $R_k^{(\omega)}$ is a.s. continuous Skorokhod's representation theorem ensures that there exists a probability space which is rich enough to accomodate

- independent random processes $\widehat{N}_k^{(1)}$, $\widehat{N}_k^{(2)}, \dots$ which are versions of N_k;
- independent random processes $\widehat{R}_k^{(\omega,1)}$, $\widehat{R}_k^{(\omega,2)}, \dots$ which are versions of $R_k^{(\omega)}$;
- random variables $\widehat{T}_1$, $\widehat{T}_2, \dots$ which are versions of T_1, $T_2, \dots$ independent of $(\widehat{N}_k^{(1)}, \widehat{R}_k^{(\omega,1)})$, $(\widehat{N}_k^{(2)}, \widehat{R}_k^{(\omega,2)}), \dots$

Furthermore,

$$\lim_{t\to\infty} \sup_{y\in[0,\,T]} \Big|\frac{\widehat{N}_k^{(r)}(ty) - V_k(ty)}{ac_{k-1}t^{\gamma+\omega(k-1)}} - \widehat{R}_k^{(\omega,r)}(y)\Big| = 0 \quad \text{a.s.} \tag{43}$$

for all $T > 0$ and $r \in \mathbb{N}$.

Set

$$\widehat{X}_{k+1}(t) := \sum_{r\in\mathbb{N}} \big(\widehat{N}_k^{(r)}(t - \widehat{T}_r) - V_k(t - \widehat{T}_r)\big)\, \mathbb{1}_{\{\widehat{T}_r \le t\}}, \quad t \ge 0.$$

The process $(\widehat{X}_{k+1}(t))_{t\ge 0}$ has the same distribution as $(X_{k+1}(t))_{t\ge 0}$. Therefore, (23) with $j = k+1$ is equivalent to

$$t^{-(\gamma+\omega k)} \sup_{y\in[0,\,T]} \widehat{X}_{k+1}(ty) \overset{\mathrm{P}}{\to} 0, \quad t \to \infty. \tag{44}$$

To prove this, write

$$\begin{aligned} \frac{\widehat{X}_{k+1}(ty)}{ac_{k-1}t^{\gamma+\omega k}} &= t^{-\omega} \sum_{r\in\mathbb{N}} \Big(\frac{\widehat{N}_k^{(r)}(ty - \widehat{T}_r) - V_k(ty - \widehat{T}_r)}{ac_{k-1}t^{\gamma+\omega(k-1)}} - \widehat{R}_k^{(\omega,r)}(y - t^{-1}\widehat{T}_r)\Big)\, \mathbb{1}_{\{\widehat{T}_r \le ty\}} \\ &+ t^{-\omega} \sum_{r\in\mathbb{N}} \widehat{R}_k^{(\omega,r)}(y - t^{-1}\widehat{T}_r)\, \mathbb{1}_{\{\widehat{T}_r \le ty\}} =: t^{-\omega}(\widehat{Z}_1(t,y) + \widehat{Z}_2(t,y)). \end{aligned}$$

For all $T > 0$,

$$t^{-\omega} \sup_{y\in[0,\,T]} |\widehat{Z}_1(t,y)| \le \sum_{r\in\mathbb{N}} \sup_{y\in[0,\,T]} \Big|\frac{\widehat{N}_k^{(r)}(ty) - V_k(ty)}{ac_{k-1}t^{\gamma+\omega(k-1)}} - \widehat{R}_k^{(\omega,r)}(y)\Big|\, \mathbb{1}_{\{\widehat{T}_r \le tT\}}\,. \tag{45}$$

For $r \in \mathbb{N}$, the random variables $\eta_r(t) := \sup_{y\in[0,\,T]} \Big|\frac{\widehat{N}_k^{(r)}(ty) - V_k(ty)}{ac_{k-1}t^{\gamma+\omega(k-1)}} - \widehat{R}_k^{(\omega,r)}(y)\Big|$ are i.i.d. and independent of $\widehat{T}_1$, $\widehat{T}_2, \dots$. Furthermore,

$$\mathbb{E}[\eta_1(t)]^2 \le 2\Big(\frac{\mathbb{E}[\sup_{s\in[0,\,Tt]}(N_k(s) - V_k(s))]^2}{(ac_{k-1}t^{\gamma+\omega(k-1)})^2} + \mathbb{E}[\sup_{s\in[0,\,T]} R_k^{(\omega)}(s)]^2\Big) = O(1), \quad t \to \infty \tag{46}$$

in view of (17) and the well-known fact that the supremum over $[0, T]$ of any a.s. continuous Gaussian process has exponential tails. Since $\lim_{t\to\infty} \eta_1(t) = 0$ a.s., inequality (46) ensures that $\lim_{t\to\infty} \mathbb{E}\eta_1(t) = 0$. The right-hand side in (45) is dominated by

$$t^{-\omega} \sum_{r\in\mathbb{N}} (\eta_r(t) - \mathbb{E}\eta_r(t))\, \mathbb{1}_{\{\widehat{T}_r \le t\}} + t^{-\omega}\widehat{N}(t)\mathbb{E}\eta_1(t),$$

where $\widehat{N}(t) := \#\{r \in \mathbb{N} : \widehat{T}_r \le t\}$. Using the last limit relation and (16) we conclude that the second summand converges to 0 a.s., as $t \to \infty$. The first summand converges to zero in probability, as $t \to \infty$, by Markov's inequality in combination with $t^{-2\omega}\mathbb{E}\Big(\sum_{r\in\mathbb{N}}(\eta_r(t) -$

$\mathbb{E}\eta_r(t))\,\mathbb{1}_{\{\widehat{T}_r\le t\}}\Big)^2 = t^{-2\omega}V(t)\mathrm{Var}\,\eta_1(t) = O(t^{-\omega})$. Thus, for all $T>0$, $t^{-\omega}\sup_{y\in[0,T]}|\widehat{Z}_1(t,y)| \overset{\mathrm{P}}{\to} 0$ as $t\to\infty$.

The process $(\widehat{Z}_2(t,y))$ has the same distribution as the process $(Z_2(t,y))$ in which the random variables involved do not carry the hats. Thus, it suffices to prove that

$$t^{-\omega}\sup_{y\in[0,\,T]}|Z_2(t,y)| \overset{\mathrm{P}}{\to} 0,\quad t\to\infty. \tag{47}$$

In what follows we write $\mathbb{E}_{(T_r)}(\cdot)$ for $\mathbb{E}(\cdot|(T_r))$ and $\mathbb{P}_{(T_r)}(\cdot)$ for $\mathbb{P}(\cdot|(T_r))$. Since $\mathbb{E}_{(T_r)}[Z_2(t,y)]^2 = \mathbb{E}[R_k^{(\omega)}(1)]^2\int_{[0,\,ty]}(y-t^{-1}x)^{2(\gamma+\omega(k-1))}\mathrm{d}N_1(x) \le \mathbb{E}[R_k^{(\omega)}(1)]^2y^{2(\gamma+\omega(k-1))}N_1(ty)$. Using the Cramér-Wold device and Markov's inequality in combination with (16) we infer that, given (T_r), with probability one finite-dimensional distributions of $(t^{-\omega}Z_2(t,y))_{y\ge0}$ converge weakly to the zero vector, as $t\to\infty$. Thus, (47) follows if we can show that the family of $\mathbb{P}_{(T_r)}$-distributions of $(t^{-\omega}Z_2(t,y))_{y\ge0}$ is tight. As a preparation, we observe that the process $R_k^{(\omega)}$ inherits the local Hölder continuity of W. Indeed, recalling (39) we obtain, for $x,y\in[0,T]$ and $k\ge2$,

$$|R_k^{(\omega)}(x)-R_k^{(\omega)}(y)| \le \omega(k-1)\int_0^{x\vee y}|W(x-z)-W(y-z)|z^{\omega(k-1)-1}\mathrm{d}z \le M_TT^{\omega(k-1)}|x-y|^{\beta}\ \text{a.s.} \tag{48}$$

It is also important that the random variable M_T has finite moments of all positive orders, see Theorem 1 in [4]. Pick now integer $n\ge2$ such that $2n\beta>1$. By Rosenthal's inequality (Theorem 3 in [38]), for $x,y\in[0,\,T]$ and a positive constant C,

$$\begin{aligned}\mathbb{E}_{(T_r)}(Z_2(t,x)-Z_2(t,y))^{2n} &\le C\Big(\Big(\sum_{r\in\mathbb{N}}\mathbb{E}_{(T_r)}(R_k^{(\omega,r)}(x-t^{-1}T_r)-R_k^{(\omega,r)}(y-t^{-1}T_r))^2\,\mathbb{1}_{\{T_r\le tT\}}\Big)^n\\ &+ \sum_{r\in\mathbb{N}}\mathbb{E}_{(T_r)}(R_k^{(\omega,r)}(x-t^{-1}T_r)-R_k^{(\omega,r)}(y-t^{-1}T_r))^{2n}\,\mathbb{1}_{\{T_r\le tT\}}\Big)\\ &\le 2CT^{2n\omega(k-1)}\mathbb{E}M_T^{2n}|x-y|^{2n\beta}N_1(tT)^n\end{aligned}$$

having utilized (48) for the second inequality. In view of (16), this entails that a classical sufficient condition for tightness (formula (12.51) on p. 95 in [10]) holds

$$t^{-2n\omega}\mathbb{E}_{(T_r)}(Z_2(t,x)-Z_2(t,y))^{2n}\le C_1(x-y)^{2n\beta}$$

for a positive constant C_1 and large enough t. Thus, we have proved that (47) holds conditionally on (T_r), hence, also unconditionally. The proof of Proposition 3.5 is complete.

4 The case of homogeneous residual allocation model

In this section we apply Theorem 2.1 to the case of fragmentation law given by homogeneous residual allocation model (1). Let $B:=(B(s))_{s\ge0}$ be a standard Brownian motion (BM) and for $q\ge0$ let

$$B_q(s):=\int_{[0,\,s]}(s-y)^q\mathrm{d}B(y),\quad s\ge0.$$

The process $B_q:=(B_q(s))_{s\ge0}$ is a centered Gaussian process called the fractionally integrated BM or the Riemann-Liouville process. Clearly $B=B_0$, and for $q\in\mathbb{N}$ the process can be obtained as a repeated integral of the BM. It is known that B_q is locally Hölder continuous with any exponent $\beta<q+1/2$ [26].

Theorem 4.1. *Let $(P_k)_{k\in\mathbb{N}}$ be given by* (1) *with iid U_i's such that*

$$\mu := \mathbb{E}|\log U_1| < \infty,\ \sigma^2 := \mathrm{Var}(\log U_1) \in (0,\infty)$$

and $\mathbb{E}|\log(1-U_1)| < \infty$. Then

$$\Bigg(\frac{(j-1)!(K_{n,j}(\cdot)-(j!)^{-1}(\mu^{-1}\log n(\cdot))^j)}{\sqrt{\sigma^2\mu^{-2j-1}(\log n)^{2j-1}}}\Bigg)_{j\in\mathbb{N}} \ \Rightarrow\ (B_{j-1}(\cdot))_{j\in\mathbb{N}}, \quad n\to\infty$$

in the product J_1-topology on $D[0,1]^{\mathbb{N}}$.

Proof. Let $(\xi_k,\eta_k)_{k\in\mathbb{N}}$ be independent copies of a random vector (ξ,η) with positive arbitrarily dependent components. Denote by $(S_k)_{k\in\mathbb{N}_0}$ the zero-delayed ordinary random walk with increments ξ_k, that is, $S_0:=0$ and $S_k:=\xi_1+\ldots+\xi_k$ for $k\in\mathbb{N}$. Consider a *perturbed* random walk

$$\tilde T_k := S_{k-1}+\eta_k, \quad k\in\mathbb{N} \tag{49}$$

and then define $\tilde N(t) := \#\{k\in\mathbb{N}: \tilde T_k \le t\}$ and $\tilde V(t):=\mathbb{E}\tilde N(t)$ for $t\ge 0$. It is clear that

$$\tilde V(t) = \mathbb{E}U((t-\eta)^+) = \int_{[0,\,t]} U(t-y)\mathrm{d}\tilde G(y), \quad t\ge 0 \tag{50}$$

where, for $t\ge 0$, $U(t):=\sum_{k\ge 0}\mathbb{P}\{S_k\le t\}$ is the renewal function and $\tilde G(t)=\mathbb{P}\{\eta\le t\}$.

For P_k written as (1), $T_k=-\log P_k$ becomes

$$T_k = |\log U_1|+\ldots+|\log U_{k-1}|+|\log(1-U_k)|, \quad k\in\mathbb{N}$$

which is a particular case of (49) with $(\xi,\eta)=(|\log U_1|,|\log(1-U_1)|)$. In view of this and Lemma 4.2 given below, the conditions of Theorem 2.1 hold with $\omega=\varepsilon_1=\varepsilon_2=1$, $\gamma=1/2$, $c=\mu^{-1}$, $W=B$ and $R_j=B_{j-1}$. □

Lemma 4.2. *Assume that $\mathtt{m}:=\mathbb{E}\xi<\infty$, $\mathtt{s}^2:=\mathrm{Var}\,\xi\in(0,\infty)$ and $\mathbb{E}\eta<\infty$. Then*

(a)

$$b_1 \le \tilde V(t) - \mathtt{m}^{-1}t \le a_0, \quad t\ge 0 \tag{51}$$

for some constants $b_1<0$ and $a_0>0$. Also,

$$\frac{\tilde N(t\cdot)-\mathtt{m}^{-1}(t\cdot)}{(\mathtt{s}^2\mathtt{m}^{-3}t)^{1/2}} \ \Rightarrow\ B(\cdot), \quad t\to\infty$$

in the J_1-topology on D.

(b) $\mathbb{E}\sup_{s\in[0,\,t]}(\tilde N(s)-\tilde V(s))^2 = O(t)$ *as $t\to\infty$.*

Proof. (a) A standard result of the renewal theory tells us that

$$0\le U(t)-\mathtt{m}^{-1}t\le a_0, \tag{52}$$

where a_0 is a known positive constant. The second inequality in combination with $\tilde V(t)\le U(t)$ proves the second inequality in (51). Using the first inequality in (52) yields

$$\begin{aligned}\tilde V(t)-\mathtt{m}^{-1}t &= \int_{[0,\,t]}(U(t-y)-\mathtt{m}^{-1}(t-y))\mathrm{d}\tilde G(y)\\ &- \mathtt{m}^{-1}\int_0^t(1-\tilde G(y))\mathrm{d}y \ge -\mathtt{m}^{-1}\int_0^t(1-\tilde G(y))\mathrm{d}y \ge -\mathtt{m}^{-1}\mathbb{E}\eta.\end{aligned}$$

For a proof of weak convergence see Theorem 3.2 in [2].
(b) We shall use a decomposition

$$\tilde{N}(t) - \tilde{V}(t) = \sum_{r\geq 0}(\mathbb{1}_{\{S_r+\eta_{r+1}\leq t\}} - \tilde{G}(t-S_r)) + \int_{[0,\,t]} \tilde{G}(t-x)\mathrm{d}(\nu(x)-U(x)),$$

where $\nu(x) := \#\{r \in \mathbb{N}_0 : S_r \leq t\}$ for $x \geq 0$, so that $U(x) = \mathbb{E}\nu(x)$. It suffices to prove that

$$\mathbb{E}\sup_{s\in[0,\,t]}\Big(\sum_{r\geq 0}(\mathbb{1}_{\{S_r+\eta_{r+1}\leq s\}} - \tilde{G}(s-S_r))\Big)^2 = O(t), \quad t\to\infty \tag{53}$$

and

$$D(t) := \mathbb{E}\sup_{s\in[0,\,t]}\Big(\int_{[0,\,s]}\tilde{G}(s-x)\mathrm{d}(\nu(x)-U(x))\Big)^2 = O(t), \quad t\to\infty. \tag{54}$$

Proof of (53). For each $j\in\mathbb{N}$, we write

$$\begin{aligned}\sup_{s\in[j,\,j+1)}\sum_{r\geq 0}(\mathbb{1}_{\{S_r+\eta_{r+1}\leq s\}} - \tilde{G}(s-S_r)) &\leq \sum_{r\geq 0}(\mathbb{1}_{\{S_r+\eta_{r+1}\leq j+1\}} - \tilde{G}(j+1-S_r))\\ &+ \sum_{r\geq 0}(\tilde{G}(j+1-S_r)-\tilde{G}(j-S_r)).\end{aligned}$$

Similarly,

$$\begin{aligned}\sup_{s\in[j,\,j+1)}\sum_{r\geq 0}(\mathbb{1}_{\{S_r+\eta_{r+1}\leq s\}} - \tilde{G}(s-S_r)) &\geq \sum_{r\geq 0}(\mathbb{1}_{\{S_r+\eta_{r+1}\leq j\}} - \tilde{G}(j-S_r))\\ &- \sum_{r\geq 0}(\tilde{G}(j+1-S_r)-\tilde{G}(j-S_r)).\end{aligned}$$

Thus, (53) is a consequence of

$$\sum_{j=0}^{[t]+1}\mathbb{E}\Big(\sum_{r\geq 0}(\mathbb{1}_{\{S_r+\eta_{r+1}\leq j\}} - \tilde{G}(j-S_r))\Big)^2 = O(t), \quad t\to\infty \tag{55}$$

and

$$\sum_{j=0}^{[t]+1}\mathbb{E}\Big(\sum_{r\geq 0}(\tilde{G}(j+1-S_r)-\tilde{G}(j-S_r))\Big)^2 = O(t), \quad t\to\infty. \tag{56}$$

The second moment in (55) is equal to $\int_{[0,\,j]}\tilde{G}(j-x)(1-\tilde{G}(j-x))\mathrm{d}U(x) \leq \int_{[0,\,j]}(1-\tilde{G}(j-x))\mathrm{d}U(x)$. In view of $\mathbb{E}\eta < \infty$, the function $x\mapsto 1-\tilde{G}(x)$ is directly Riemann integrable on $[0,\infty)$. According to Lemma 6.2.8 in [27] this implies that the right-hand side of the last inequality is $O(1)$, as $j\to\infty$, thereby proving (55).

Further, set $K(j) := \int_{[0,\,j]}(\tilde{G}(j+1-x)-\tilde{G}(j-x))\mathrm{d}\nu(x)$ for $j\in\mathbb{N}_0$. Then

$$\mathbb{E}\Big(\sum_{r\geq 0}(\tilde{G}(j+1-S_r)-\tilde{G}(j-S_r))\Big)^2 \leq 2(\mathbb{E}K(j)^2+\mathbb{E}(\nu(j+1)-\nu(j))^2) \leq 2(\mathbb{E}K(j)^2+\mathbb{E}\nu(1)^2),$$

where the last inequality is a consequence of distributional subadditivity of ν, that is, $\mathbb{P}\{\nu(t+s)-\nu(s)>x\} \leq \mathbb{P}\{\nu(t)>x\}$ for $t,s,x\geq 0$. Recall that $\nu(1)$ has finite exponential moments, so that trivially $\mathbb{E}\nu(1)^2 < \infty$. Left with estimating $\mathbb{E}K(j)^2$ we infer

$$\begin{aligned}
\mathbb{E}K(j)^2 &= \mathbb{E}\Big(\tilde{G}(j+1) - \tilde{G}(j) + \sum_{k=0}^{j-1}\int_{[k,\,k+1)} \tilde{G}(j+1-x) - \tilde{G}(j-x)\mathrm{d}\nu(x)\Big)^2 \\
&\leq \mathbb{E}\Big(1 + \sum_{k=0}^{j-1}(\tilde{G}(j+1-k) - \tilde{G}(j-k))(\nu(k+1) - \nu(k))\Big)^2 \\
&\leq 2\Big(1 + (\tilde{G}(j) + \tilde{G}(j+1) - \tilde{G}(1))^2 \sum_{k=0}^{j-1} \frac{\tilde{G}(j+1-k) - \tilde{G}(j-k)}{\tilde{G}(j) + \tilde{G}(j+1) - \tilde{G}(1)} \mathbb{E}(\nu(k+1) - \nu(k))^2\Big) \\
&\leq 2(1 + (\tilde{G}(j) + \tilde{G}(j+1) - \tilde{G}(1))^2 \mathbb{E}\nu(1)^2) \leq 2(1 + 4\mathbb{E}\nu(1)^2).
\end{aligned}$$

Here, the second inequality is implied by convexity of $x \mapsto x^2$. Combining the obtained estimates together we arrive at (56).

Proof of (54). Assuming that

$$\mathbb{E} \sup_{s\in[0,\,t]} (\nu(s) - U(s))^2 = O(t), \tag{57}$$

integration by parts in (54) yields

$$D(t) = \mathbb{E} \sup_{s\in[0,\,t]} \Big(\int_{[0,\,s]} (\nu(s-x) - U(s-x))\mathrm{d}\tilde{G}(x)\Big)^2 \leq \tilde{G}(t)^2 \mathbb{E} \sup_{s\in[0,\,t]} (\nu(s) - U(s))^2 = O(t)$$

which proves (54).

Passing to the proof of (57) we first observe that in view of (52) relation (57) is equivalent to

$$\mathbb{E} \sup_{s\in[0,\,t]} (\nu(s) - \mathtt{m}^{-1}s)^2 = O(t), \quad t \to \infty. \tag{58}$$

Since $s \mapsto \nu(s) - \mathtt{m}^{-1}s$ is a (random) piecewise linear function with slope $-\mathtt{m}^{-1}$ having unit jumps at times $S_0, S_1, \ldots$ we conclude that

$$\begin{aligned}
\sup_{s\in[0,\,t]} (\nu(s) - \mathtt{m}^{-1}s)^2 &\leq \max\big(\max_{0\leq k\leq \nu(t)} (k - \mathtt{m}^{-1}S_k)^2, \max_{0\leq k\leq \nu(t)-1} (k + 1 - \mathtt{m}^{-1}S_k)^2\big) \\
&\leq 2\big(1 + \max_{0\leq k\leq \nu(t)} (k - \mathtt{m}^{-1}S_k)^2\big).
\end{aligned}$$

Applying Doob's inequality to the martingale $(S_{\nu(t)\wedge n} - \mathtt{m}(\nu(t)\wedge n))_{n\in\mathbb{N}_0}$ (this is a martingale with respect to the filtration generated by the ξ_k because $\nu(t)$ is a stopping time with respect to the same filtration) we obtain

$$\begin{aligned}
\mathbb{E} \max_{0\leq k\leq \nu(t)\wedge n} (S_k - \mathtt{m}k)^2 &= \mathbb{E} \max_{0\leq k\leq n} (S_{\nu(t)\wedge k} - \mathtt{m}(\nu(t)\wedge k))^2 \\
&\leq 4\mathbb{E}(S_{\nu(t)\wedge n} - \mathtt{m}(\nu(t)\wedge n))^2 = 4\mathtt{s}^2\mathbb{E}(\nu(t)\wedge n)
\end{aligned}$$

for each $n \in \mathbb{N}$. Here, the last equality is nothing else but Wald's identity. An application of Lévy's monotone convergence theorem yields

$$\mathbb{E} \max_{0\leq k\leq \nu(t)} (S_k - \mathtt{m}k)^2 \leq 4\mathtt{s}^2 U(t).$$

In view of (52) the right-hand side is $O(t)$, as $t \to \infty$, and (58) follows.

□

Recall that $(P_k)_{k\in\mathbb{N}}$ follows the GEM distribution with parameter $\theta > 0$ when U_i's in (1) are beta$(\theta, 1)$-distributed, in which case $\mu = \mathbb{E}|\log U_1| = \theta^{-1}, \sigma^2 = \mathrm{Var}(\log U_1) = \theta^{-2}$ and $\mathbb{E}|\log(1-U_1)| = \theta \sum_{n\geq 1} n^{-1}(n+\theta)^{-1} < \infty$.

Corollary 4.3. *For $\theta > 0$ let $(P_k)_{k\in\mathbb{N}}$ be* GEM*-distributed with parameter θ, or any random sequence such that the sequence of P_k's arranged in decreasing order follows the* PD *distribution with parameter θ. Then*

$$\left(\frac{(j-1)!(K_{n,j}(\cdot) - (j!)^{-1}(\theta \log n(\cdot))^j)}{\sqrt{(\theta \log n)^{2j-1}}}\right)_{j\in\mathbb{N}} \Rightarrow (B_{j-1}(\cdot))_{j\in\mathbb{N}}, \quad n \to \infty. \tag{59}$$

in the product J_1-topology on $D[0,1]^{\mathbb{N}}$.

5 Some regenerative models

For $(X(t))_{t\geq 0}$ a drift-free subordinator with $X(0) = 0$ and a nonzero Lévy measure ν supported by $(0, \infty)$ let

$$\Delta X(t) = X(t) - X(t-),\ t \geq 0,$$

be the associated process of jumps. The process $\Delta X(\cdot)$ assumes nonzero values on a countable set, which is dense in case $\nu(0, \infty) = \infty$. The transformed process (multiplicative subordinator) $F(t) = 1 - e^{-X(t)}$, $t \geq 0$, has the associated process of jumps

$$\Delta F(t) = e^{-X(t-)}(1 - e^{-\Delta X(t)}),\ t \geq 0.$$

In this section we identify the fragmentation law $(P_k)_{k\in\mathbb{N}}$ with nonzero jumps $\Delta F(\cdot)$ arranged in some order (for instance by decrease). Note that multiplying the Lévy measure by a positive factor corresponds to a time-change for F, hence does not affect the derived fragmentation law.

We shall assume that the Lévy measure ν is infinite and has the right tail $\nu([x, \infty))$ satisfying

$$\beta_0 + \beta_1 |\log x|^{q-r_2} \leq \nu([x, \infty)) - c_0 |\log x|^q \leq \alpha_0 + \alpha_1 |\log x|^{q-r_1} \tag{60}$$

for small enough $x > 0$ and some $q, c_0, \alpha_0, \alpha_1 > 0$, $1/2 < r_1, r_2 \leq q + 1$ and $\beta_0, \beta_1 < 0$.

Theorem 5.1. *Assume that* (60) *holds with $q, r_1, r_2 \geq 1$ and*

$$m := \mathbb{E}X(1) = \int_{[0,\infty)} x\nu(\mathrm{d}x) < \infty, \quad s^2 := \mathrm{Var}\, X(1) = \int_{[0,\infty)} x^2\nu(\mathrm{d}x) < \infty.$$

Then

$$\left(\frac{K_{n,j}(\cdot) - c_j^*(\log n(\cdot))^{(q+1)j}}{q\mathrm{B}(q, (q+1)j - q)sm^{-3/2}c_{j-1}^*(\log n)^{(q+1)j-1/2}}\right)_{j\in\mathbb{N}} \Rightarrow (B_{(q+1)j-1}(\cdot))_{j\in\mathbb{N}}, \quad n \to \infty$$

in the product J_1-topology on $D[0,1]^{\mathbb{N}}$, where

$$c_j^* := \frac{c_0\Gamma(q+2)}{m(q+1)\Gamma((q+1)j+1)}, \quad j \in \mathbb{N}.$$

Theorem 5.1 applies to the gamma subordinator with the Lévy measure

$$\nu(\mathrm{d}x) = \theta x^{-1} e^{-\lambda x}\, \mathbb{1}_{(0,\infty)}(x)\mathrm{d}x$$

and to the subordinator with

$$\nu(\mathrm{d}x) = \theta(1-e^{-x})^{-1} e^{-\lambda x}\, \mathbb{1}_{(0,\infty)}(x)\mathrm{d}x,$$

where $\theta, \lambda > 0$. In both cases $s^2 < \infty$ and (60) holds with $c_0 = q = r_1 = r_2 = 1$.

Theorem 5.1 is a consequence of Theorem 2.1, the easily checked formula

$$\int_{[0,\,u]} (u-y)^\alpha \mathrm{d}B_q(y) = q\mathrm{B}(q, \alpha+1) \int_{[0,\,u]} (u-y)^{q+\alpha} \mathrm{d}B(y), \quad u \geq 0,\ \alpha, q > 0$$

which we use for $\alpha = (q+1)(j-1)$ and the next lemma.

Lemma 5.2. *Assume that* (60) *holds and* $s^2 < \infty$. *Then the following is true:*

(a)

$$b_0 + b_1 t^{q-\min(r_2-1,0)} \leq V(t) - c_0(m(q+1))^{-1} t^{q+1} \leq a_0 + a_1 t^{q-\min(r_1-1,0)}, \quad t > 0 \quad (61)$$

for some constants a_0, $a_1 > 0$ *and* b_0, $b_1 \leq 0$, *where* $m = \mathbb{E}X(1) < \infty$,

(b)

$$\frac{N(t\cdot) - c_0(m(q+1))^{-1}(t\cdot)^{q+1}}{sm^{-3/2}t^{q+1/2}} \ \Rightarrow\ B_q(\cdot), \quad t \to \infty$$

in the J_1*-topology on* D.

(c) *Assume that* $q, r_1, r_2 \geq 1$. *Then* $\mathbb{E}\sup_{s\in[0,\,t]}(N(s) - V(s))^2 = O(t^{2q+1})$ *as* $t \to \infty$.

Proof. (a) Set $f(x) := \nu([-\log(1-e^{-x}), \infty))$ for $x \geq 0$. Inequality (60) in combination with $\lim_{x\to\infty} \nu([x,\infty)) = 0$ entails

$$\beta_0 + \beta_1 x^{q-r_2} \leq f(x) - c_0 x^q \leq \alpha_0 + \alpha_1 x^{q-r_1} \quad (62)$$

for all $x > 0$ and some constants α_0, α_1, β_0 and β_1 which are not necessarily the same as in (60).

Since

$$N(t) = \sum \mathbb{1}_{\{X(s-)-\log(1-e^{-\Delta X(s)})\leq t\}} = \sum \mathbb{1}_{\{\Delta X(s)\geq -\log(1-e^{-(t-X(s-))})\}},$$

where the summation extends to all $s > 0$ with $\Delta X(s) > 0$, we conclude that $V(x) = \mathbb{E}N(x) = \int_{[0,\,x]} f(x-y)\mathrm{d}U^*(y)$, where $U^*(x) := \int_0^\infty \mathbb{P}\{X(t) \leq x\}\mathrm{d}x = \mathbb{E}T(x)$ is the renewal function and $T(x) := \inf\{t > 0 : X(t) > x\}$ for $x \geq 0$.

Similarly to (52) we have

$$0 \leq U^*(t) - m^{-1}t \leq a_0^*, \quad t \geq 0, \quad (63)$$

where a_0^* is a known positive constant. Using this and (62) we infer

$$\begin{aligned} V(t) - c_0(m(q+1))^{-1}t^{q+1} &= \int_{[0,\,t]} (U^*(t-y) - m^{-1}(t-y))\mathrm{d}f(y) \\ &+ m^{-1}\int_0^t (f(y) - c_0 y^q)\mathrm{d}y \leq a_0^* f(t) + m^{-1}\int_0^t (\alpha_0 + \alpha_1 y^{q-r_1})\mathrm{d}y \\ &\leq a_0(\alpha_0 + \alpha_1 t^{q-r_1} + c_0 t^q) + m^{-1}(\alpha_0 t + \alpha_1(q-r_1+1)^{-1}t^{q-r_1+1}). \end{aligned}$$

This proves the second inequality in (61). Arguing analogously we obtain

$$\begin{aligned} V(t)-c_0(m(q+1))^{-1}t^{q+1} &\geq m^{-1}\int_0^t (f(y)-c_0y^q)\mathrm{d}y \geq m^{-1}\int_0^t(\beta_0+\beta_1 y^{q-r_2})\mathrm{d}y \\ &\geq m^{-1}(\beta_0 t+\beta_1(q-r_2+1)^{-1}t^{q+1-r_2}), \end{aligned}$$

thereby proving the first inequality in (61).
(b) Write

$$\begin{aligned} N(t) &= \sum \big(\mathbb{1}_{\{\Delta X(s)\geq -\log(1-e^{-(t-X(s-))})\}} - f(t-X(s-))\big)\,\mathbb{1}_{\{X(s-)\leq t\}} \\ &+ \sum f(t-X(s-))\,\mathbb{1}_{\{X(s-)\leq t\}} =: N_1(t)+N_2(t). \end{aligned} \tag{64}$$

As a preparation for the proof of part (b) we intend to show that

$$\lim_{t\to\infty} t^{-q-1/2}N_1(t)=0 \quad \text{a.s.,} \quad t\to\infty. \tag{65}$$

Proof of (65). To reduce technicalities to a minimum we only consider the case $q>1$. Since $\mathbb{E}[N_1(t)]^2=V(t)$ and $V(t)\sim c_0(m(q+1))^{-1}t^{q+1}$ as $t\to\infty$ we conclude that

$$\lim_{\mathbb{N}\ni\ell\to\infty} \ell^{(q+1/2)}N_1(\ell)=0 \quad \text{a.s.}$$

by the Borel-Cantelli lemma. For each $t\geq 0$, there exists $\ell\in\mathbb{N}_0$ such that $t\in[\ell,\ell+1)$. Now we use a.s. monotonicity of $N(t)$ and $N_2(t)$ to obtain

$$\begin{aligned} &(\ell+1)^{-(q+1/2)}(N_1(\ell)-(N_2(\ell+1)-N_2(\ell)))\leq t^{-(q+1/2)}N_1(t) \\ \leq\; &\ell^{-(q+1/2)}(N_1(\ell+1)+N_2(\ell+1)-N_2(\ell)) \quad \text{a.s.} \end{aligned}$$

Thus, it remains to check that

$$\lim_{\ell\to\infty}\ell^{-(q+1/2)}(N_2(\ell+1)-N_2(\ell))=0, \quad \text{a.s.}$$

In view of (62), f satisfies a counterpart of (15), whence

$$\begin{aligned} N_2(\ell+1)-N_2(\ell) &= \int_{[0,\,\ell]}(f(\ell+1-y)-f(\ell-y))\mathrm{d}T(y)+\int_{(\ell,\,\ell+1]}f(\ell+1-y)\mathrm{d}T(y) \\ &\leq (c_0(q-1)(\ell+1)^{q-1}+\alpha_0+\alpha_1(\ell+1)^{q-r_1}-\beta_0+|\beta_1|\ell^{q-r_2}+f(1))T(\ell+1) \\ &= O(\ell^{q+\max(1-r_1,1-r_2,0)})=o(\ell^{q+1/2}) \end{aligned} \tag{66}$$

a.s. as $\ell\to\infty$. For the penultimate equality we have used the strong law of large numbers for $T(y)$. The last equality follows from $r_1,r_2>1/2$.

We are ready to prove part (b). We shall use representation (64). Relation (65) entails

$$t^{-q-1/2}\sup_{y\in[0,\,T]}N_1(ty)\ \stackrel{\mathrm{P}}{\to}\ 0, \quad t\to\infty. \tag{67}$$

for each $T>0$. Thus, we are left with showing that

$$\frac{N_2(t\cdot)-c_0(m(q+1))^{-1}(t\cdot)^{q+1}}{sm^{-3/2}t^{q+1/2}}\ \Rightarrow\ B_q(\cdot), \quad t\to\infty$$

in the J_1-topology on D. The proof of this is similar to that of weak convergence of the jth coordinate, $j \geq 2$, in (22). The only difference is that, instead of (12), we use

$$\frac{T(t\cdot) - m^{-1}(t\cdot)}{sm^{-3/2}t^{1/2}} \Rightarrow B(\cdot), \quad t \to \infty$$

in the J_1-topology on D, where B is a Brownian motion, see Theorem 2a in [11].
(c) Since the proof is analogous to that of Lemma 4.2(b) we only give a sketch. In view of (64) it suffices to prove that, as $t \to \infty$,

$$\mathbb{E} \sup_{s\in[0,\,t]} \Big(\sum \big(\mathbb{1}_{\{\Delta X(v) \geq -\log(1-e^{-(s-X(v-))})\}} - f(s - X(v-))\big) \mathbb{1}_{\{X(v-)\leq s\}} \Big)^2 = O(t^{2q+1}) \qquad (68)$$

and

$$\mathbb{E} \sup_{s\in[0,\,t]} \Big(\int_{[0,\,s]} f(s-x)\mathrm{d}(T(x) - U^*(x)) \Big)^2 = O(t^{2q+1}). \qquad (69)$$

Proof of (68). Arguing as in the proof of Lemma 4.2(b) we conclude that (68) is a consequence of

$$\sum_{\ell=0}^{[t]+1} \mathbb{E}\Big(\sum \big(\mathbb{1}_{\{\Delta X(v) \geq -\log(1-e^{-(\ell-X(v-))})\}} - f(\ell - X(v-))\big) \mathbb{1}_{\{X(v-)\leq \ell\}} \Big)^2 = O(t^{2q+1}) \qquad (70)$$

and

$$\sum_{\ell=0}^{[t]+1} \mathbb{E}(N_2(\ell+1) - N_2(\ell))^2 = O(t^{2q+1}). \qquad (71)$$

The second moment in (70) is equal to $V(\ell) = O(\ell^{q+1})$. This entails that the left-hand side of (70) is $O(t^{q+2})$, hence $O(t^{2q+1})$ because of the assumption $q \geq 1$. Finally, since $r_1, r_2 \geq 1$ by assumption and $\mathbb{E}[T(\ell)]^2 = O(\ell^2)$, inequality (66) entails $\mathbb{E}(N_2(\ell+1) - N_2(\ell))^2 = O(\ell^{2q})$ and thereupon (71).
Proof of (69). Set $\hat{\nu}(x) := \inf\{k \in \mathbb{N} : X(k) > x\}$ for $x \geq 0$. Since $T(x) \leq \hat{\nu}(x) \leq T(x) + 1$ a.s. and, according to (57), $\mathbb{E}\sup_{s\in[0,\,t]}(\hat{\nu}(s) - \mathbb{E}\hat{\nu}(s))^2 = O(t)$ as $t \to \infty$, we infer $\mathbb{E}\sup_{s\in[0,\,t]}(T(s) - U^*(s))^2 = O(t)$ as $t \to \infty$. With this at hand, relation (69) readily follows.

The proof of Lemma 5.2 is complete. □

6 The Poisson-Kingman model

Let $(X(t))_{t\geq 0}$ be a subordinator as in Section 5 with the only differences that the parameters in (60) satisfy $q \in (0, 2)$, $q/2 < r_1, r_2 \leq q$ and that we additionally assume

$$\int_{(1,\infty)} (\log x)^s \nu(\mathrm{d}x) < \infty, \qquad (72)$$

where $s = 2q$ when $q \in (0, 3/2)$ and $s = \varepsilon + q/(2-q)$ for some $\varepsilon > 0$ when $q \in [3/2, 2)$.

The ranked sequence of jumps of the process $(X(t)/X(1))_{t\in[0,1]}$ can be represented as $P_j := L_j/L$, where $L_1 \geq L_2 \geq \cdots > 0$ is the sequence of atoms of a non-homogeneous Poisson random measure with mean measure ν, and $L := \sum_{j\geq 1} L_j \overset{\mathrm{d}}{=} X(1)$. This is the Poisson-Kingman construction [32, Section 3] of probabilities $(P_j)_{j\in\mathbb{N}}$, which we regard as fragmentation law underlying a nested occupancy scheme.

Theorem 6.1. *Assume that the function $x \mapsto \nu((x,\infty))$ is strictly increasing and continuous on $[0,\infty)$. For the fragmentation law as described above limit relation* (10) *holds with $\omega = q$, $\gamma = q/2$, $c = c_0$, $a = c_0^{1/2}$ and $W(s) := B(s^q)$ for $s \geq 0$ being a time changed Brownian motion.*

Theorem 6.1 is a consequence of Theorem 2.1 and Lemma 6.2 given next.

Lemma 6.2. *Under the assumptions of Theorem 6.1 the following is true:*

(a)

$$\beta_2 + \beta_3 t^{q-r_4} \leq V(t) - c_0 t^q \leq \alpha_2 + \alpha_3 t^{q-r_3}, \quad t > 0 \tag{73}$$

for some constants $\alpha_2, \alpha_3 > 0$, $q \in (0,2)$, $q/2 < r_3, r_4 \leq q$ and β_2, $\beta_3 < 0$.

(b)

$$\mathbb{E} \sup_{s\in[0,\,t]} (N(s) - V(s))^2 = O(t^q), \quad t \to \infty.$$

(c)

$$\frac{N(t\cdot) - c_0(t\cdot)^q}{(c_0 t^q)^{1/2}} \;\Rightarrow\; W(\cdot), \quad t \to \infty$$

in the J_1-topology on D, where $W(s) = B(s^q)$ for $s \geq 0$.

Condition (72) *is not needed for part (c).*

Proof. For $t \in \mathbb{R}$, set $\widehat{N}(t) := \#\{k \in \mathbb{N} : L_k \geq e^{-t}\}$ so that $N(t) = \#\{k \in \mathbb{N} : L_k/L \geq e^{-t}\} = \widehat{N}(t - \log L)$. Note that $N(t) = 0$ for $t < 0$. Further, put $m(t) := \nu((e^{-t},\infty))$ for $t \in \mathbb{R}$ and note that m is a strictly increasing and continuous function with $m(-\infty) = 0$. In view of (60)

$$\beta_0 + \beta_1 t^{q-r_2} \leq m(t) - c_0 t^q \leq \alpha_0 + \alpha_1 t^{q-r_1} \tag{74}$$

for[2] $t \geq 0$, where $\alpha_0, \alpha_1 > 0$, $q \in (0,2)$, $q/2 < r_1, r_2 \leq q$ and $\beta_0, \beta_1 < 0$. Later, we shall need the following consequences of (74):

$$m(t) \;\sim\; c_0 t^q, \quad t \to \infty \tag{75}$$

and

$$\lim_{t\to\infty} \sup_{s\in[0,\,s_0]} \Big| \frac{m(ts)}{c_0 t^q} - s^q \Big| = 0 \tag{76}$$

for all $s_0 > 0$. For the latter we have also used Dini's theorem.

The random process $(\widehat{N}(t))_{t\in\mathbb{R}}$ is non-homogeneous Poisson. In particular, $\widehat{N}(t)$ has a Poisson distribution of mean $m(t)$. Let $\mathcal{P} := (\mathcal{P}(t))_{t\geq 0}$ denote a homogeneous Poisson process of unit intensity. Throughout the proof we use the representation $(\widehat{N}(t))_{t\in\mathbb{R}} = (\mathcal{P}(m(t))_{t\in\mathbb{R}}$ which gives us a transition from $\mathcal{P}$ to $\widehat{N}$. The converse transition, namely that the arrival times of $\mathcal{P}$ are $m(-\log L_1)$, $m(-\log L_2), \ldots$ is secured by our assumption that m is strictly increasing and continuous (this assumption is not needed to guarantee the direct transition).

(a) Write

$$N(t) - \widehat{N}(t) \;=\; (\widehat{N}(t - \log L) - \widehat{N}(t))\, \mathbb{1}_{\{L\leq 1\}} - (\widehat{N}(t) - \widehat{N}(t - \log L))\, \mathbb{1}_{\{L>1\}} =: N_1(t) - N_2(t)$$

[2]Actually, (60) only ensures that (74) holds for large enough t. However, adjusting α_i and β_i, $i = 0,1$ properly one obtains (74) for all $t \geq 0$. Of course, α_i and β_i in (74) are not necessarily the same as in (60).

and observe that

$$\begin{aligned} N_1(t) &\le (\widehat{N}(t-\log L_1)-\widehat{N}(t))\,\mathbb{1}_{\{L_1\le 1\}} \le (1+\mathcal{P}^*(m(t-\log L_1)-m(-\log L_1)) \\ &- \mathcal{P}^*(m(t)-m(-\log L_1)))\,\mathbb{1}_{\{L_1\le 1\}}, \end{aligned} \tag{77}$$

where $\mathcal{P}^* := (\mathcal{P}^*(t))_{t\ge 0}$ is a homogeneous Poisson process of unit intensity which is independent of L_1. More precisely, the arrival times of $\mathcal{P}^*$ are $m(-\log L_2)-m(-\log L_1)$, $m(-\log L_3)-m(-\log L_1),\ldots$. For later use we note that

$$\begin{aligned} &((\mathcal{P}^*(m(t-\log L_1)-m(-\log L_1))-\mathcal{P}^*(m(t)-m(-\log L_1)))\,\mathbb{1}_{\{e^{-t}\le L_1\le 1\}})_{t\ge 0} \\ \overset{\mathrm{d}}{=}\ &((\mathcal{P}^*(m(t-\log L_1))-\mathcal{P}^*(m(t)))\,\mathbb{1}_{\{e^{-t}\le L_1\le 1\}})_{t\ge 0}, \end{aligned} \tag{78}$$

where $\overset{\mathrm{d}}{=}$ means that the distributions of the processes are the same. Inequality (77) entails

$$\mathbb{E}N_1(t) \le (1+\mathbb{E}(m(t-\log L_1)-m(t\vee(-\log L_1)))\,\mathbb{1}_{\{L_1\le 1\}} \le (1+\mathbb{E}(m(t-\log L_1)-m(t)))\,\mathbb{1}_{\{L_1\le 1\}}.$$

In view of (74) for $t, x\ge 0$

$$\begin{aligned} &m(t+x)-m(t) \\ \le\ &c_0((t+x)^q-t^q)+\alpha_0+\alpha_1(x+t)^{q-r_1}+|\beta_0|+|\beta_1|t^{q-r_2} \le c_0(x^q\,\mathbb{1}_{\{q\in(0,1]\}} \\ +\ &q(x^{q-1}+t^{q-1})x\,\mathbb{1}_{\{q\in(1,2)\}})+\alpha_0+\alpha_1(x^{q-r_1}+t^{q-r_1})+|\beta_0|+|\beta_1|t^{q-r_2}. \end{aligned} \tag{79}$$

We have used subadditivity of $x\mapsto x^\kappa$ on $\mathbb{R}_+ := [0,\infty)$ when $\kappa\in(0,1]$ and the mean value theorem for differentiable functions to obtain $(t+x)^\kappa-t^\kappa = \kappa x(t+x)^{\kappa-1}$ when $\kappa>1$. We infer

$$\mathbb{E}(\log_- L_1)^\alpha < \infty \quad \text{for any } \alpha>0 \tag{80}$$

as a consequence of $\int_1^\infty y^{\alpha-1}\mathbb{P}\{-\log L_1>y\}\mathrm{d}y = \int_1^\infty y^{\alpha-1}e^{-m(y)}\mathrm{d}y<\infty$, where the finiteness is justified by (74). Hence,

$$\begin{aligned} \mathbb{E}N_1(t) &\le 1+c_0(\mathbb{E}(\log_- L_1)^q\,\mathbb{1}_{\{q\in(0,1]\}}+q(\mathbb{E}(\log_- L_1)^q+t^{q-1}\mathbb{E}(\log_- L_1)\,\mathbb{1}_{\{q\in(1,2)\}})+\alpha_0 \\ &+ \alpha_1(\mathbb{E}(\log_- L_1)^{q-r_1}+t^{q-r_1})+|\beta_0|+|\beta_1|t^{q-r_2}. \end{aligned}$$

Thus, the right-hand inequality in part (a) holds with $r_3 = r_1\wedge r_2$ when $q\in(0,1]$ and $r_3 = r_1\wedge r_2\wedge 1$ when $q\in(1,2)$.

To analyse $N_2(t)$, set $\theta := q$ if $q\in(0,1]$ and $\theta := q/(2-q)$ if $q\in(1,2)$ and then pick $\varepsilon>0$ such that $\theta+\varepsilon\le 2q$ when $q\in(0,3/2)$ and take the same ε as in (72) when $q\in[3/2,2)$. Further, choose $\delta<1-(q\vee 1)/2$ and $\varrho_1>1$ sufficiently close to one to ensure that $r_5 := (\theta+\varepsilon)\delta/\varrho_1 > q/2$. Put $\varrho_2 := \varrho_1/(\varrho_1-1)$. It holds that

$$\begin{aligned} N_2(t) &= (\widehat{N}(t)-\widehat{N}(t-\log L))\,\mathbb{1}_{\{1<L\le \exp(t^\delta)\}}+(\widehat{N}(t)-\widehat{N}(t-\log L))\,\mathbb{1}_{\{L>\exp(t^\delta)\}} \\ &\le (\widehat{N}(t)-\widehat{N}(t-t^\delta))+\widehat{N}(t)\,\mathbb{1}_{\{L>\exp(t^\delta)\}}. \end{aligned}$$

Condition (72) ensures that $\mathbb{E}(\log_+ L)^{\theta+\varepsilon}<\infty$ by Theorem 25.3 in [39]. A combination of Hölder's and Markov's inequalities yields

$$\begin{aligned} \mathbb{E}\widehat{N}(t)\,\mathbb{1}_{\{L>\exp(t^\delta)\}} &\le (\mathbb{E}(\widehat{N}(t))^{\varrho_2})^{1/\varrho_2}(\mathbb{P}\{\log L>t^\delta\})^{1/\varrho_1} \\ &\le (\mathbb{E}(\widehat{N}(t))^{\varrho_2})^{1/\varrho_2}(\mathbb{E}(\log_+ L)^{\theta+\varepsilon})^{1/\varrho_1}t^{-(\theta+\varepsilon)\delta/\varrho_1}. \end{aligned}$$

Since $\widehat{N}(t)$ has a Poisson distribution of mean $m(t)$, and $m(t)$ satisfies (75), the right-hand side does not exceed $\alpha_5 + \alpha_4 t^{q-r_5}$ for $t \geq 0$ and some $\alpha_4, \alpha_5 > 0$.

Further, using (74) we obtain for $t \geq 0$

$$\begin{aligned}\mathbb{E}(\widehat{N}(t) - \widehat{N}(t-t^\delta)) &= m(t) - m(t-t^\delta) \leq c_0(t^{\delta q}\,\mathbb{1}_{\{q\in(0,1]\}} + qt^{q-1+\delta}\,\mathbb{1}_{\{q\in(1,2)\}}) + \alpha_0 + \alpha_1 t^{q-r_1}\\ &+ |\beta_0| + |\beta_1| t^{q-r_2} \leq \alpha_7 + \alpha_6 t^{q-r_6}.\end{aligned}$$

By our choice of δ, r_6 satisfies $r_6 > q/2$. We have proved the left-hand inequality in part (a) with $r_4 := r_5 \wedge r_6$.

(b) Having written

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,\,t]}(N(s) - V(s))^2\,\mathbb{1}_{\{L>1\}}\\ \leq\ & 3\Big(\mathbb{E}\sup_{s\in[0,\,t]}(\mathcal{P}(m(s-\log L)) - m(s-\log L))^2\,\mathbb{1}_{\{L>1\}}\\ +\ & \mathbb{E}\sup_{s\in[0,\,t]}(m(s-\log L) - m(s))^2\,\mathbb{1}_{\{L>1\}} + \sup_{s\in[0,\,t]}(m(s) - V(s))^2\Big),\end{aligned}$$

we intend to show that each of the three terms on the right-hand side is $O(t^q)$.

1st summand. Recall that $(\mathcal{P}(t) - t)_{t\geq 0}$ is a martingale with respect to the natural filtration. Using

$$\sup_{s\in[0,\,t]}(\mathcal{P}(m(s-\log L)) - m(s-\log L))^2\,\mathbb{1}_{\{L>1\}} \leq \sup_{s\in(-\infty,\,t]}(\mathcal{P}(m(s)) - m(s))^2 \leq \sup_{s\in[0,\,m(t)]}(\mathcal{P}(s) - s)^2$$

and then invoking Doob's inequality we obtain

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,\,t]}(\mathcal{P}(m(s-\log L)) - m(s-\log L))^2\,\mathbb{1}_{\{L>1\}} &\leq \mathbb{E}\sup_{s\in[0,\,m(t)]}(\mathcal{P}(s) - s)^2\\ &\leq 4\mathbb{E}(\mathcal{P}(m(t)) - m(t))^2 = 4m(t) = O(t^q).\end{aligned}$$

2nd summand. The following inequalities hold

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,\,t]}(m(s) - m(s-\log L))^2\,\mathbb{1}_{\{L>1\}}\\ \leq\ & (m(t) - m(0))^2\mathbb{P}\{\log L > t\} + \mathbb{E}\sup_{s\in[0,\,t-\log L]}(m(s+\log L) - m(s))^2\,\mathbb{1}_{\{0<\log L\leq t\}}\\ \leq\ & (m(t) - m(0))^2\mathbb{P}\{\log L > t\} + \mathbb{E}\sup_{s\in[0,\,t]}(m(s+\log L) - m(s))^2\,\mathbb{1}_{\{\log L>0\}}\,. \qquad (81)\end{aligned}$$

Note that (72) entails $\mathbb{E}(\log_+ L)^{2q} < \infty$. In view of (75) the first summand on the right-hand side of (81) is $O(1)$ by Markov's inequality. Using (79) in combination with $\mathbb{E}(\log_+ L)^{2q} < \infty$ we conclude that the second summand on the right-hand side of (81) is $O(t^q)$.

3rd summand. Using (73) and (74) yields $\sup_{s\in[0,\,t]}(m(s) - V(s))^2 \leq \sup_{s\in[0,\,t]}(C_1 + C_2 s^{q-r})^2 = O(t^{2q-2r})$ for appropriate constants C_1, C_2 and $q/2 < r \leq q$. The latter inequality ensures that $\sup_{s\in[0,\,t]}(m(s) - V(s))^2 = O(t^q)$.

To deal with the expectation in question on the event $\{L \leq 1\}$ we write

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,\,t]}(N(s) - V(s))^2\,\mathbb{1}_{\{L\leq 1\}}\\ \leq\ & 3\Big(\mathbb{E}\sup_{s\in[0,\,t]}(\mathcal{P}(m(s-\log L)) - \mathcal{P}(m(s)))^2\,\mathbb{1}_{\{L\leq 1\}}\\ +\ & \mathbb{E}\sup_{s\in[0,\,t]}(\mathcal{P}(m(s)) - m(s))^2 + \sup_{s\in[0,\,t]}(m(s) - V(s))^2\Big).\end{aligned}$$

We already know from the previous part of the proof, that the second and the third summand on the right-hand side are $O(t^q)$. As for the first summand, we use (77) and (78) to obtain

$$\begin{aligned} & \mathbb{E} \sup_{s\in[0,\,t]} (\mathcal{P}(m(s-\log L))-\mathcal{P}(m(s)))^2\,\mathbb{1}_{\{L\le 1\}} \\ \le\ & \mathbb{E}(1+\mathcal{P}^*(m(t-\log L_1))-m(-\log L_1))^2\,\mathbb{1}_{\{-\log L_1>t\}} \\ +\ & \mathbb{E} \sup_{s\in[-\log L_1,\,t]} (1+\mathcal{P}^*(m(s-\log L_1))-\mathcal{P}(m(s)))^2\,\mathbb{1}_{\{0\le -\log L_1\le t\}} \end{aligned}$$

The principal asymptotic term of the first summand is $\mathbb{E}(m(t-\log L_1)-m(t))^2\,\mathbb{1}_{\{-\log L_1>t\}}$. Invoking (79) and (80) we infer that the last expression is $o(1)$. To estimate the second summand we write

$$\begin{aligned} & \mathbb{E} \sup_{s\in[-\log L_1,\,t]} (\mathcal{P}^*(m(s-\log L_1))-\mathcal{P}(m(s)))^2\,\mathbb{1}_{\{0\le -\log L_1\le t\}} \\ \le\ & 3\Big(\mathbb{E} \sup_{s\in[-\log L_1,\,t]} (\mathcal{P}^*(m(s-\log L_1))-m(s-\log L_1))^2\,\mathbb{1}_{\{0\le -\log L_1\le t\}} \\ +\ & \mathbb{E}\sup_{s\in[0,\,t]} (\mathcal{P}^*(m(s))-m(s))^2+\mathbb{E}\sup_{s\in[0,\,t]}(m(s-\log L_1)-m(s))^2\,\mathbb{1}_{\{-\log L_1\ge 0\}}\Big) \\ \le\ & 3\Big(2\mathbb{E}\sup_{s\in[0,\,2t]} (\mathcal{P}^*(m(s))-m(s))^2+\mathbb{E}\sup_{s\in[0,\,t]}(m(s-\log L_1)-m(s))^2\,\mathbb{1}_{\{-\log L_1\ge 0\}}\Big). \end{aligned}$$

The last expression is $O(t^q)$ which can be seen by mimicking the arguments used in the previous part of the proof.

(c) A specialization of the functional limit theorem for the renewal processes with finite variance (see, for instance, Theorem 3.1 on p. 162 in [25]) yields

$$\frac{\mathcal{P}(t\cdot)-(t\cdot)}{t^{1/2}} \ \Rightarrow\ B(\cdot), \quad t\to\infty \tag{82}$$

in the J_1-topology on D.

It is well-known (see, for instance, Lemma 2.3 on p. 159 in [25]) that the composition mapping $(x,\varphi)\mapsto(x\circ\varphi)$ is continuous at continuous functions $x:\mathbb{R}_+\to\mathbb{R}$ and nondecreasing continuous functions $\varphi:\mathbb{R}_+\to\mathbb{R}_+$. This observation taken together with (82) and (76) enables us to conclude that

$$\frac{\widehat{N}(t\cdot)-m(t\cdot)}{(c_0t^q)^{1/2}} \ \Rightarrow\ W(\cdot), \quad t\to\infty \tag{83}$$

in the J_1-topology on D. Further, we have for all $s_0,t>0$

$$\sup_{s\in[0,\,s_0]} |m(ts)-c_0ts| \le \max(\alpha_0+\alpha_1(ts_0)^{q-r_1}, |\beta_0|+|\beta_1|(ts_0)^{q-r_2})$$

whence

$$\lim_{t\to\infty} t^{-q/2}\sup_{s\in[0,\,s_0]}|m(ts)-c_0ts|=0,$$

where the assumption $r_1,r_2>q/2$ has to be recalled. This enables us to replace in (83) $m(t\cdot)$ with $c_0(t\cdot)^q$ thereby giving

$$\frac{\widehat{N}(t\cdot)-c_0(t\cdot)^q}{(c_0t^q)^{1/2}} \ \Rightarrow\ W(\cdot), \quad t\to\infty \tag{84}$$

in the J_1-topology on D.

By Skorohod's theorem it is possible to define $\widetilde{N}$ and W on the same probability space, so that limit relation (84) holds in the J_1-topology on D almost surely, hence also for $t - \log L$ in place of t. Now, in the centering and normalization functions for $\widetilde{N}(t - \log L)$ we wish to replace $t - \log L$ by t. This is trivial as far as the normalization is concerned, for $(t - \log L)^{q/2} \sim t^{q/2}$ a.s. as $t \to \infty$. As for the centering, it suffices to show that, for all $s_0 > 0$,

$$t^{-q/2} \sup_{s \in [0,\, s_0]} |(ts)^q - ((t - \log L)s)^q| = t^{-q/2} s_0^q |t^q - (t - \log L)^q| \overset{\mathrm{P}}{\to} 0, \quad t \to \infty. \tag{85}$$

If $q \in (0, 1]$, then $|t^q - (t - \log L)^q| \leq |\log L|^q$ by subadditivity of $x \mapsto x^q$ on $\mathbb{R}_+$. If $q \in (1, 2)$, then $|t^q - (t - \log L)^q| \leq q |\log L| (t + |\log L|)^{q-1}$ by the mean value theorem for differentiable functions. These inequalities entail (85) which completes the proof of part (c) and the lemma. □

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