Applied Probability Trust (2 October 2017)

FUNCTIONAL LIMIT THEOREMS FOR THE NUMBER OF BUSY SERVERS IN A $G/G/\infty$ QUEUE

ALEXANDER IKSANOV,* *Taras Shevchenko National University of Kyiv and University of Wrocław*

WISSEM JEDIDI,** *King Saud University and Université de Tunis El Manar*

FETHI BOUZEFFOUR,*** *King Saud University*

Abstract

We discuss weak convergence of the number of busy servers in a $G/G/\infty$ queue in the J_1-topology on the Skorokhod space. We prove two functional limit theorems, with random and nonrandom centering, respectively, thereby solving two open problems stated in [18]. A new integral representation for the limit Gaussian process is given.

Keywords: functional limit theorem; $G/G/\infty$ queue; perturbed random walk; tightness

2010 Mathematics Subject Classification: Primary 60F17
Secondary 60K05

1. Introduction

Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η) where both ξ and η are positive. No condition is imposed on the dependence structure between ξ and η.

* Postal address: Faculty of Computer Science and Cybernetics, 01601 Kyiv, Ukraine and Institute of Mathematics, 50-384 Wrocław, Poland; e-mail: iksan@univ.kiev.ua

** Postal address: Department of Statistics & OR, P.O. Box 2455, Riyadh 11451, Saudi Arabia and Faculté des Sciences de Tunis, LR11ES11 Laboratoire d'Analyse Mathématiques et Applications, 2092 Tunis, Tunisia; e-mail: wissem_jedidi@yahoo.fr

*** Postal address: Department of Mathematics, College of Sciences, Riyadh 11451, Saudi Arabia; e-mail: fbouzaffour@ksu.edu.sa

Define

$$K(t) := \sum_{k\geq 0} \mathbb{1}_{\{S_k+\eta_{k+1}\leq t\}} \quad \text{and} \quad Z(t) := \sum_{k\geq 0} \mathbb{1}_{\{S_k\leq t<S_k+\eta_{k+1}\}}, \quad t\geq 0,$$

where $(S_k)_{k\in\mathbb{N}_0}$ is the zero-delayed ordinary random walk with increments ξ_k for $k\in\mathbb{N}$, i.e., $S_0=0$ and $S_k=\xi_1+\ldots+\xi_k$, $k\in\mathbb{N}$, and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. In a G/G/$\infty$-queuing system, where customers arrive at times $S_0=0<S_1<S_2<\ldots$ and are immediately served by one of infinitely many idle servers, the service time of the kth customer being η_{k+1}, $K(t)$ gives the number of customers served up to and including time $t\geq 0$, whereas $Z(t)$ gives the number of busy servers at time t. Some other interpretations of $Z(t)$ can be found in [12]. The process $(Z(t))_{t\geq 0}$ was also used to model the number of active sources in a communication network (for instance, active sessions in a computer network) [13, 18, 19].

From a more theoretical viewpoint, $K(t)$ is the number of visits to the interval $[0,t]$ of a *perturbed random walk* $(S_k+\eta_{k+1})_{k\in\mathbb{N}_0}$ and $Z(t)$ is the difference between the number of visits to $[0,t]$ of the ordinary random walk $(S_k)_{k\in\mathbb{N}_0}$ and $(S_k+\eta_{k+1})_{k\in\mathbb{N}_0}$. To proceed, we need a definition. Denote by $X:=(X(t))_{t\geq 0}$ a random process arbitrarily dependent on ξ. Let $(X_k,\xi_k)_{k\in\mathbb{N}}$ be i.i.d. copies of the pair (X,ξ). Following [8] we call *random process with immigration* the random process $(Y(t))_{t\geq 0}$ defined by

$$Y(t):=\sum_{k\geq 0} X_{k+1}(t-S_k)\,\mathbb{1}_{\{S_k\leq t\}}, \quad t\geq 0.$$

If X is deterministic, the random process with immigration becomes a classical renewal shot noise process. Getting back to the mainstream we conclude that both $(K(t))_{t\geq 0}$ and $Z:=(Z(t))_{t\geq 0}$ are particular instances of the random process with immigration which correspond to $X(t)=\mathbb{1}_{\{\eta\leq t\}}$ and $X(t)=\mathbb{1}_{\{\eta>t\}}$, respectively.

Let $D:=D[0,\infty)$ be the Skorokhod space of real-valued functions on $[0,\infty)$, which are right-continuous on $[0,\infty)$ with finite limits from the left at each positive point. The Skorokhod spaces $D(\mathbb{R})$, $D(0,\infty)$ and $D[0,T]$ for $T>0$ which appear below are defined similarly. We shall write $\overset{\mathrm{J_1}}{\Rightarrow}$ and $\overset{\mathbb{P}}{\to}$ to denote weak convergence in the J_1-topology on D and convergence in probability, respectively. The classical references concerning the J_1-topology are [2, 11, 15].

According to Example 2.1 in [16], when $\mathbb{E}\xi<\infty$, $\mathbb{E}\eta<\infty$ and some additional assumptions hold, $(Z(u+t))_{u\in\mathbb{R}}$ converges weakly as $t\to\infty$ in the J_1-topology on

$D(\mathbb{R})$ to a stationary limit process. The article [12] is an important earlier contribution dealing with weak convergence of the one-dimensional distributions. Assuming that the distribution tails of ξ and η are regularly varying at ∞ of negative indices larger than -1 Proposition 1 in [17] proves weak convergence of $(Z(ut))_{u>0}$, properly normalized without centering, as $t\to\infty$ in the J_1-topology on $D(0,\infty)$ to the convolution of a power function and an inverse stable subordinator. Corollary 2.7 in [18] is a weaker result of this flavor. The problem of weak convergence of $(K(ut))_{u\geq 0}$ which is much simpler than that of $(Z(ut))_{u\geq 0}$ was solved in [1] under the assumptions that the distribution of ξ belongs to the domain of attraction of an α-stable distribution, $\alpha\in(0,2]\backslash\{1\}$ and that $\mathbb{E}\eta^a<\infty$ for some $a>0$ when $\alpha\in(1,2]$.

In this paper under regular variation assumption (1) which particularly implies that $\mathbb{E}\eta=\infty$ we shall prove weak convergence of $(Z(ut))_{u\geq 0}$, properly centered and normalized, in the J_1-topology on D as $t\to\infty$. Let us interpret S_0, $S_1,\ldots$ as successive times when some source of a communication system turns on and begins a transmission of random length η_1, $\eta_2,\ldots$. As discussed in Section 1 of [19] and on pp. 132-133 in [18] condition (1) is then well justified by empirical studies of transmission durations in the World Wide Web.

We start with a functional limit theorem with a random centering.

Theorem 1.1. *Assume that $\mu:=\mathbb{E}\xi\in(0,\infty)$ and that*

$$1-F(t)=\mathbb{P}\{\eta>t\}\sim t^{-\beta}\ell(t),\quad t\to\infty \tag{1}$$

for some $\beta\in[0,1)$ and some ℓ slowly varying at ∞. Then

$$\frac{\sum_{k\geq 0}\big(\mathbb{1}_{\{S_k\leq ut<S_k+\eta_{k+1}\}}-(1-F(ut-S_k))\,\mathbb{1}_{\{S_k\leq ut\}}\big)}{\sqrt{\mu^{-1}\int_0^t(1-F(y))\,\mathrm{d}y}}\overset{\mathrm{J_1}}{\Rightarrow} V_\beta(u),\quad t\to\infty, \tag{2}$$

where $V_\beta:=(V_\beta(u))_{u\geq 0}$ is a centered Gaussian process with

$$\mathbb{E}\,V_\beta(u)V_\beta(s)=u^{1-\beta}-(u-s)^{1-\beta},\quad 0\leq s\leq u. \tag{3}$$

In the case where ξ and η are independent, weak convergence of the finite-dimensional distributions in (2) was proved in Proposition 3.2 of [18]. In the general case treated here where ξ and η are arbitrarily dependent the aforementioned convergence outside zero (i.e., weak convergence of $(Z_t^*(u_1),\ldots,Z_t^*(u_n))$ for any $n\in\mathbb{N}$ and any $0<$

$u_1 < \ldots < u_n < \infty$, where $Z_t^*(u)$ denotes the left-hand side in (2)) follows from a specialization of Proposition 2.1 in [8]. In Section 5.2 of [18] the authors write: 'We suspect that the' finite-dimensional 'convergence can be considerably strengthened'. Our Theorem 1.1 confirms their conjecture.

Also, the authors of [18] ask on p. 154: 'When can the random centering' in (2) 'be replaced by a non-random centering?' Our second result states that such a replacement is possible whenever ξ possesses finite moments of sufficiently large positive orders. Our approach is essentially based on the decomposition

$$\begin{aligned}
&\sum_{k\geq 0} \mathbb{1}_{\{S_k\leq ut<S_k+\eta_{k+1}\}} - \mu^{-1}\int_0^{ut}(1-F(y))\,\mathrm{d}y \\
= &\Big(\sum_{k\geq 0} \mathbb{1}_{\{S_k\leq ut<S_k+\eta_{k+1}\}} - \sum_{k\geq 0}\mathbb{E}(\mathbb{1}_{\{S_k\leq ut<S_k+\eta_{k+1}\}}|S_k)\Big) \\
+ &\Big(\sum_{k\geq 0}\mathbb{E}(\mathbb{1}_{\{S_k\leq ut<S_k+\eta_{k+1}\}}|S_k) - \mu^{-1}\int_0^{ut}(1-F(y))\,\mathrm{d}y\Big) \\
= &\sum_{k\geq 0}\Big(\mathbb{1}_{\{S_k\leq ut<S_k+\eta_{k+1}\}} - (1-F(ut-S_k))\,\mathbb{1}_{\{S_k\leq ut\}}\Big) \\
+ &\Big(\sum_{k\geq 0}(1-F(ut-S_k))\,\mathbb{1}_{\{S_k\leq ut\}} - \mu^{-1}\int_0^{ut}(1-F(y))\,\mathrm{d}y\Big).
\end{aligned} \tag{4}$$

Weak convergence on D of the first summand on the right-hand side, normalized by $\sqrt{\mu^{-1}\int_0^t(1-F(y))\,\mathrm{d}y}$, was treated in Theorem 1.1. Thus, we are left with analyzing weak convergence of the second summand.

Theorem 1.2. *Suppose that condition* (1) *holds. If*

$$\mathbb{E}\xi^r < \infty \quad \text{for some } r > 2(1-\beta)^{-1}, \tag{5}$$

then

$$\frac{\sum_{k\geq 0} \mathbb{1}_{\{S_k\leq ut<S_k+\eta_{k+1}\}} - \mu^{-1}\int_0^{ut}(1-F(y))\,\mathrm{d}y}{\sqrt{\mu^{-1}\int_0^t(1-F(y))\,\mathrm{d}y}} \overset{\mathrm{J}_1}{\Rightarrow} V_\beta(u), \quad t\to\infty, \tag{6}$$

where $\mu = \mathbb{E}\xi < \infty$ *and* V_β *is a centered Gaussian process with covariance* (3).

It is worth stressing that investigating Z directly, i.e., not using (4), seems to be a formidable task unless ξ and η are independent, and the distribution of ξ is exponential (for the latter situation, see [19] and references therein). We note in passing that our Theorem 1.2 includes Theorem 1 in [19] as a particular case.

Under the assumption that ξ and η are independent, weak convergence of the *one-dimensional* distributions in (6) was proved in Theorem 2 of [12]. It turns out that regular variation condition (1) is not needed for this convergence to hold. Weak convergence of the *finite-dimensional* distributions in (6) takes place under (1) and the weaker assumption $\mathbb{E}\xi^2 < \infty$. Assumption (5) is of principal importance for our proof which relies upon a strong approximation result given in Lemma 3.1. We do not know whether (1) and the second moment assumption are sufficient for weak convergence on D, that is, it is unclear whether assumption (5) can be weakened. More generally, weak convergence of the finite-dimensional distributions of $Z(ut)$, properly normalized and centered (the normalization is not necessarily of the form $\sqrt{\mu^{-1}\int_0^t (1-F(y))\,\mathrm{d}y}$, and the limit process is not necessarily V_β), holds whenever the distribution of ξ belongs to the domain of attraction of an α-stable distribution, $\alpha \in (0,2]\backslash\{1\}$, see Theorem 3.3.21 in [7] which is a specialization of Theorems 2.4 and 2.5 in [8]. We do not state these results here because in this paper we are only interested in weak convergence on D.

The rest of the paper is structured as follows. Theorems 1.1 and 1.2 are proved in Sections 2 and 3, respectively. In Section 4 we discuss an integral representation of the limit process V_β which seems to be new.

2. Proof of Theorem 1.1

We first formulate two technical assertions which are needed for the proof of Theorem 1.1. The first of these can be found in the proof of Lemma 7.3 in [1].

Lemma 2.1. *Let $G : [0,\infty) \to [0,\infty)$ be a locally bounded function. Then, for any $l \in \mathbb{N}$*

$$\mathbb{E}\Big(\sum_{k\geq 0} G(t-S_k)\,\mathbb{1}_{\{S_k\leq t\}}\Big)^l \leq \Big(\sum_{j=0}^{[t]} \sup_{y\in[j,j+1)} G(y)\Big)^l \mathbb{E}(\nu(1))^l, \quad t \geq 0. \tag{7}$$

The second auxiliary result is well-known. See, for instance, Theorem 2.1 (b) in [9]. It is of principal importance here that ξ is a.s. positive rather than nonnegative.

Lemma 2.2. *For all $a > 0$ and all $t > 0$ $\mathbb{E}e^{a\nu(t)} < \infty$.*

We proceed by observing that

$$a(t) := \sum_{k=0}^{[t]+1} (1-F(k)) \sim \int_0^t (1-F(y))\mathrm{d}y \sim (1-\beta)^{-1} t^{1-\beta} \ell(t) \tag{8}$$

as $t \to \infty$, where the second equivalence follows from Karamata's theorem (Proposition 1.5.8 in [3]). In particular, the first equivalence enables us to replace the integral in the denominator of (2) with the sum. For each $t, u \geq 0$, denote by $\widehat{Z}(ut)$ the first summand in decomposition (4), i.e.,

$$\begin{aligned} \widehat{Z}(ut) &:= \sum_{k\geq 0} \big(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - (1-F(ut-S_k))\, \mathbb{1}_{\{S_k \leq ut\}} \big) \\ &= \sum_{k\geq 0} \big(\mathbb{1}_{\{S_k + \eta_{k+1} \leq ut\}} - F(ut-S_k)\, \mathbb{1}_{\{S_k \leq ut\}} \big) \end{aligned}$$

and then set

$$Z_t(u) := \frac{\sum_{k\geq 0} \big(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - (1-F(ut-S_k))\, \mathbb{1}_{\{S_k \leq ut\}} \big)}{\sqrt{a(t)}} = \frac{\widehat{Z}(ut)}{\sqrt{a(t)}}, \quad u \geq 0.$$

Our proof of Theorem 1.1 is similar to the proof of Theorem 1 in [19] which treats the case where ξ and η are independent, and the distribution of ξ is exponential (Poisson case). Lemma 2.3 given below is concerned with inevitable technical complications that appear outside the Poisson case. Put

$$\nu(t) := \inf\{k \in \mathbb{N}_0 : S_k > t\}, \quad t \in \mathbb{R}$$

and note that the random variable $\nu(1)$ has finite moments of all positive orders by Lemma 2.2.

Lemma 2.3. *Let $l \in \mathbb{N}$ and $0 \leq v < u$. For any chosen $A > 1$ and $\rho \in (0, 1-\beta)$ there exist $t_1 > 1$ such that*

$$\mathbb{E}|Z_t(u) - Z_t(v)|^{2l} \leq c(l)(u-v)^{l(1-\beta-\rho)}$$

whenever $u - v < 1$ and $(u-v)t \geq t_1$, where $c(l) := 2C_l(4A)^l(u-v)^{l(1-\beta-\rho)}\mathbb{E}(\nu(1))^l$ and C_l is a finite positive constant.

Proof. With $u, v \geq 0$ fixed, $\widehat{Z}(ut) - \widehat{Z}(vt)$ equals the terminal value of the martingale $(R(k,t), \mathcal{F}_k)_{k\in\mathbb{N}_0}$, where $R(0,t) := 0$,

$$\begin{aligned} R(k,t) &:= \sum_{j=0}^{k-1} \big((\mathbb{1}_{\{S_j + \eta_{j+1} \leq ut\}} - F(ut-S_j)\, \mathbb{1}_{\{S_j \leq ut\}}) \\ &- (\mathbb{1}_{\{S_j + \eta_{j+1} \leq vt\}} - F(vt-S_j)\, \mathbb{1}_{\{S_j \leq vt\}}) \big), \end{aligned}$$

$\mathcal{F}_0 := \{\Omega, \oslash\}$ and $\mathcal{F}_k := \sigma((\xi_j, \eta_j) : 1 \leq j \leq k)$. We use the Burkholder-Davis-Gundy inequality (Theorem 11.3.2 in [4]) to obtain for any $l \in \mathbb{N}$

$$\begin{aligned}
& \mathbb{E}(\widehat{Z}(ut) - \widehat{Z}(vt))^{2l} \\
\leq \;& C_l\bigg(\mathbb{E}\Big(\sum_{k\geq 0}\mathbb{E}\big((R(k+1,t) - R(k,t))^2|\mathcal{F}_k\big)\Big)^l + \sum_{k\geq 0}\mathbb{E}\big(R(k+1,t) - R(k,t)\big)^{2l}\bigg) \\
=: \;& C_l(I_1(t) + I_2(t)) \qquad (9)
\end{aligned}$$

for a positive constant C_l. Next we show that

$$I_1(t) \leq 2^l\mathbb{E}(\nu(1))^l(a((u-v)t))^l, \quad t \geq 0 \qquad (10)$$

and that

$$I_2(t) \leq 2^{2l}\mathbb{E}\nu(1)a((u-v)t), \quad t \geq 0. \qquad (11)$$

Proof of (10). We first observe that

$$\begin{aligned}
& \sum_{k\geq 0}\mathbb{E}\big((R(k+1,t) - R(k,t))^2|\mathcal{F}_k\big) \\
= \;& \int_{(vt,\, ut]} F(ut-y)(1-F(ut-y))\mathrm{d}\nu(y) \\
+ \;& \int_{[0,\, vt]} (F(ut-y) - F(vt-y))(1 - F(ut-y) + F(vt-y))\mathrm{d}\nu(y) \\
\leq \;& \int_{(vt,\, ut]} (1-F(ut-y))\mathrm{d}\nu(y) + \int_{[0,\, vt]} (F(ut-y) - F(vt-y))\mathrm{d}\nu(y)
\end{aligned}$$

whence

$$\begin{aligned}
I_1(t) \;\leq\; & 2^{l-1}\bigg(\mathbb{E}\Big(\int_{(vt,\, ut]}(1-F(ut-y))\mathrm{d}\nu(y)\Big)^l \\
+ \;& \mathbb{E}\Big(\int_{[0,\, vt]}(F(ut-y) - F(vt-y))\mathrm{d}\nu(y)\Big)^l\bigg)
\end{aligned}$$

having utilized $(x+y)^l \leq 2^{l-1}(x^l + y^l)$ for nonnegative x and y. Using Lemma 2.1 with $G(y) = (1-F(y))\,\mathbb{1}_{[0,(u-v)t)}(y)$ and $G(y) = F((u-v)t+y) - F(y)$, respectively, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\int_{(vt,\,ut]}(1-F(ut-y))\mathrm{d}\nu(y)\Big)^l\\
=\ &\mathbb{E}\Big(\int_{[0,\,ut]}(1-F(ut-y))\,\mathbb{1}_{[0,\,(u-v)t)}(ut-y)\mathrm{d}\nu(y)\Big)^l\\
\leq\ &\mathbb{E}(\nu(1))^l\Big(\sum_{n=0}^{[ut]}\sup_{y\in[n,\,n+1)}((1-F(y))\,\mathbb{1}_{[0,(u-v)t)}(y))\Big)^l\\
\leq\ &\mathbb{E}(\nu(1))^l\Big(\sum_{n=0}^{[(u-v)t]}(1-F(n))\Big)^l\leq\mathbb{E}(\nu(1))^l(a((u-v)t))^l \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big(\int_{[0,\,vt]}(F(ut-y)-F(vt-y))\mathrm{d}\nu(y)\Big)^l\\
\leq\ &\mathbb{E}(\nu(1))^l\Big(\sum_{n=0}^{[vt]}\sup_{y\in[n,\,n+1)}(F((u-v)t+y)-F(y))\Big)^l\\
\leq\ &\mathbb{E}(\nu(1))^l\Big(\sum_{n=0}^{[vt]}(1-F(n))-\sum_{n=0}^{[vt]}(1-F((u-v)t+n+1))\Big)^l\\
\leq\ &\mathbb{E}(\nu(1))^l\Big(\sum_{n=0}^{[vt]}(1-F(n))-\sum_{n=0}^{[ut]+2}(1-F(n))+\sum_{n=0}^{[(u-v)t]+1}(1-F(n))\Big)^l\\
\leq\ &\mathbb{E}(\nu(1))^l(a((u-v)t))^l. \qquad (13)
\end{aligned}$$

Combining (12) and (13) yields (10).

Proof of (11). Let us calculate

$$\begin{aligned}
&\mathbb{E}((R(k+1,t)-R(k,t))^{2l}|\mathcal{F}_k)\\
\leq\ &2^{2l-1}((1-F(ut-S_k))^{2l}F(ut-S_k)\\
+\ &(F(ut-S_k))^{2l}(1-F(ut-S_k)))\,\mathbb{1}_{\{vt<S_k\leq ut\}}\\
+\ &((1-F(ut-S_k)+F(vt-S_k))^{2l}(F(ut-S_k)-F(vt-S_k))\\
+\ &(F(ut-S_k)-F(vt-S_k))^{2l}(1-F(ut-S_k)+F(vt-S_k)))\,\mathbb{1}_{\{S_k\leq vt\}}\\
\leq\ &2^{2l-1}((1-F(ut-S_k))\,\mathbb{1}_{\{vt<S_k\leq ut\}}+(F(ut-S_k)-F(vt-S_k))\,\mathbb{1}_{\{S_k\leq vt\}}).
\end{aligned}$$

Therefore,

$$I_2(t)\leq 2^{2l-1}\Big(\mathbb{E}\int_{(vt,\,ut]}(1-F(ut-y))\mathrm{d}\nu(y)+\mathbb{E}\int_{[0,\,vt]}(F(ut-y)-F(vt-y))\mathrm{d}\nu(y)\Big).$$

Using now formulae (12) and (13) with $l = 1$ yields (11).

In view of (8) we can invoke Potter's bound (Theorem 1.5.6(iii) in [3]) to conclude that for any chosen $A > 1$ and $\rho \in (0, 1-\beta)$ there exists $t_1 > 1$ such that

$$\frac{a((u-v)t)}{a(t)} \le A(u-v)^{1-\beta-\rho}$$

whenever $u - v < 1$ and $(u-v)t \ge t_1$. Note that $u - v < 1$ and $(u-v)t \ge t_1$ together imply $t \ge t_1$. Hence

$$\frac{I_1(t)}{(a(t))^l} \quad \le \quad 2^l \mathbb{E}(\nu(1))^l \left(\frac{a((u-v)t)}{a(t)}\right)^l \le (4A)^l \mathbb{E}(\nu(1))^l (u-v)^{l(1-\beta-\rho)}. \qquad (14)$$

Increasing t_1 if needed we can assume that $t^{1-\beta-\rho}/a(t) \le 1$ for $t \ge t_1$ whence

$$\begin{aligned} \frac{1}{\sum_{n=0}^{[t]+1}(1-F(n))} &= \frac{((u-v)t)^{1-\beta-\rho}}{((u-v)t)^{1-\beta-\rho}\sum_{n=0}^{[t]+1}(1-F(n))} \\ &\le \frac{(u-v)^{1-\beta-\rho}}{((u-v)t)^{1-\beta-\rho}} \le (u-v)^{1-\beta-\rho} \end{aligned}$$

because $((u-v)t)^{1-\beta-\rho} \ge t_1^{1-\beta-\rho} > 1$. This implies

$$\frac{I_2(t)}{(a(t))^l} \quad \le \quad 2^{2l}\mathbb{E}\nu(1)\frac{a((u-v)t)}{a(t)}\frac{1}{(a(t))^{l-1}} \le (4A)^l \mathbb{E}(\nu(1))^l (u-v)^{l(1-\beta-\rho)}, \quad (15)$$

where we have used $\mathbb{E}\nu(1) \le \mathbb{E}(\nu(1))^l$ which is a consequence of $\nu(1) \ge 1$ a.s.

Now the claim follows from (9), (14) and (15). □

We are ready to prove Theorem 1.1. As discussed in the paragraph following Theorem 1.1 weak convergence of $(Z_t(u_1), \ldots, Z_t(u_n))$ for any $n \in \mathbb{N}$ and any $0 < u_1 < \ldots, u_n < \infty$ was proved in earlier works. In view of $V_\beta(0) = 0$ a.s., this immediately extends to $0 \le u_1 < \ldots, u_n < \infty$. Thus, it remains to prove tightness on $D[0, T]$ for any $T > 0$. Since the normalization in (2) is regularly varying it is enough to investigate the case $T = 1$ only. Suppose we can prove that for any $\varepsilon > 0$ and $\gamma > 0$ there exist $t_0 > 0$ and $\delta > 0$ such that

$$\mathbb{P}\big\{\sup_{0\le u,v\le 1, |u-v|\le\delta} |Z_t(u) - Z_t(v)| > \varepsilon\big\} \le \gamma \qquad (16)$$

for all $t \ge t_0$. Then, by Theorem 15.5 in [2] the desired tightness follows along with continuity of the paths of (some version of) the limit process.

On pp. 763-764 in [19] it is shown that (the specific form of Z_t plays no role here)

$$\begin{aligned}\sup_{0\le u,v\le 1, |u-v|\le 2^{-i}} |Z_t(u)-Z_t(v)| &\le 2\sum_{j=i}^{I}\max_{1\le k\le 2^j}|Z_t(k2^{-j})-Z_t((k-1)2^{-j})|\\ &+ 2\max_{0\le k\le 2^I-1}\sup_{0\le w\le 2^{-I}}|Z_t(k2^{-I}+w)-Z_t(k2^{-I})|\end{aligned}$$

for any positive integers i and I, $i\le I$. Hence (16) follows if we can check that for any $\varepsilon>0$ and $\gamma>0$ there exist $t_0>0$, $i\in\mathbb{N}$ and $I\in\mathbb{N}$, $i\le I$ such that

$$\mathbb{P}\Big\{\sum_{j=i}^{I}\max_{1\le k\le 2^j}|Z_t(k2^{-j})-Z_t((k-1)2^{-j})|>\varepsilon\Big\}\le\gamma,\quad t\ge t_0 \tag{17}$$

and that

$$\max_{0\le k\le 2^I-1}\sup_{0\le w\le 2^{-I}}|Z_t(k2^{-I}+w)-Z_t(k2^{-I})| \overset{\mathbb{P}}{\to} 0,\quad t\to\infty. \tag{18}$$

Proof of (17). By Lemma 2.3, for any chosen $A>1$ and $\rho\in(0,1-\beta)$ there exists $t_1>1$ such that

$$\mathbb{E}|Z_t(k2^{-j})-Z_t((k-1)2^{-j})|^{2l}\le c(l)2^{-jl(1-\beta-\rho)} \tag{19}$$

whenever $2^{-j}t\ge t_1$. Let $I=I(t)$ denote the integer number satisfying

$$2^{-I}t\ge t_1>2^{-I-1}t.$$

Then the inequalities (19) and

$$\begin{aligned}\mathbb{E}(\max_{1\le k\le 2^j}|Z_t(k2^{-j})-Z_t((k-1)2^{-j})|)^{2l} &\le \sum_{k=1}^{2^j}\mathbb{E}|Z_t(k2^{-j})-Z_t((k-1)2^{-j})|^{2l}\\ &\le c(l)2^{-j(l(1-\beta-\rho)-1)}\end{aligned}$$

hold whenever $j\le I$. Pick now minimal $l\in\mathbb{N}$ such that $l(1-\beta-\rho)>1$. Given positive ε and γ choose minimal $i\in\mathbb{N}$ satisfying

$$2^{-i(l(1-\beta-\rho)-1)}\le\varepsilon^{2l}(1-2^{-(l(1-\beta-\rho)-1)/(2l)})^{2l}\gamma/c(l).$$

Increase t if needed to ensure that $i \le I$. Invoking Markov's inequality and then the triangle inequality for the L_{2l}-norm gives

$$\begin{aligned}
& \mathbb{P}\{\sum_{j=i}^{I} \max_{1\le k\le 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})| > \varepsilon\} \\
\le\ & \varepsilon^{-2l}\mathbb{E}\bigg(\sum_{j=i}^{I} \max_{1\le k\le 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})|\bigg)^{2l} \\
\le\ & \varepsilon^{-2l}\bigg(\sum_{j=i}^{I}(\mathbb{E}(\max_{1\le k\le 2^j} |Z_t(k2^{-j}) - Z_t((k-1)2^{-j})|)^{2l})^{1/2l}\bigg)^{2l} \\
\le\ & \varepsilon^{-2l}c(l)\bigg(\sum_{j\ge i} 2^{-j(l(1-\beta-\rho)-1)/(2l)}\bigg)^{2l} \\
=\ & \varepsilon^{-2l}c(l)\frac{2^{-i(l(1-\beta-\rho)-1)}}{(1-2^{-(l(1-\beta-\rho)-1)/(2l)})^{2l}} \le \gamma
\end{aligned}$$

for all t large enough.

Proof of (18). We shall use a decomposition

$$\begin{aligned}
& (a(t))^{1/2}(Z_t(k2^{-I}+w) - Z_t(k2^{-I})) \\
=\ & \sum_{j\ge 0}\big(\mathbb{1}_{\{S_j+\eta_{j+1}\le (k2^{-I}+w)t\}} - F((k2^{-I}+w)t - S_j)\big)\,\mathbb{1}_{\{k2^{-I}t < S_j \le (k2^{-I}+w)t\}} \\
+\ & \sum_{j\ge 0}\big(\mathbb{1}_{\{k2^{-I}t < S_j+\eta_{j+1}\le (k2^{-I}+w)t\}} \\
-\ & (F((k2^{-I}+w)t - S_j) - F(k2^{-I}t - S_j))\big)\,\mathbb{1}_{\{S_j\le k2^{-I}t\}} \\
=:\ & J_1(t,k,w) + J_2(t,k,w).
\end{aligned}$$

It suffices to prove that for $i = 1, 2$

$$(a(t))^{-1/2}\max_{0\le k\le 2^I-1}\sup_{0\le w\le 2^{-I}} |J_i(t,k,w)| \overset{\mathbb{P}}{\to} 0, \quad t\to\infty. \tag{20}$$

Proof of (20) for $i = 1$. Since $|J_1(t,k,w)| \le \nu((k2^{-I}+w)t) - \nu(k2^{-I}t)$ and $\nu(t)$ is a.s. nondecreasing we infer $\sup_{0\le w\le 2^{-I}} |J_1(t,k,w)| \le \nu((k+1)2^{-I}t) - \nu(k2^{-I}t)$. By Boole's inequality and distributional subadditivity of $\nu(t)$ (see formula (5.7) on p. 58 in [6])

$$\begin{aligned}
& \mathbb{P}\big\{\max_{0\le k\le 2^I-1}(\nu((k+1)2^{-I}t) - \nu(k2^{-I}t)) > \delta(a(t))^{1/2}\big\} \\
\le\ & \sum_{k=0}^{2^I-1}\mathbb{P}\big\{\nu((k+1)2^{-I}t) - \nu(k2^{-I}t) > \delta(a(t))^{1/2}\big\} \\
\le\ & 2^I\mathbb{P}\big\{\nu(2^{-I}t) > \delta(a(t))^{1/2}\big\} \le 2^I\mathbb{P}\big\{\nu(2t_1) > \delta(a(t))^{1/2}\big\}
\end{aligned}$$

for any $\delta > 0$. The right-hand side converges to zero as $t \to \infty$ because $\nu(2t_1)$ has finite exponential moments of all positive orders (see Lemma 2.2).

Proof of (20) for $i = 2$. We have

$$\begin{aligned}
& \sup_{0 \le w \le 2^{-I}} |J_2(t,k,w)| \\
\le\ & \sup_{0 \le w \le 2^{-I}} \Big(\sum_{j \ge 0} \mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \le (k2^{-I}+w)t\}} \mathbb{1}_{\{S_j \le k2^{-I}t\}} \\
+\ & \sum_{j \ge 0} (F((k2^{-I}+w)t - S_j) - F(k2^{-I}t - S_j))) \mathbb{1}_{\{S_j \le k2^{-I}t\}} \Big) \\
\le\ & \sum_{j \ge 0} \mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \le (k+1)2^{-I}t\}} \mathbb{1}_{\{S_j \le k2^{-I}t\}} \\
+\ & \sum_{j \ge 0} (F(((k+1)2^{-I})t - S_j) - F(k2^{-I}t - S_j))) \mathbb{1}_{\{S_j \le k2^{-I}t\}} \Big) \\
\le\ & \Big| \sum_{j \ge 0} \big(\mathbb{1}_{\{k2^{-I}t < S_j + \eta_{j+1} \le (k+1)2^{-I}t\}} \\
-\ & (F(((k+1)2^{-I})t - S_j) - F(k2^{-I}t - S_j))\big) \mathbb{1}_{\{S_j \le k2^{-I}t\}} \Big| \\
+\ & 2 \sum_{j \ge 0} (F(((k+1)2^{-I})t - S_j) - F(k2^{-I}t - S_j))) \mathbb{1}_{\{S_j \le k2^{-I}t\}} \\
=:\ & J_{21}(t,k) + 2J_{22}(t,k).
\end{aligned}$$

Pick minimal $r \in \mathbb{N}$ satisfying $r(1-\beta) > 1$ so that $\lim_{t\to\infty} t^{-1}(a(t))^r = \infty$. Using (13) with $u = (k+1)2^{-I}$ and $v = k2^{-I}$ we obtain

$$\mathbb{E}(J_{22}(t,k))^{2r} \le \mathbb{E}(\nu(1))^{2r}(a(2^{-I}t))^{2r} \le \mathbb{E}(\nu(1))^{2r}(a(2t_1))^{2r}$$

which implies

$$\begin{aligned}
(a(t))^{-r}\mathbb{E}(\max_{0 \le k \le 2^I - 1} J_{22}(t,k))^{2r} &\le (a(t))^{-r} 2^I \max_{0 \le k \le 2^I - 1} \mathbb{E}(J_{22}(t,k))^{2r} \\
&\le (a(t))^{-r} 2^I \mathbb{E}(\nu(1))^{2r}(a(2t_1))^{2r}.
\end{aligned}$$

The right-hand side converges to zero as $t \to \infty$ by our choice of r. Consequently, $(a(t))^{-1/2} \max_{0 \le k \le 2^I - 1} J_{22}(t,k) \overset{\mathbb{P}}{\to} 0$ as $t \to \infty$ by Markov's inequality.

Using a counterpart of the first inequality in (9) for the martingale $(R^*(l,t), \mathcal{F}_l)_{l \in \mathbb{N}_0}$, where $R^*(0,t) := 0$ and

$$R^*(l,t) := \sum_{j=0}^{l-1} \big(\mathbb{1}_{\{vt < S_j + \eta_{j+1} \le ut\}} - (F(ut - S_j) - F(vt - S_j))\big) \mathbb{1}_{\{S_j \le vt\}}, \quad l \in \mathbb{N}$$

for $u=(k+1)2^{-I}t$ and $v=k2^{-I}t$, one can check that

$$\begin{aligned}\mathbb{E}(J_{21}(t,k))^{2r} &\leq C_r\bigg(\mathbb{E}\Big(\int_{[0,\,k2^{-I}t]}(F((k+1)2^{-I}t-y)-F(k2^{-I}t-y))\mathrm{d}\nu(y)\Big)^r \\ &+ \mathbb{E}\int_{[0,\,k2^{-I}t]}(F((k+1)2^{-I}t-y)-F(k2^{-I}t-y))\mathrm{d}\nu(y)\bigg).\end{aligned}$$

In view of (13) the right-hand side does not exceed

$$C_r(\mathbb{E}(\nu(1))^r(a(2^{-I}t))^r+\mathbb{E}\nu(1)a(2^{-I}t))\leq C_r(\mathbb{E}(\nu(1))^r(a(2t_1))^r+\mathbb{E}\nu(1)a(2t_1)).$$

Arguing as above we conclude that $(a(t))^{-1/2}\max_{0\leq k\leq 2^I-1}J_{21}(t,k)\overset{\mathbb{P}}{\to}0$ as $t\to\infty$, and (20) for $i=2$ follows. The proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.2

Our argument is essentially based on the following strong approximation result, see Corollary 3.1 (ii) in [5].

Lemma 3.1. *Suppose that $\mathbb{E}\xi^r<\infty$ for some $r>2$. Then there exists a standard Brownian motion W such that*

$$\lim_{t\to\infty}t^{-1/r}\sup_{0\leq s\leq t}\left|\nu(s)-\mu^{-1}s-\sigma\mu^{-3/2}W(s)\right|=0\quad\text{a.s.},$$

where $\mu=\mathbb{E}\xi$ and $\sigma^2=\operatorname{Var}\xi$.

Passing to the proof of Theorem 1.2, set $f(t):=\sqrt{t(1-F(t))}$ for $t>0$. In view of (8)

$$\sqrt{\int_0^t(1-F(y))\mathrm{d}y}\ \sim\ (1-\beta)^{-1/2}t^{1/2-\beta/2}(\ell(t))^{1/2}\ \sim\ (1-\beta)^{-1/2}f(t) \qquad (21)$$

as $t\to\infty$. Assuming that $\mathbb{E}\xi^r<\infty$ for some $r>2(1-\beta)^{-1}$ we intend to show that

$$\frac{\sup_{0\leq u\leq T}\left|\sum_{k\geq0}(1-F(ut-S_k))\,\mathbb{1}_{\{S_k\leq ut\}}-\mu^{-1}\int_0^{ut}(1-F(y))\mathrm{d}y\right|}{f(t)}\ \overset{\mathbb{P}}{\to}\ 0,\quad t\to\infty$$

for any $T>0$. This in combination with (21) and Theorem 1.1 is sufficient for the proof of the J_1-convergence.

We proceed by observing that

$$\sum_{k\geq0}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\leq t\}}-\mu^{-1}\int_0^t(1-F(y))\mathrm{d}y=\int_{[0,\,t]}(1-F(t-y))\mathrm{d}(\nu(y)-\mu^{-1}y).$$

Integration by parts yields

$$\begin{aligned}
&\int_{[0,\,t]}(1-F(t-y))\mathrm{d}(\nu(y)-\mu^{-1}y)+\mathbb{P}\{\eta=t\}\\
=\;& \nu(t)-\mu^{-1}t-\int_{[0,\,t)}(\nu(t-y)-\mu^{-1}(t-y))\mathrm{d}F(y)=\Big(\nu(t)-\mu^{-1}t-\sigma\mu^{-3/2}W(t)\\
-\;& \int_{[0,\,t)}(\nu(t-y)-\mu^{-1}(t-y)-\sigma\mu^{-3/2}W(t-y))\mathrm{d}F(y)\Big)\\
+\;& \sigma\mu^{-3/2}\Big(W(t)-\int_{[0,\,t)}W(t-y)\mathrm{d}F(y)\Big)=:R_1(t)+\sigma\mu^{-3/2}R_2(t),
\end{aligned}$$

where $\sigma^2=\mathrm{Var}\,\xi<\infty$ and W is a standard Brownian motion as defined in Lemma 3.1. For any $T>0$

$$\begin{aligned}
\sup_{0\le u\le T}|R_1(ut)| &\le \sup_{0\le u\le T}|\nu(ut)-\mu^{-1}ut-\sigma\mu^{-3/2}W(ut)|\\
&+\sup_{0\le u\le T}\int_{[0,\,ut)}|\nu(ut-y)-\mu^{-1}(ut-y)-\sigma\mu^{-3/2}W(ut-y)|\mathrm{d}F(y)\\
&\le \sup_{0\le u\le Tt}|\nu(u)-\mu^{-1}u-\sigma\mu^{-3/2}W(u)|\\
&+\sup_{0\le u\le T}\sup_{0\le y\le ut}|\nu(y)-\mu^{-1}y-\sigma\mu^{-3/2}W(y)|\\
&\le 2\sup_{0\le u\le Tt}|\nu(u)-\mu^{-1}u-\sigma\mu^{-3/2}W(u)|.
\end{aligned}$$

By Lemma 3.1 the right-hand side is $o(t^{1/r})$ a.s. as $t\to\infty$. Hence, our choice of r in combination with (21) ensure that

$$\lim_{t\to\infty}\frac{\sup_{0\le u\le T}|R_1(ut)|}{f(t)}=0\quad\text{a.s.}$$

Further, we note that

$$R_2(t)=W(t)(1-F(t))+\int_{[0,\,t]}(W(t)-W(t-y))\mathrm{d}F(y)=:R_{21}(t)+R_{22}(t).$$

Pick now $\varepsilon\in(0,(1-\beta)/2)$ if $\beta\in[1/2,1)$ and $\varepsilon\in(0,1/2-\beta)$ if $\beta\in[0,1/2)$. With this ε, we have for any $T>0$

$$\begin{aligned}
\sup_{0\le u\le T}|R_{22}(ut)| &\le \sup_{0\le u\le T}\int_{[0,\,ut]}\frac{|W(ut)-W(ut-y)|}{y^{1/2-\varepsilon}}y^{1/2-\varepsilon}\mathrm{d}F(y)\\
&\le \sup_{0\le u\le T}\sup_{0\le x\le ut}\frac{|W(ut)-W(ut-x)|}{x^{1/2-\varepsilon}}\int_{[0,\,ut]}y^{1/2-\varepsilon}\mathrm{d}F(y)\\
&\le \sup_{0\le v<u\le tT}\frac{|W(u)-W(v)|}{(u-v)^{1/2-\varepsilon}}\int_{[0,\,Tt]}y^{1/2-\varepsilon}\mathrm{d}F(y)\\
&\overset{\mathrm{d}}{=} \sup_{0\le v<u\le T}\frac{|W(u)-W(v)|}{(u-v)^{1/2-\varepsilon}}t^{\varepsilon}\int_{[0,\,Tt]}y^{1/2-\varepsilon}\mathrm{d}F(y).
\end{aligned}$$

Here,

$$\sup_{0\leq v<u\leq T}\frac{|W(u)-W(v)|}{(u-v)^{1/2-\varepsilon}}<\infty \quad \text{a.s.}$$

because the Brownian motion W is locally Hölder continuous with exponent $1/2-\varepsilon$ (for any $\varepsilon\in(0,1/2)$), and the distributional equality denoted by $\overset{\mathrm{d}}{=}$ is a consequence of self-similarity of W with index $1/2$. Now it is convenient to treat two cases separately.

Case $\beta\in[1/2,1)$ in which

$$\frac{t^{\varepsilon}\int_{[0,\,Tt]}y^{1/2-\varepsilon}\mathrm{d}F(y)}{f(t)}\sim\frac{\mathbb{E}\eta^{1/2-\varepsilon}}{t^{1/2-\beta/2-\varepsilon}(\ell(t))^{1/2}}\to 0,\quad t\to\infty$$

by (21) and our choice of ε. This proves

$$\frac{\sup_{0\leq u\leq T}|R_{22}(ut)|}{f(t)}\overset{\mathbb{P}}{\to}0,\quad t\to\infty. \tag{22}$$

Case $\beta\in[0,1/2)$. Here, we conclude that

$$\frac{t^{\varepsilon}\int_{[0,\,Tt]}y^{1/2-\varepsilon}\mathrm{d}F(y)}{f(t)}\sim\frac{T^{1/2-\beta-\varepsilon}(\ell(t))^{1/2}}{(1/2-\beta-\varepsilon)t^{\beta/2}}\to 0,\quad t\to\infty$$

having utilized (21), Theorem 1.6.4 in [3] which is applicable by our choice of ε and the fact that $\ell(t)\sim\mathbb{P}\{\eta>t\}\to 0$ as $t\to\infty$ when $\beta=0$. Thus, (22) holds in this case, too.

It remains to check weak convergence on D of $R_{21}(\cdot t)/f(t)$ to the zero function or equivalently

$$\frac{\sup_{0\leq u\leq T}|R_{21}(ut)|}{f(t)}\overset{\mathbb{P}}{\to}0,\quad t\to\infty \tag{23}$$

for each $T>0$. We note that weak convergence on $D(0,\infty)$ follows immediately from the fact that $\lim_{t\to\infty}(1-F(ut))/(1-F(t))=u^{-\beta}$ locally uniformly in u on $(0,\infty)$. A longer proof is needed to treat weak convergence on $D[0,\infty)$, i.e., with 0 included. We shall only consider the case where $T>1$, the case $T\in(0,1]$ being analogous and simpler. By Potter's bound (Theorem 1.5.6 (iii) in [3]), for any chosen $A>1$ and $\delta>0$ there exists $t_0>0$ such that $1-F(ut)/(1-F(t))\leq Au^{-\beta-\delta}$ whenever $u\in(0,1]$ and $ut\geq t_0$. With this t_0, write

$$\sup_{0\leq u\leq T}|R_{21}(ut)|\leq\sup_{0\leq u\leq t_0/t}|R_{21}(ut)|\vee\sup_{t_0/t\leq u\leq 1}|R_{21}(ut)|\vee\sup_{1\leq u\leq T}|R_{21}(ut)|.$$

For the first supremum on the right-hand side we have

$$\sup_{0\leq u\leq t_0/t}|W(ut)|(1-F(ut))\leq\sup_{0\leq u\leq t_0}|W(u)|$$

which converges to zero a.s. when divided by $f(t)$.

For the third supremum,

$$\begin{aligned}\sup_{1\le u\le T}|W(ut)|(1-F(ut)) &\le (1-F(t))\sup_{0\le u\le T}|W(ut)|\\ &\overset{\mathrm{d}}{=} t^{1/2}(1-F(t))\sup_{0\le u\le T}|W(u)|,\end{aligned}$$

and the right hand-side divided by $f(t)$ converges to zero a.s. in view of (21).

Finally,

$$\frac{\sup_{t_0/t\le u\le 1}|W(ut)|(1-F(ut))}{1-F(t)}\le A\sup_{t_0/t\le u\le 1}|W(ut)|u^{-\beta-\delta}. \tag{24}$$

As before we distinguish the two cases.

Case $\beta\in[1/2,1)$. Choose δ satisfying $\delta\in(0,(1-\beta)/2)$. The law of the iterated logarithm for $|W|$ at large times guarantees that $\lim_{t\to\infty}|W(t)|t^{-\beta-\delta}=0$ a.s. whence $\sup_{u\ge t_0}|W(u)|u^{-\beta-\delta}<\infty$ a.s. With this at hand we continue (24) as follows:

$$\begin{aligned}\frac{\sup_{t_0/t\le u\le 1}|W(ut)|(1-F(ut))}{f(t)} &\le \frac{At^{\beta+\delta}\sqrt{1-F(t)}\sup_{t_0\le u\le t}\big(|W(u)|u^{-\beta-\delta}\big)}{t^{1/2}}\\ &\sim \frac{A\sup_{u\ge t_0}\big(|W(u)|u^{-\beta-\delta}\big)(\ell(t))^{1/2}}{t^{1/2-\beta/2-\delta}}\quad \text{a.s.}\end{aligned}$$

having utilized (21) for the last asymptotic equivalence. The right-hand side converges to zero a.s.

Case $\beta\in[0,1/2)$. Pick δ so small that $\beta+\delta<1/2$. The law of the iterated logarithm for $|W|$ at small times entails $\lim_{t\to 0+}|W(t)|t^{-\beta-\delta}=0$ a.s. and thereupon $\sup_{0\le u\le 1}|W(u)|u^{-\beta-\delta}<\infty$ a.s. Continuing (24) with the help of self-similarity of W we further infer

$$\frac{\sup_{t_0/t\le u\le 1}|W(ut)|(1-F(ut))}{f(t)}\le A\sup_{0\le u\le 1}|W(u)|u^{-\beta-\delta}\sqrt{1-F(t)}.$$

It remains to note that the right-hand side trivially converges to zero a.s.

Combining pieces together we conclude that (23) holds. The proof of Theorem 1.2 is complete.

4. Integral representation of the limit process V_β

The definition of the process V_β given in Theorem 1.1 provides only little insight into the pathwise structure of V_β. The purpose of this section is to find a pathwise representation of V_β thereby eliminating the aforementioned drawback.

Since V_0 is a standard Brownian motion, we assume throughout the rest of the section that $\beta \in (0,1)$. Denote by $B := (B(u,v))_{u,v\geq 0}$ a standard Brownian sheet, i.e., a two-parameter continuous centered Gaussian field with $\mathbb{E}B(u_1,v_1)B(u_2,v_2) = (u_1 \wedge u_2)(v_1 \wedge v_2)$. In particular, B is a Brownian motion in u (in v) for each fixed v (u). See Section 3 in [20] for more properties of B. It turns out that the limit process V_β can be represented as the integral of a deterministic function with respect to the Brownian sheet. Such integrals are constructed in [10]. Also, these can be thought of as particular instances of the integrals of the first kind with respect to the Brownian sheet, see Section 4 in [20]. Set

$$V_\beta^*(u) = \sqrt{1-\beta} \int_{[0,\,u]} \int_{[0,\infty)} \mathbb{1}_{\{x+z^{-1/\beta}>u\}} \,\mathrm{d}B(x,z), \quad u \geq 0. \tag{25}$$

Clearly, the process $V_\beta^* := (V_\beta^*(u))_{u\geq 0}$ is centered Gaussian. Since

$$\begin{aligned}
& \mathbb{E}V_\beta^*(u)V_\beta^*(s) \\
= \;& (1-\beta) \int_{[0,\infty)} \int_{[0,\infty)} \mathbb{1}_{\{x+z^{-1/\beta}>u\}} \mathbb{1}_{[0,\,u]}(x) \mathbb{1}_{\{x+z^{-1/\beta}>s\}} \mathbb{1}_{[0,\,s]}(x) \mathrm{d}z\mathrm{d}x \\
= \;& (1-\beta) \int_0^s \int_0^\infty \mathbb{1}_{\{x+z^{-1/\beta}>u\}} \,\mathrm{d}z\mathrm{d}x = (1-\beta) \int_0^s (u-x)^{-\beta}\mathrm{d}x \\
= \;& u^{1-\beta} - (u-s)^{1-\beta}
\end{aligned}$$

for $0 \leq s \leq u$, we conclude that V_β^* is a version of V_β.

The discussion above does not give a clue on where equality (25) comes from. Here is a non-rigorous argument based on the idea from [14] which allows one to guess (25). We start with an integral representation

$$\begin{aligned}
& \frac{\sum_{k\geq 0} \left(\mathbb{1}_{\{S_k \leq ut < S_k + \eta_{k+1}\}} - (1-F(ut-S_k)) \mathbb{1}_{\{S_k \leq ut\}} \right)}{\sqrt{\mu^{-1} \int_0^t (1-F(y))\mathrm{d}y}} \\
= \;& \int_{[0,\,u]} \int_{[0,\infty)} \mathbb{1}_{\{x+z>u\}} \,\mathrm{d}\left(\frac{\sum_{k=1}^{\nu(xt)} \mathbb{1}_{\{\eta_k \leq zt\}} - \nu(xt)F(zt)}{\sqrt{\mu^{-1} \int_0^t (1-F(y))\mathrm{d}y}} \right)
\end{aligned} \tag{26}$$

where $\nu(t) = \inf\{k \in \mathbb{N} : S_k > t\}$ for $t \geq 0$. It is likely that

$$\frac{\sum_{k=1}^{[xt]} \mathbb{1}_{\{\eta_k \leq zt\}} - [xt]F(zt)}{\sqrt{t(1-F(t))}}$$

converges weakly as $t \to \infty$ to $B(x, z^{-\beta})$ on some appropriate space of functions $g : [0,\infty) \times [0,\infty) \to \mathbb{R}$ equipped with some topology which is strong enough to ensure continuity of composition (since a proof of this is beyond our reach we cannot make the argument rigorous). The latter together with (8) and the well-known relation $t^{-1}\nu(tx) \overset{\mathrm{J_1}}{\Rightarrow} \mu^{-1}x$ as $t \to \infty$ should entail that

$$\frac{\sum_{k=1}^{\nu(xt)} \mathbb{1}_{\{\eta_k \le zt\}} - \nu(xt)F(zt)}{\sqrt{\mu^{-1}\int_0^t (1-F(y))\mathrm{d}y}}$$

converges weakly to $\sqrt{1-\beta}B(x, z^{-\beta})$. One may expect that the right-hand side of (26) converges weakly to the right-hand side of (25). On the other hand, the left-hand side of (26) converges weakly to V_β by Theorem 1.1.

Acknowledgements

The authors thank Professor Peter W. Glynn, the Editor-in-Chief and the anonymous referee for several useful comments which helped improving the presentation of our work. The second and third authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding their Research group No. (RG-1437-020).

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