Functional limit theorems for Galton-Watson processes with very active immigration

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Abstract

We prove weak convergence on the Skorokhod space of Galton-Watson processes with immigration, properly normalized, under the assumption that the tail of the immigration distribution has a logarithmic decay. The limits are extremal shot noise processes. By considering marginal distributions, we recover the results of Pakes [*Adv. Appl. Probab.*, 11(1979), 31-62].

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1 Introduction and main result

In this paper we are concerned with the Galton-Watson processes (GW processes, in short) with immigration. Below we outline the setting and refer to the classical treatises [1, 2] for more details on the GW processes with and without immigration.

Let $(X_{i,k})_{i\in\mathbb{N},k\in\mathbb{N}_0}$ and $(J_k)_{k\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, be mutually independent families of independent and identically distributed (i.i.d.) random objects, where each $X_{i,k} := (X_{i,k}(n))_{n\in\mathbb{N}_0}$ is a GW process with $X_{i,k}(0) = 1$ and $\mathbb{E}\, X_{i,k}(1) = \mu \in (0,\infty)$, and each J_k is a nonnegative integer-valued random variable with $\mathbb{P}\{J_k = 0\} < 1$. The random sequence $Y := (Y_n)_{n\in\mathbb{N}_0}$ defined by

$$Y_n := \sum_{k=0}^{n}\sum_{i=1}^{J_k} X_{i,k}(n-k), \quad n\in\mathbb{N}_0$$

is called a *Galton-Watson process with immigration*. The random variable J_k represents the number of immigrants which arrived at time k, while the GW process $(X_{i,k}(n))_{n\in\mathbb{N}_0}$ represents the number of descendants of the ith immigrant which arrived at time k, for $1\le i\le J_k$.

Let J denote a random variable with the same law as the J_k's. It is known that the asymptotic behavior of Y_n depends heavily upon the finiteness of the logarithmic moment $\mathbb{E}\log^+ J$, where $\log^+ x = \max(\log x, 0)$. In the supercritical case $\mu > 1$, the a.s. limit $\lim_{n\to\infty} Y_n/\mu^n$ exists and is finite a.s. provided that $\mathbb{E}\log^+ J < \infty$,

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whereas $\lim_{n\to\infty} Y_n/c^n = \infty$ a.s. for every $c > 0$ if $\mathbb{E}\log^+ J = \infty$; see [10], [11]. In the subcritical case $\mu < 1$, Y_n converges in distribution to a non-degenerate random variable provided that $\mathbb{E}\log^+ J < \infty$, whereas Y_n diverges to $+\infty$ in probability if $\mathbb{E}\log^+ J = \infty$; see [6]. These and more refined results can be found in [1], Theorems 6.1 and 6.4.

Let $D := D[0,\infty)$ denote the Skorokhod space of right-continuous functions defined on $[0,\infty)$ with finite limits from the left at positive points. We intend to prove functional limit theorems for the process $\log^+(Y_{[n\cdot]})$ in D as $n \to \infty$ under the assumption

$$\mathbb{P}\{\log J > x\} \sim x^{-\alpha}\ell(x), \quad x \to \infty \tag{1.1}$$

for some $\alpha \in (0,1]$ and some ℓ slowly varying at ∞, which justifies the term "very active immigration".

To state our results we need to introduce certain Poisson random measures which appear as limits for the extremal order statistics of the sequence $(\log^+ J_k)_{k\in\mathbb{N}_0}$. Denote by M_p the set of point measures ν on $[0,\infty) \times (0,\infty]$ which satisfy

$$\nu([0,T] \times [\delta,\infty]) < \infty \tag{1.2}$$

for all $T > 0$ and all $\delta > 0$. For positive a and b, let

$$N^{(a,b)} := \sum_{k=1}^{\infty} \varepsilon_{(t_k^{(a,b)},\, j_k^{(a,b)})}$$

be a Poisson random measure on $[0,\infty) \times (0,\infty]$ with mean measure $\mathbb{LEB} \times \mu_{a,b}$, where $\varepsilon_{(t,x)}$ is the probability measure concentrated at $(t,x) \in [0,\infty) \times (0,\infty]$, $\mathbb{LEB}$ is the Lebesgue measure on $[0,\infty)$, and $\mu_{a,b}$ is a measure on $(0,\infty]$ defined by

$$\mu_{a,b}\big((x,\infty]\big) = ax^{-b}, \quad x > 0.$$

Throughout the paper we use $\Rightarrow$ to denote weak convergence on the Skorokhod space D equipped with the J_1-topology (see [3, 7] for the necessary background) and on M_p endowed with the vague topology.

Theorem 1.1 treats the situation in which the behaviour in mean of the GW processes $X_{i,k}$ affects the limit behavior of Y (except in the less interesting case $\mu = 1$), whereas in the situation of Theorem 1.3 any traces of the $X_{i,k}$ disappear in the limit. We stipulate hereafter that the supremum over the empty set is equal to zero.

Theorem 1.1. *Assume that for some $c > 0$,*

$$\lim_{x\to\infty} x\,\mathbb{P}\{\log J > x\} = c. \tag{1.3}$$

Then, as $n \to \infty$,

$$\frac{\log^+(Y_{[n\cdot]})}{n} \quad \Rightarrow \quad \sup_{t_k^{(c,1)} \le \cdot} \big(j_k^{(c,1)} + (\cdot - t_k^{(c,1)})\log\mu\big). \tag{1.4}$$

Remark 1.2. Realizations of the limit processes, which are called extremal shot noise processes [5], are shown on Figure 1. In the critical case $\mu = 1$, the limit is the well-known extremal process; see [9], Sections 4.3–4.4. We shall see in the course of the proof that in the supercritical case $\mu > 1$, we also have

$$\frac{\log^+(\mu^{-[n\cdot]}Y_{[n\cdot]})}{n} \quad \Rightarrow \quad \sup_{t_k^{(c,1)} \le \cdot} \big(j_k^{(c,1)} - t_k^{(c,1)}\log\mu\big), \tag{1.5}$$

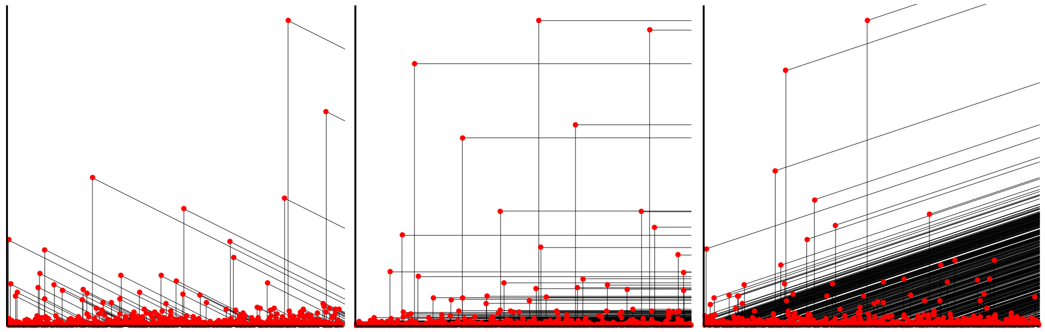

Figure 1: The limit processes appearing in Theorem 1.1. Left: subcritical case $\mu < 1$. Middle: critical case $\mu = 1$. Right: supercritical case $\mu > 1$.

in which case the marginal distributions of the limit process have support equal to $[0, \infty)$.

Theorem 1.3. *Suppose that* (1.1) *holds. If* α *in* (1.1) *equals* 1*, suppose additionally that* $\lim_{x\to\infty} \ell(x) = \infty$*. Let* $(b_n)_{n\in\mathbb{N}}$ *be a sequence of positive numbers satisfying*

$$\lim_{n\to\infty} n\,\mathbb{P}\{\log J > b_n\} = 1.$$

Then, as $n \to \infty$*,*

$$\frac{\log^+(Y_{[n\cdot]})}{b_n} \quad \Rightarrow \quad \sup_{t_k^{(1,\,\alpha)} \leq \cdot} j_k^{(1,\alpha)}. \tag{1.6}$$

Remark 1.4. Let us derive closed formulae for the marginal distributions of the limit processes. We claim that, with $r, s > 0$ and $u \geq 0$,

$$\begin{aligned}\mathbb{P}\left\{\sup_{t_k^{(r,1)} \leq u} \left(j_k^{(r,1)} + s(t_k^{(r,1)} - u)\right) \leq x\right\} \quad &= \quad \mathbb{P}\left\{\sup_{t_k^{(r,1)} \leq u} \left(j_k^{(r,1)} - st_k^{(r,1)}\right) \leq x\right\} \\ &= \quad \left(\frac{x}{x+su}\right)^{r/s} \end{aligned} \tag{1.7}$$

for all $x \geq 0$. We only provide details for the second probability. Since

$$N := N^{(r,1)}\big((t,y) : 0 \leq t \leq u, y - st > x\big)$$

is a Poisson random variable, we have $\mathbb{P}\big\{N = 0\big\} = e^{-\mathbb{E}\,N}$ and it remains to note that

$$\mathbb{E}\,N = \int_0^u \int_{[0,\infty)} \mathbb{1}_{\{y-st>x\}}\,\mu_{r,\,1}(\mathrm{d}y)\mathrm{d}t = r\int_0^u (x+st)^{-1}\mathrm{d}t = \frac{r}{s}\log\frac{x+su}{x}.$$

Similarly, for the marginals of the extremal process appearing in (1.6) we obtain, with $a, b > 0$ and $u \geq 0$,

$$\mathbb{P}\left\{\sup_{t_k^{(a,\,b)} \leq u} j_k^{(a,\,b)} \leq x\right\} = \mathbb{P}\left\{N^{(a,b)}\left((t,y) : 0 \leq t \leq u, y > x\right) = 0\right\} = e^{-uax^{-b}}$$

for all $x \geq 0$. Armed with these observations we conclude that relations (1.4) and (1.5) include the results obtained by Pakes [8] concerning weak convergence of the one-dimensional distributions (Theorem 2, Theorem 6 and Theorem 12 for the subcritical $\mu < 1$, supercritical $\mu > 1$ and critical $\mu = 1$ cases, respectively). Similarly, the one-dimensional version of our relation (1.6) is equivalent to the limit relations of Theorem 3 (case $\mu < 1$), Theorem 7 (case $\mu > 1$) and Theorem 12 (case $\mu = 1$) of [8]. To be more precise, Pakes states in Theorems 3 and 7 that

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{\log(1+Y_n)}{a_n} \le x\right\} = \exp(-(|\log\mu|)^{-1}x^{-\alpha}), \quad x \ge 0 \tag{1.8}$$

whenever

$$1 - \mathbb{E}(1-e^{-x})^J \sim x^{-\alpha}\ell(x) \quad x \to \infty \tag{1.9}$$

with a_n defined by $1 - \mathbb{E}(1-e^{-a_n})^J \sim (|\log\mu| n)^{-1}$ as $n \to \infty$. Formula (1.8) is misleading because it contains $|\log\mu|$ thereby suggesting that the contribution of the $X_{i,k}$ persists in the limit. However, the relation

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{\log(1+Y_n)}{b_n} \le x\right\} = \exp(-x^{-\alpha}), \quad x \ge 0 \tag{1.10}$$

which is a consequence of (1.6) shows this is not the case. Having observed that (1.9) is equivalent to (1.1) we infer $b_n \sim (|\log\mu|)^{-1/\alpha}a_n$ which implies that (1.8) and (1.10) are actually equivalent.

Finally, we note that unlike us, Pakes [8] imposed a regular variation assumption on the tail of $X_{1,1}(1)$ for the critical case and used the Seneta-Heyde norming rather than μ^{-n} in the supercritical case in (1.5).

Remark 1.5. One may expect that, under (1.1), Y_n is well approximated by $Z_n := \mathbb{E}(Y_n|(J_k)_{k\in\mathbb{N}_0}) = \sum_{k=0}^n \mu^{n-k}J_k$ for large n. Although this turns out to be true, it is worth stressing that both the behavior in mean and the survival probability (especially in the subcritical case) of the underlying GW processes affect the asymptotics of Y_n. The sequence $(Z_n)_{n\in\mathbb{N}_0}$ is a rather particular case[1] of the much studied Markov chain $(X_n)_{n\in\mathbb{N}_0}$ defined by

$$X_0 := B_0 \quad \text{and} \quad X_n = A_nX_{n-1} + B_n, \quad n \in \mathbb{N},$$

where (A_n, B_n) are i.i.d. $\mathbb{R}^2$-valued random vectors independent of B_0. Functional limit theorems for $\log^+(X_{[n\cdot]})$ were obtained in [4] under the assumption that (1.1) holds with B_1 replacing J and that $\lim_{n\to\infty}(A_1\cdot\ldots\cdot A_n) = 0$ a.s. (this corresponds to the subcritical GW processes).

2 Preparatory results

We start with a lemma that might have been known. Recall that $(X_{1,1}(n))_{n\in\mathbb{N}_0}$ is a GW process with $X_{1,1}(0) = 1$ and mean $\mu \in (0,\infty)$.

Lemma 2.1. *If $\mu \le 1$, then for all $\delta > 0$,*

$$\lim_{n\to\infty} e^{\delta n}\mu^{-n}\,\mathbb{P}\{X_{1,1}(n) \ge 1\} = \infty. \tag{2.1}$$

Proof. Put $p_n := \mathbb{P}\{X_{1,1}(n) \ge 1\}$, $n \in \mathbb{N}$. If $\mathbb{P}\{X_{1,1}(1) = 1\} = 1$, then $\mu = 1$ and $p_n = 1$, and (2.1) holds trivially. To deal with the remaining cases $\mu = 1$ and

[1] Just take $A_n = \mu$ and $B_n = J_n$, $n \in \mathbb{N}_0$.

$\mathbb{P}\{X_{1,1}=1\}<1$ or $\mu<1$ in which $\lim_{n\to\infty} p_n=0$ we set $f_n(s):=\mathbb{E}\, s^{X_{1,1}(n)}$, $n\in\mathbb{N}$, $s\in[0,1]$. Then

$$\mu=\lim_{s\to 1-}\frac{1-f_1(s)}{1-s}=\lim_{n\to\infty}\frac{1-f_1(f_n(0))}{1-f_n(0)}=\lim_{n\to\infty}\frac{p_{n+1}}{p_n}$$

having utilized $1-p_n=\mathbb{P}\{X_{1,1}(n)=0\}=f_n(0)\to 1-$ as $n\to\infty$ for the second equality. This implies $\lim_{n\to\infty}\frac{\mu^{-(n+k)}p_{n+k}}{\mu^{-n}p_n}=1$ for each $k\in\mathbb{N}$ and thereupon (2.1). $\square$

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying $\lim_{n\to\infty} n^{-1}c_n=\infty$ or $c_n=n$ for all $n\in\mathbb{N}$. For each $0<\gamma<1$, set

$$Y_{[n\cdot]}^{(\le\gamma)}:=\sum_{k=0}^{[n\cdot]}\mathbb{1}_{\{J_k\le e^{\gamma c_n}\}}\sum_{i=1}^{J_k}X_{i,k}([n\cdot]-k). \tag{2.2}$$

The next lemma shows that $Y_{[n\cdot]}^{\le\gamma}$, the contribution coming from times in which immigration is not extremely active, is negligible as $n\to\infty$ and $\gamma\to 0+$. No assumptions on the tail of J are imposed in this lemma.

Lemma 2.2. *Fix $T>0$ and $\gamma>0$. For every $\delta>0$,*

$$\sum_{n\ge 1}\mathbb{P}\left\{\sup_{0\le t\le T}\frac{1}{c_n}\log^+\left(m([nt])Y_{[nt]}^{(\le\gamma)}\right)>\gamma+\delta\right\}<\infty,$$

where $m(n):=\mu^{-n}\wedge 1$, $n\in\mathbb{N}_0$ when $c_n=n$ and $m(n)=1$, $n\in\mathbb{N}_0$ when $\lim_{n\to\infty} n^{-1}c_n=\infty$.

Proof. Suppose first that $c_n=n$. For $r\in\mathbb{N}_0$, set $d(r):=(\mu^{-r}\wedge 1)\sum_{k=0}^r\mu^k$ and note that $d(r)=r+1$ if $\mu=1$ and $d(r)\le(1-\mu\wedge\mu^{-1})^{-1}<\infty$ if $\mu\ne 1$. For all $\delta>0$,

$$\begin{aligned}
&\sum_{n\ge 1}\mathbb{P}\left\{\sup_{0\le t\le T}\frac{1}{n}\log^+\left(\left(\mu^{-[nt]}\wedge 1\right)Y_{[nt]}^{(\le\gamma)}\right)>\gamma+\delta\right\}\\
\le\ &\sum_{n\ge 1}\sum_{r=0}^{[nT]}\mathbb{P}\left\{\left(\mu^{-r}\wedge 1\right)\sum_{k=0}^{r}\mathbb{1}_{\{J_k\le e^{\gamma n}\}}\sum_{i=1}^{J_k}X_{i,k}(r-k)>e^{(\gamma+\delta)n}\right\}\\
\le\ &\sum_{n\ge 1}\sum_{r=0}^{[nT]}\mathbb{P}\left\{\left(\mu^{-r}\wedge 1\right)\sum_{k=0}^{r}\sum_{i=1}^{[e^{\gamma n}]}X_{i,k}(r-k)>e^{(\gamma+\delta)n}\right\}\\
\le\ &\sum_{n\ge 1}e^{-\delta n}\sum_{r=0}^{[nT]}d(r)<\infty.
\end{aligned}$$

We have used Boole's inequality for the second line and Markov's inequality in combination with $\mathbb{E}\,X_{i,k}(r-k)=\mu^{r-k}$ for the fourth line.

When $\lim_{n\to\infty} n^{-1}c_n = \infty$, the proof is similar: for all $\delta > 0$,

$$\begin{aligned}
&\sum_{n\geq 1} \mathbb{P}\left\{\sup_{0\leq t\leq T} \frac{1}{c_n} \log^+\left(Y^{(\leq\gamma)}_{[nt]}\right) > \gamma+\delta\right\} \\
\leq\; & \sum_{n\geq 1}\sum_{r=0}^{[nT]} \mathbb{P}\left\{\sum_{k=0}^{r} \mathbb{1}_{\{J_k\leq e^{\gamma c_n}\}} \sum_{i=1}^{J_k} X_{i,k}(r-k) > e^{(\gamma+\delta)c_n}\right\} \\
\leq\; & \sum_{n\geq 1}\sum_{r=0}^{[nT]} \mathbb{P}\left\{\sum_{k=0}^{r} \sum_{i=1}^{[e^{\gamma c_n}]} X_{i,k}(r-k) > e^{(\gamma+\delta)c_n}\right\} \\
\leq\; & \sum_{n\geq 1} e^{-\delta c_n}([nT]+1)\sum_{k=0}^{[nT]} \mu^k < \infty,
\end{aligned}$$

because $([nT]+1)\sum_{k=0}^{[nT]}\mu^k$ grows at most exponentially whereas $e^{-\delta c_n}$ decreases superexponentially in view of $\lim_{n\to\infty} n^{-1}c_n = \infty$. $\square$

In the next lemma, which is needed to prove Theorem 1.1, we identify the response functions of the limit extremal shot noise process. Roughly speaking, this lemma states that a GW process with finite mean μ starting at time 0 with approximately $e^{an+o(n)}$ particles has at time nt approximately $\mu^{nt}e^{an+o(n)}$ particles. However, there is one exception: if the process is subcritical, then the population dies out approximately at time $na/|\log\mu|$, and the number of particles after this time is 0.

Lemma 2.3. *Let $(A_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ be a sequence satisfying $\lim_{n\to\infty} n^{-1}\log A_n = a$ for some $a > 0$. Then, for every $T > 0$ and every sequence $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}_0$,*

$$\lim_{n\to\infty} \sup_{0\leq t\leq T}\left|\frac{1}{n}\log^+\left(\sum_{i=1}^{A_n} X_{i,k_n}([nt])\right) - (a+t\log\mu)^+\right| = 0 \quad \text{a.s.} \tag{2.3}$$

Proof. According to the Borel-Cantelli lemma in combination with

$$\lim_{n\to\infty}\sup_{t\in[0,T]} |[nt]/n - t| = 0$$

it suffices to check that for all $\varepsilon \in (0,a)$,

$$\sum_{n\geq 1}\mathbb{P}\left\{\sup_{0\leq t\leq T}\left|\log^+\left(\sum_{i=1}^{A_n} X_{i,0}([nt])\right) - (an+[nt]\log\mu)^+\right| > \varepsilon n\right\} < \infty.$$

We write $X^*_{i,0}(l) := \mu^{-l}X_{i,0}(l)$, so that $\mathbb{E}\,X^*_{i,0}(l) = 1$. By Boole's inequality the last probability is bounded from above by $I_1(n) + I_2(n)$ with

$$\begin{aligned}
I_1(n) &:= \sum_{r=0}^{[nT]} \mathbb{P}\left\{\sum_{i=1}^{A_n} X_{i,0}(r) > e^{(an+r\log\mu)^+ + \varepsilon n}\right\}, \\
I_2(n) &:= \sum_{r=0}^{r_n} \mathbb{P}\left\{\sum_{i=1}^{A_n} X^*_{i,0}(r) < e^{(a-\varepsilon)n}\right\},
\end{aligned}$$

where $r_n := [nT] \wedge [nT_0]$ and

$$T_0 := \begin{cases} (a-\varepsilon)/|\log \mu|, & \text{if } \mu < 1, \\ +\infty, & \text{if } \mu \geq 1. \end{cases}$$

To prove that $\sum_{n\geq 1} I_1(n)$ is finite, note that $|\log A_n - an| \leq \frac{\varepsilon}{2}n$ for sufficiently large n. Using Markov's inequality yields

$$\sum_{n\geq 1} I_1(n) \leq \sum_{n\geq 1}\sum_{r=0}^{[nT]} \frac{e^{\log A_n + r\log\mu}}{e^{(an+r\log\mu)^+ + \varepsilon n}} \leq C + \sum_{n\geq 1}([nT]+1)e^{-\frac{1}{2}\varepsilon n} < \infty.$$

In the following, we prove that $\sum_{n\geq 1} I_2(n)$ is finite. Let $p_n := \mathbb{P}\{X_{1,0}(n) \geq 1\}$, $n \in \mathbb{N}$, be the probability that a GW process starting with a single particle at time 0 does not die out at time n. We fix $u > 0$ and use Markov's inequality in combination with the fact that $(e^{-uX^*_{1,0}(l)})_{l\in\mathbb{N}_0}$ is a submartingale w.r.t. the natural filtration to infer that for large enough n and all $r \leq r_n$,

$$\begin{aligned} \log \mathbb{P}\left\{\sum_{i=1}^{A_n} X^*_{i,0}(r) < e^{(a-\varepsilon)n}\right\} &\leq ue^{(a-\varepsilon)n} + A_n \log \mathbb{E}\, e^{-uX^*_{1,0}(r_n)} &(2.4)\\ &\leq ue^{(a-\varepsilon)n} + A_n \log(1 - p_{r_n} + p_{r_n} e^{-\mu^{-r_n}u}) \\ &\leq ue^{(a-\varepsilon)n} - A_n(1 - e^{-\mu^{-r_n}u})p_{r_n} \\ &\leq ue^{(a-\varepsilon)n} - e^{(a-\frac{1}{2}\varepsilon)n}(1 - e^{-\mu^{-r_n}u})p_{r_n}. &(2.5) \end{aligned}$$

Further, we consider the three cases separately.

Supercritical case $\mu > 1$. It is well known (see, for instance, Theorem 5.1 on p. 83 together with Corollary 5.3 on p. 85 in [1]) that there exists a function L slowly varying at ∞ with $\liminf_{x\to\infty} L(x) > 0$ such that, as $n \to \infty$, $X^*_{1,0}(n)/L(\mu^n)$ converges a.s. to a random variable W, say, which is positive with positive probability[2]. In particular, by the dominated convergence,

$$c := \lim_{n\to\infty} \log \mathbb{E}\, e^{-X^*_{1,0}(n)/L(\mu^n)} = \log \mathbb{E}\, e^{-W} < 0.$$

With $u = 1/L(\mu^{[nT]})$, inequality (2.4) takes the form

$$\log \mathbb{P}\left\{\sum_{i=1}^{A_n} X^*_{i,0}(r) < e^{(a-\varepsilon)n}\right\} < \frac{e^{(a-\varepsilon)n}}{L(\mu^{[nT]})} + (c + o(1))A_n.$$

The right-hand side goes to $-\infty$ as $n \to \infty$ exponentially fast because $A_n > e^{(a-\frac{1}{2}\varepsilon)n}$ for large n and L is slowly varying, thereby proving $\sum_{n\geq 1} I_2(n) < \infty$.

Critical case $\mu = 1$. With $u > 0$ fixed, inequality (2.5) takes the form

$$\log \mathbb{P}\left\{\sum_{i=1}^{A_n} X^*_{i,0}(r) < e^{(a-\varepsilon)n}\right\} \leq ue^{(a-\varepsilon)n} - e^{(a-\frac{1}{2}\varepsilon)n}(1 - e^{-u})p_{[nT]}.$$

In view of (2.1) this goes to $-\infty$ as $n \to \infty$ exponentially fast, whence $\sum_{n\geq 1} I_2(n) < \infty$.

Subcritical case $\mu < 1$. With $u = e^{-(a-\varepsilon)n}$, expression (2.5) takes the form

$$1 - (1 - e^{-\mu^{-[nT_0]}e^{-(a-\varepsilon)n}})p_{[nT_0]}e^{(a-\frac{1}{2}\varepsilon)n} \leq 1 - (1 - e^{-\mu})p_{[nT_0]}\mu^{-[nT_0]}e^{\frac{1}{2}\varepsilon n},$$

[2]The sequence $(\mu^n L(\mu^n))$ is known as the Seneta-Heyde norming.

because $\mu < \mu^{-[nT_0]} e^{-(a-\varepsilon)n} \leq 1$. In view of (2.1) this goes to $-\infty$ as $n \to \infty$ exponentially fast, thus proving that $\sum_{n\geq 1} I_2(n) < \infty$ in this case, too. $\square$

In the next lemma, we consider a GW process starting at time 0 with $e^{(a+o(1))c_n}$ particles, where, as before, $(c_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers satisfying $\lim_{n\to\infty} n^{-1} c_n = \infty$. At time nt, the number of particles in such a process is approximately $\mu^{nt} e^{(a+o(1))c_n} = e^{(a+o(1))c_n}$, so that we do not see any changes on the logarithmic scale. The subcritical case plays no special role here, because the process is very unlikely to die out on the time scale n.

Lemma 2.4. *Let $(A_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ be a sequence satisfying $\lim\limits_{n\to\infty} c_n^{-1} \log A_n = a$ for some $a > 0$. Then, for every $T > 0$ and every sequence $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}_0$,*

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \left| \frac{1}{c_n} \log^+ \left(\sum_{i=1}^{A_n} X_{i,k_n}([nt]) \right) - a \right| = 0 \quad \text{a.s.} \tag{2.6}$$

Proof. The proof is similar to that of Lemma 2.3. Therefore we only give an outline. It suffices to prove that for all $\varepsilon \in (0, a)$,

$$\sum_{n\geq 1} \mathbb{P} \left\{ \sup_{0\leq t\leq T} \left| \log^+ \left(\sum_{i=1}^{A_n} X_{i,0}([nt]) \right) - ac_n \right| > \varepsilon c_n \right\} < \infty.$$

The last probability is bounded from above by $J_1(n) + J_2(n)$, where

$$J_1(n) := \sum_{r=0}^{[nT]} \mathbb{P} \left\{ \sum_{i=1}^{A_n} X_{i,0}(r) > \exp((a+\varepsilon)c_n) \right\}$$
$$J_2(n) := \sum_{r=0}^{[nT]} \mathbb{P} \left\{ \sum_{i=1}^{A_n} X^*_{i,0}(r) < \mu^{-r} \exp((a-\varepsilon)c_n) \right\}.$$

For sufficiently large n, we have $|\log A_n - ac_n| \leq \frac{\varepsilon}{2} c_n$. We use Markov's inequality to obtain

$$\sum_{n\geq 1} J_1(n) \leq \sum_{n\geq 1} A_n e^{-(a+\varepsilon)c_n} \sum_{r=0}^{[nT]} \mu^r \leq C + \sum_{n\geq 1} e^{-\frac{1}{2}\varepsilon c_n} \sum_{r=0}^{[nT]} \mu^r < \infty,$$

where the finiteness follows from the fact that $e^{-\frac{1}{2}\varepsilon c_n}$ decreases superexponentially in view of $\lim\limits_{n\to\infty} n^{-1} c_n = \infty$. While analyzing $J_2(n)$ we only treat the subcritical case. A counterpart of (2.5) reads

$$\log \mathbb{P} \left\{ \sum_{i=1}^{A_n} X^*_{i,0}(r) < \mu^{-r} e^{(a-\varepsilon)c_n} \right\} \leq u\mu^{-[nT]} e^{(a-\varepsilon)c_n} - e^{(a-\frac{1}{2}\varepsilon)c_n} (1 - e^{-\mu^{-[nT]} u}) p_{[nT]}$$

for large enough n, $r \leq [nT]$ and any $u > 0$. On setting $u = \mu^{[nT]}$ the right-hand side takes the form

$$e^{(a-\varepsilon)c_n} (1 - (1 - e^{-1}) e^{\frac{1}{2}\varepsilon c_n} p_{[nT]}).$$

As $n \to \infty$, this goes to $-\infty$ superexponentially fast by (2.1). The proof of Lemma 2.4 is complete. $\square$

Lemma 2.5. *For all $x, y \geq 0$, we have*

$$\log^+ x \leq \log^+(x+y) \leq \log^+ x + \log^+ y + 2\log 2. \tag{2.7}$$

Proof. While the left-hand inequality follows by monotonicity, the right-hand inequality is a consequence of

$$\log^+ x \leq \log(1+x) \leq \log^+ x + \log 2, \quad x \geq 0$$

and the subadditivity of $x \mapsto \log(1+x)$, namely,

$$\log^+(x+y) \leq \log(1+x+y) \leq \log(1+x) + \log(1+y) \leq \log^+ x + \log^+ y + 2\log 2.$$

□

3 Proofs of the main results

Proof of Theorem 1.1. It is a standard fact of the extreme-value theory that condition (1.3) entails the point processes convergence

$$N_n := \sum_{k\geq 0} \mathbb{1}_{\{J_k \neq 0\}}\, \varepsilon_{(\frac{k}{n},\, \frac{1}{n}\log J_k)} \quad \Rightarrow \quad N^{(c,1)}$$

weakly on M_p, as $n \to \infty$; see, for instance, Corollary 4.19 (ii) on p. 210 in [9].

Step 1: Passing to a.s. convergence. By the Skorokhod representation theorem there are versions $\widehat{N}_n$ and $\widehat{N}^{(c,1)}$ of N_n and $N^{(c,1)}$ (defined on some new probability space) which converge a.s. That is, with probability 1,

$$\widehat{N}_n := \sum_{k\geq 0} \mathbb{1}_{\{y_k^{(n)} \neq -\infty\}}\, \varepsilon_{(\frac{k}{n},\, y_k^{(n)})} \quad \to \quad \widehat{N}^{(c,1)} = \sum_{m \geq 0} \varepsilon_{(\tau_m,\, y_m)}, \tag{3.1}$$

vaguely on M_p, as $n \to \infty$. Extending, if necessary, the probability space on which $(\widehat{N}_n)_{n\in\mathbb{N}}$ and $\widehat{N}^{(c,1)}$ are defined, we can independently construct GW processes $(\widehat{X}_{i,k})_{i\in\mathbb{N}, k\in\mathbb{N}_0}$ having the same law as $(X_{i,k})_{i\in\mathbb{N}, k\in\mathbb{N}_0}$. Write

$$\widehat{Z}_n(t) := \sum_{k=0}^{[nt]} \sum_{i=1}^{\exp(n y_k^{(n)})} \widehat{X}_{i,k}([nt]-k), \quad n \in \mathbb{N}_0, \quad t \geq 0,$$

so that for each $n \in \mathbb{N}$, the distributions of the processes $(\widehat{Z}_n(t))_{t\geq 0}$ and $(Y_{[nt]})_{t\geq 0}$ coincide. Fix some $T > 0$ and let d_T be the standard J_1-metric on the Skorokhod space $D[0,T]$. Our aim is to prove that with probability 1,

$$\lim_{n\to\infty} d_T\left(\frac{1}{n}\log^+ \widehat{Z}_n(\cdot), \sup_{\tau_k \leq \cdot}\big(y_k + (\cdot - \tau_k)\log\mu\big)\right) = 0. \tag{3.2}$$

Step 2: Estimate for non-extremal order statistics. We shall decompose the process $(\widehat{Z}_n(t))_{t\geq 0}$ into the contribution coming from times with extremely active immigration, and the contribution of all the other times. For a truncation parameter $0 < \gamma < 1$, put

$$\widehat{Z}_n^{(\leq\gamma)}(t) := \sum_{k=0}^{[nt]} \mathbb{1}_{\{y_k^{(n)} \leq \gamma\}} \sum_{i=1}^{\exp(n y_k^{(n)})} \widehat{X}_{i,k}([nt]-k),$$

$$\widehat{Z}_n^{(>\gamma)}(t) := \sum_{k=0}^{[nt]} \mathbb{1}_{\{y_k^{(n)} > \gamma\}} \sum_{i=1}^{\exp(n y_k^{(n)})} \widehat{X}_{i,k}([nt]-k),$$

so that

$$\widehat{Z}_n(t) = \widehat{Z}_n^{(\le\gamma)}(t) + \widehat{Z}_n^{(>\gamma)}(t), \quad n \in \mathbb{N}_0,\ t \ge 0. \tag{3.3}$$

Suppose for a moment that with probability 1,

$$\lim_{\gamma\to 0+} \limsup_{n\to\infty} d_T\left(\frac{1}{n}\log^+ \widehat{Z}_n^{(>\gamma)}(\cdot),\ \sup_{\tau_k\le\cdot}\big(y_k + (\cdot - \tau_k)\log\mu\big)\right) = 0, \tag{3.4}$$

where γ is restricted to the set $\{1/m\colon m = 2,3,\ldots\}$. Let us argue that (3.4) implies (3.2).

Proof in the case $\mu \le 1$. Using (3.3) in combination with Lemma 2.5 yields

$$\log^+(\widehat{Z}_n^{(>\gamma)}(t)) \le \log^+(\widehat{Z}_n(t)) \le \log^+(\widehat{Z}_n^{(>\gamma)}(t)) + \log^+(\widehat{Z}_n^{(\le\gamma)}(t)) + 2\log 2.$$

Since the processes $(\widehat{Z}_n^{(\le\gamma)}(t))_{t\ge 0}$ and $(Y_{[nt]}^{(\le\gamma)})_{t\ge 0}$ (c.f. (2.2)) have the same distribution, we can use Lemma 2.2 and the Borel-Cantelli lemma to conclude that with probability 1,

$$\limsup_{n\to\infty} \sup_{0\le t\le T} \frac{1}{n}\log^+(\widehat{Z}_n^{(\le\gamma)}(t)) \le \gamma.$$

It follows from the last two inequalities that with probability 1,

$$\limsup_{n\to\infty} \sup_{0\le t\le T} \left|\frac{1}{n}\log^+(\widehat{Z}_n(t)) - \frac{1}{n}\log^+(\widehat{Z}_n^{(>\gamma)}(t))\right| \le \gamma.$$

The Skorokhod distance is majorized by the sup-distance, whence we conclude that with probability 1,

$$\limsup_{n\to\infty} d_T\left(\frac{1}{n}\log^+ \widehat{Z}_n(\cdot), \frac{1}{n}\log^+ \widehat{Z}_n^{(>\gamma)}(\cdot)\right) \le \gamma.$$

The triangle inequality entails that (3.4) implies (3.2).

Proof in the case $\mu > 1$. Our aim is to obtain an upper bound for $\log^+(\widehat{Z}_n(t))$. Since the processes $(\widehat{Z}_n^{(\le\gamma)}(t))_{t\ge 0}$ and $(Y_{[nt]}^{(\le\gamma)})_{t\ge 0}$ have the same distribution, Lemma 2.2 in conjunction with the Borel-Cantelli lemma allows us to conclude that with probability 1,

$$\limsup_{n\to\infty} \sup_{0\le t\le T} \frac{1}{n}\log^+\left(\mu^{-[nt]}\widehat{Z}_n^{(\le\gamma)}(t)\right) \le \gamma.$$

Since $\log x \le \log^+ x$ and $[nt] \le nt$, it follows that with probability 1, for sufficiently large n,

$$\widehat{Z}_n^{(\le\gamma)}(t) \le e^{n(t\log\mu + 2\gamma)}, \quad 0 \le t \le T. \tag{3.5}$$

Noting that $e^{n(t\log\mu+2\gamma)} \ge e^{2n\gamma} > 1$ for sufficiently large n and all $t \ge 0$, we obtain the estimate

$$\begin{aligned}
\log^+(\widehat{Z}_n(t)) &\le \log^+\left(\widehat{Z}_n^{(>\gamma)}(t) + e^{n(t\log\mu+2\gamma)}\right)\\
&\le \log\left(\widehat{Z}_n^{(>\gamma)}(t) + e^{n(t\log\mu+2\gamma)}\right)\\
&\le n(t\log\mu - 2\sqrt{\gamma}) + \log\left(e^{-n(t\log\mu-2\sqrt{\gamma})}\widehat{Z}_n^{(>\gamma)}(t) + e^{4n\sqrt{\gamma}}\right).
\end{aligned}$$

Using again the inequality $\log x \le \log^+ x$ and then Lemma 2.5, we arrive at

$$\log^+(\widehat{Z}_n(t)) \le n(t\log\mu - 2\sqrt{\gamma}) + \log^+\left(e^{-n(t\log\mu-2\sqrt{\gamma})}\widehat{Z}_n^{(>\gamma)}(t)\right) + 4n\sqrt{\gamma} + 2\log 2. \tag{3.6}$$

Below we shall prove that with probability 1 there exist a random $0 < \gamma_0 < 1$ and $n_1 \in \mathbb{N}$ such that for all $0 < \gamma < \gamma_0$ and $n > n_1$, we have

$$\widehat{Z}_n^{(>\gamma)}(t) \geq e^{n(t \log \mu - 2\sqrt{\gamma})}, \quad \sqrt{\gamma} \leq t \leq T. \tag{3.7}$$

Given (3.7), we conclude that $\log^+$ on the right-hand side of (3.6) can be replaced by log, thus yielding for all $\sqrt{\gamma} \leq t \leq T$ the estimate

$$\log^+(\widehat{Z}_n^{(>\gamma)}(t)) \leq \log^+(\widehat{Z}_n(t)) \leq 4n\sqrt{\gamma} + \log^+(\widehat{Z}_n^{(>\gamma)}(t)) + 2\log 2,$$

where the first inequality is an immediate consequence of (3.3). For $0 \leq t \leq \sqrt{\gamma}$, Lemma 2.5 and (3.5) yield

$$\log^+(\widehat{Z}_n^{(>\gamma)}(t)) \leq \log^+(\widehat{Z}_n(t)) \leq n(\sqrt{\gamma}\log\mu + 2\gamma) + \log^+(\widehat{Z}_n^{(>\gamma)}(t)) + 2\log 2.$$

From now on, we can argue as in the case $\mu \leq 1$ to conclude that (3.4) implies (3.2)

Proof of (3.7). First we prove that with probability 1 there is a $\gamma_0 > 0$ such that for all $0 < \gamma < \gamma_0$ at least one atom (τ, y) of the point process $\widehat{N}^{(c,1)}$ satisfies $0 < \tau < \sqrt{\gamma}$ and $y > \gamma$. Indeed, the number of points of $\widehat{N}^{(c,1)}$ in the set $(0, 1/\sqrt{2k}) \times (1/k, \infty)$ is Poisson-distributed with parameter $c\sqrt{k/2}$, for all $k \in \mathbb{N}$. Since $\sum_{k\geq 1} e^{-c\sqrt{k/2}}$ is finite, the Borel-Cantelli lemma implies that with probability 1 we can find $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ at least one atom (τ, y) of $\widehat{N}^{(c,1)}$ satisfies $0 < \tau < 1/\sqrt{2k}$ and $y > 1/k$. If $1/(k+1) \leq \gamma \leq 1/k$, then it follows $0 < \tau < \sqrt{\gamma}$ and $y > \gamma$, so that we can take $\gamma_0 = 1/k_0$.

Since with probability 1, $\widehat{N}_n$ converges to $\widehat{N}^{(c,1)}$ vaguely on M_p, there is a random $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ at least one atom, say $(k_n/n, z_n)$, of $\widehat{N}_n$ satisfies $k_n/n < \sqrt{\gamma}$ and $z_n > \gamma$. In the following, we condition on the σ-field generated by $\widehat{N}^{(c,1)}$ and $\widehat{N}_n$, so that we can view γ_0 and n_0 as deterministic quantities.

Recall that we consider the case $\mu > 1$. As has already been mentioned in the proof of Lemma 2.3 (case $\mu > 1$), there exists a function L slowly varying at ∞ such that

$$\frac{X_{1,0}(m)}{\mu^m L(\mu^m)} \to W \quad \text{a.s. as } m \to \infty,$$

the limit random variable W being a.s. positive on the survival event of the GW process $X_{1,0}$. It follows that there is $\varepsilon > 0$ such that

$$\mathbb{P}\left\{\inf_{m\in\mathbb{N}} \frac{\widehat{X}_{i,k}(m)}{\mu^m L(\mu^m)} > \varepsilon\right\} > \varepsilon, \quad i \in \mathbb{N},\ k \in \mathbb{N}_0\,.$$

Since $z_n > \gamma$ for $n \geq n_0$ we have

$$\sum_{n\geq n_0} \mathbb{P}\left\{\inf_{m\in\mathbb{N}} \frac{\widehat{X}_{i,k_n}(m)}{\mu^m L(\mu^m)} \leq \varepsilon \text{ for all } i = 1, \ldots, e^{nz_n}\right\} < \sum_{n\geq n_0} (1-\varepsilon)^{e^{nz_n}} < \infty.$$

By the Borel-Cantelli lemma, for sufficiently large n, there is an immigrant i_n arriving at time $k_n < n\sqrt{\gamma}$ whose offspring numbers satisfy $\widehat{X}_{i_n,k_n}(m) > \varepsilon\mu^m L(\mu^m)$ for all $m \in \mathbb{N}$. For all $\sqrt{\gamma} \leq t \leq T$ and sufficiently large $n \geq n_1$ we have

$$\widehat{Z}_n^{(>\gamma)}(t) \geq \widehat{X}_{i_n,k_n}([nt] - k_n) > \varepsilon\mu^{[nt]-k_n} L(\mu^{[nt]-k_n}) > e^{n(t\log\mu - 2\sqrt{\gamma})},$$

thereby completing the proof of (3.7).

STEP 3: ENUMERATING THE POINTS. In the following we prove (3.4). Let $\mathcal{F}$ be the σ-field generated by $(y_k^{(n)})_{k\in\mathbb{N}_0,n\in\mathbb{N}}$ and $(\tau_k,\, y_k)_{k\in\mathbb{N}_0}$. Until further notice we work conditionally on $\mathcal{F}$, so that all $\mathcal{F}$-measurable variables can be treated as deterministic constants. After discarding an event of probability 0, we can assume that the points (τ_k, y_k), $k\in\mathbb{N}_0$, have the following properties:

PROPERTY 1: $\tau_k\notin\{0,T\}$ and $y_k\notin\{1/2,1/3,\dots,\infty\}$ for all $k\in\mathbb{N}_0$.

PROPERTY 2: $\sup_{\tau_k\le t}\big(y_k+(t-\tau_k)\log\mu\big)\ge 0$ for all $t\ge 0$.

PROPERTY 3: $\tau_k\neq\tau_j$ a.s. for $k\neq j$.

PROPERTY 4: $(0,0)$ is an accumulation point of (τ_k,y_k), $k\in\mathbb{N}_0$.

Relation (3.1) implies that for large enough n and some $p\in\mathbb{N}$,

$$\widehat{N}_n([0,T]\times(\gamma,\infty])=\widehat{N}^{(c,1)}([0,T]\times(\gamma,\infty])=p,$$

where $p\neq 0$ if γ is sufficiently small. Denote by $(\bar\tau_j,\bar y_j)_{1\le j\le p}$ an enumeration of the points of $\widehat{N}^{(c,1)}$ in $[0,T]\times(\gamma,\infty]$ with $0<\bar\tau_1<\bar\tau_2<\ldots<\bar\tau_p<T$ and by $(\bar\tau_j^{(n)},\bar y_j^{(n)})_{1\le j\le p}$ the analogous enumeration of the points of $\widehat{N}_n$ in $[0,T]\times(\gamma,\infty]$. Then, relation (3.1) implies that (possibly, after renumbering the points),

$$\lim_{n\to\infty}\bar\tau_j^{(n)}=\bar\tau_j,\quad \lim_{n\to\infty}\bar y_j^{(n)}=\bar y_j,\quad j=1,\dots,p. \tag{3.8}$$

For sufficiently large $n\in\mathbb{N}$, $j=1,\dots,p$, and $t\ge 0$, set

$$U_{n,j}(t):=\sum_{i=1}^{\exp(n\bar y_j^{(n)})}\widehat{X}_{i,\,n\bar\tau_j^{(n)}}([nt]),$$

so that $\widehat{Z}_n^{(>\gamma)}(t)=\sum_{\bar\tau_j{}^{(n)}\le t}U_{n,j}(t-\bar\tau_j^{(n)})$. Put also

$$Z_{n,j}(t):=n^{-1}\log^+U_{n,j}(t),\quad Z_j(t):=(\bar y_j+t\log\mu)^+.$$

For later needs, we also define these functions to be zero for $t<0$. We rewrite (3.4) in the following form:

$$\lim_{\gamma\to0+}\limsup_{n\to\infty}d_T\Bigg(\frac1n\log^+\Bigg(\sum_{\bar\tau_j{}^{(n)}\le\cdot}U_{n,j}(\cdot-\bar\tau_j^{(n)})\Bigg),\sup_{\tau_k\le\cdot}\big(y_k+(\cdot-\tau_k)\log\mu\big)\Bigg)=0. \tag{3.9}$$

STEP 4: PROOF OF (3.9). By the triangle inequality, we have

$$\begin{aligned} d_T\Bigg(\frac1n\log^+\Bigg(\sum_{\bar\tau_j^{(n)}\le\cdot}U_{n,j}(\cdot-\bar\tau_j^{(n)})\Bigg),\sup_{\tau_k\le\cdot}\big(y_k+(\cdot-\tau_k)\log\mu\big)\Bigg)&\\ \le d_T\Bigg(\frac1n\log^+\Bigg(\sum_{\bar\tau_j^{(n)}\le\cdot}U_{n,j}(\cdot-\bar\tau_j^{(n)})\Bigg),\sup_{\bar\tau_j\le\cdot}Z_j(\cdot-\bar\tau_j)\Bigg)&\\ +\sup_{0\le t\le T}\Big|\sup_{\tau_k\le t}\big(y_k+(t-\tau_k)\log\mu\big)-\sup_{\bar\tau_j\le t}Z_j(t-\bar\tau_j)\Big|,& \end{aligned} \tag{3.10}$$

where for the last term we have used the fact that the Skorokhod metric d_T is dominated by the uniform metric on $[0,T]$. In the following, we estimate both terms on the right-hand side.

First term in (3.10). We intend to check that

$$\lim_{n\to\infty} d_T\bigg(\frac{1}{n}\log^+\bigg(\sum_{\bar\tau_j^{(n)}\le\cdot} U_{n,j}(\cdot-\bar\tau_j^{(n)})\bigg), \sup_{\bar\tau_j\le\cdot} Z_j(\cdot-\bar\tau_j)\bigg)=0 \quad \text{a.s.} \tag{3.11}$$

In view of

$$\begin{aligned}\sup_{\bar\tau_j^{(n)}\le t}\ \log^+\big(U_{n,j}(t-\bar\tau_j^{(n)})\big) &\le \log^+\bigg(\sum_{\bar\tau_j^{(n)}\le t} U_{n,j}(t-\bar\tau_j^{(n)})\bigg)\\ &\le \log^+\#\{j:\bar\tau_j^{(n)}\le t\}+\sup_{\bar\tau_j^{(n)}\le t}\ \log^+\big(U_{n,j}(t-\bar\tau_j^{(n)})\big)\\ &\le \log p+\sup_{\bar\tau_j^{(n)}\le t}\ \log^+\big(U_{n,j}(t-\bar\tau_j^{(n)})\big)\end{aligned}$$

for $t\in[0,T]$, Equation (3.11) is equivalent to

$$\lim_{n\to\infty} d_T\bigg(\sup_{\bar\tau_j^{(n)}\le\cdot} Z_{n,j}(\cdot-\bar\tau_j^{(n)}), \sup_{\bar\tau_j\le\cdot} Z_j(\cdot-\bar\tau_j)\bigg)=0 \quad \text{a.s.} \tag{3.12}$$

In view of (3.8), an application of Lemma 2.3 yields

$$Z_{n,j}(t) \quad\to\quad Z_j(t),\quad j=1,\ldots,p \tag{3.13}$$

a.s. uniformly on $[0,T]$ (recall that we work conditionally on $\mathcal{F}$). Define λ_n to be continuous and strictly increasing functions on $[0,T]$ with $\lambda_n(0)=0$, $\lambda_n(T)=T$, $\lambda_n(\bar\tau_j)=\bar\tau_j^{(n)}$ for $j=1,\ldots,p$, and let λ_n be linearly interpolated elsewhere on $[0,T]$. It is easily seen that $\lim_{n\to\infty}\sup_{0\le t\le T}|\lambda_n(t)-t|=0$. This implies that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\big|Z_{n,j}\big(\lambda_n(t)-\bar\tau_j^{(n)}\big)-Z_j(t-\bar\tau_j)\big|=0,\quad j=1,\ldots,p \tag{3.14}$$

a.s. Indeed, for $t\in[0,\bar\tau_j)$ we have $Z_{n,j}(\lambda_n(t)-\bar\tau_j^{(n)})=Z_j(t-\bar\tau_j)=0$. Also, as a consequence of (3.8) and (3.13) we obtain the relation

$$\lim_{n\to\infty}\sup_{t\in[0,T-\bar\tau_j]}|Z_{n,j}(\lambda_n(t+\bar\tau_j)-\bar\tau_j^{(n)})-Z_j(t)|=0,$$

which proves (3.14). Now (3.12) follows from

$$\begin{aligned}&\sup_{t\in[0,T]}\Big|\sup_{\bar\tau_j^{(n)}\le\lambda_n(t)} Z_{n,j}\big(\lambda_n(t)-\bar\tau_j^{(n)}\big)-\sup_{\bar\tau_j\le t} Z_j(t-\bar\tau_j)\Big|\\ =\ &\sup_{t\in[0,T]}\Big|\sup_{\bar\tau_j\le t} Z_{n,j}\big(\lambda_n(t)-\bar\tau_j^{(n)}\big)-\sup_{\bar\tau_j\le t} Z_j(t-\bar\tau_j)\Big|\\ \le\ &\sup_{t\in[0,T]}\sum_{\bar\tau_j\le t}\big|Z_{n,j}\big(\lambda_n(t)-\bar\tau_j^{(n)}\big)-Z_j(t-\bar\tau_j)\big|\\ \le\ &\sum_{j=1}^{p}\sup_{t\in[0,T]}\big|Z_{n,j}\big(\lambda_n(t)-\bar\tau_j^{(n)}\big)-Z_j(t-\bar\tau_j)\big|\end{aligned}$$

because the right-hand side converges to zero a.s. by (3.14).

Second term in (3.10). Left with proving that

$$\lim_{\gamma\to 0+}\sup_{0\le t\le T}\Big|\sup_{\tau_k\le t}\big(y_k+(t-\tau_k)\log\mu\big)-\sup_{\bar\tau_j\le t}\big(\bar y_j+(t-\bar\tau_j)\log\mu\big)^+\Big|=0, \tag{3.15}$$

we first recall that the points $(\bar\tau_1,\bar y_1),\dots,(\bar\tau_p,\bar y_p)$ belong to the collection $(\tau_k,y_k)_{k\in\mathbb{N}_0}$. Hence, for $t\in[0,T]$,

$$\sup_{\tau_k\le t}\big(y_k+(t-\tau_k)\log\mu\big)\ge\sup_{\bar\tau_j\le t}\big(\bar y_j+(t-\bar\tau_j)\log\mu\big)^+, \tag{3.16}$$

where we also used that the supremum on the left-hand side is nonnegative by Property 2. Pick now $\tau_k\notin\{\bar\tau_1,\dots,\bar\tau_p\}$ satisfying $\tau_k\le t$. Recall that all y_k other than $\bar y_1,\dots,\bar y_p$ do not exceed γ. If $\mu\le 1$, we infer

$$y_k+(t-\tau_k)\log\mu\le y_k\le\gamma\le\gamma+\sup_{\bar\tau_j\le t}(\bar y_j+(t-\bar\tau_j)\log\mu)^+.$$

Together with (3.16) this proves (3.15). In the following, let $\mu>1$. Fix some $\delta>0$. It suffices to show that for sufficiently small $\gamma>0$ we have

$$y_k+(t-\tau_k)\log\mu\le\sup_{\bar\tau_j\le t}\big(\bar y_j+(t-\bar\tau_j)\log\mu\big)^++\gamma+\delta \tag{3.17}$$

for all $k\in\mathbb{N}_0$ such that $\tau_k\le t$. If $y_k>\gamma$, then (τ_k,y_k) is one of the points $(\bar\tau_1,\bar y_1),\dots,(\bar\tau_p,\bar y_p)$, and (3.17) is evident. Let therefore $y_k\le\gamma$. Then,

$$y_k+(t-\tau_k)\log\mu\le\gamma+t\log\mu.$$

This immediately implies (3.17) if $t\le\delta/\log\mu$. Therefore, let $t>\delta/\log\mu$. If $\gamma>0$ is sufficiently small, then by Property 4 we can find a point $(\bar\tau_{j'},\bar y_{j'})$ such that $\bar\tau_{j'}<\delta/\log\mu$. We infer

$$\sup_{\bar\tau_j\le t}\big(\bar y_j+(t-\bar\tau_j)\log\mu\big)^+\ge\bar y_{j'}+(t-\bar\tau_{j'})\log\mu\ge t\log\mu-\delta.$$

Taking the last two inequalities together we arrive at (3.17), which proves (3.16). The proof of Theorem 1.1 is complete. □

The proof of Theorem 1.3 runs the same path as that of Theorem 1.1. We note that the proof is essentially based on the convergence

$$\sum_{k\ge 0}\mathbb{1}_{\{J_k\ne 0\}}\,\varepsilon_{\left(\frac{k}{n},\,\frac{\log J_k}{b_n}\right)}\quad\Rightarrow\quad N^{(1,\alpha)},\quad n\to\infty$$

on M_p and uses the corresponding part of Lemma 2.2 together with Lemma 2.4 in which we take $c_n=b_n$. We refrain from discussing the details which are much simpler here.

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