Alexander Iksanov, Konrad Kolesko and Matthias Meiners

ASYMPTOTIC FLUCTUATIONS IN SUPERCRITICAL CRUMP-MODE-JAGERS PROCESSES

Abstract. Consider a supercritical Crump–Mode–Jagers process $(\mathcal{Z}_t^\varphi)_{t\geq 0}$ counted with random characteristic φ. Nerman's celebrated law of large numbers [*Z. Wahrsch. Verw. Gebiete* 57, 365–395, 1981] states that, under some mild assumptions, $e^{-\alpha t}\mathcal{Z}_t^\varphi$ converges almost surely as $t \to \infty$ to aW. Here, $\alpha > 0$ is the Malthusian parameter, a is a constant and W is the limit of Nerman's martingale, which is positive on the survival event. In this general situation, under additional (second moment) assumptions, we prove a central limit theorem for $(\mathcal{Z}_t^\varphi)_{t\geq 0}$. More precisely, we show that there exist a constant $k \in \mathbb{N}_0$ and a function $H(t)$, a finite random linear combination of functions of the form $t^j e^{\lambda t}$ with $\alpha/2 \leq \mathrm{Re}(\lambda) < \alpha$, such that $(\mathcal{Z}_t^\varphi - ae^{\alpha t}W - H(t))/\sqrt{t^k e^{\alpha t}}$ converges in distribution to a normal law with random variance. This result unifies and extends various central limit theorem-type results for specific branching processes.

Contents

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1. Introduction

A general (Crump-Mode-Jagers) branching process starts at time 0 with a single individual, the ancestor, who is alive in the random time interval $[0, \zeta)$ for a random variable ζ, the life span, taking values in $[0, \infty]$. The ancestor produces offspring born at the points of a reproduction point process ξ on $[0, \infty)$. No particular assumption about the dependence structure between ξ and ζ is made. For each individual u that is ever born there is an independent copy (ξ_u, ζ_u) of the pair (ξ, ζ) that determines the birth times of the individual's offspring relative to u's time of birth and its life span.

The general branching process encompasses e.g. the Bienaymé-Galton-Watson process, the Yule process, the continuous-time Markov branching process, the Sevast'yanov process, and the Bellman-Harris process. We refer to [21] for a more detailed account of the history of the general branching process and its predecessors.

The general branching process counted with random characteristic at time t is the sum over all individuals ever born where the contribution of each individual to the sum is determined by some random characteristic that may take into account all aspects of the individual's life such as its age at time t, its life span, etc. This formulation makes it possible to treat at one go various quantities of interest derived from the general branching process such as the number of births up to time t, the number of individuals alive at time t, the number of individuals alive at time t younger than a given threshold $a > 0$, etc. A formal description of the model will be given in Section 2.

General branching processes serve as models of biological populations such as humans, cells or plants [15, 21, 28, 35], as models for tumor growth [13, 28], but also for neutron chain reactions [2] or fragmentation [26] (after a change of time) to name but a few. The general branching process is also an important tool within related fields of applied probability or theoretical computer science. In fact, its applications in these fields are numerous and any attempt to give a complete survey here is hopeless. We confine ourselves to mentioning its successful application in the study of asymptotic properties of random graph growth models driven by preferential attachment dynamics [4, 8, 32, 39] and particularly random tree growth models [12, 18, 19, 27, 29, 36]. It is also used as an approximation for epidemic models [9, 41] and as a model of the initial phase of epidemics such as SARS, Ebola and SARS-CoV-2 [7, 10, 11], during which the disease spreads exponentially fast but the impact of population structure and preventive measures is still small [41].

The laws of large numbers of the supercritical general branching process counted with random characteristic are due to Nerman [33, 34] in the single-type, non-lattice case, that is, when the reproduction point process is not concentrated on any lattice. There were earlier results for special cases, but here we refrain from sketching the history and instead refer to the introduction of [34]. The lattice version of Nerman's law of large numbers was proved by Gatzouras [14].

In view of the relevance of the general branching process in applications and the fact that the laws of large numbers date back as far as 1981, it is remarkable, and rather surprising, that there is no comprehensive central limit theorem for the general process counted with random characteristic in the literature. However, there are partial results indicating the intricate nature of the fluctuations that can occur. For the multi-type continuous-time Markov branching process with finite type space where individuals give birth only at the time of their death Athreya [5, 6] proved a central limit theorem and Janson [24] proved a functional central limit theorem. Asmussen and Hering [2, Section VIII.3] provide results for the asymptotic fluctuations of multi-type Markov branching processes with rather general type space. In principle, these results contain the single-type case of the general branching process since such a process can be seen as a Markov process in which the type of an individual at time t is its entire life history up to time t. However, this type space is large, and the assumptions of [2] are typically not satisfied except in special cases such as the case of the Galton-Watson process. Recently, Janson studied the asymptotic fluctuations of single-type supercritical general branching processes in the lattice case [25]. For the non-lattice case, there is a second-order result by Janson and Neininger [26] for Kolmogorov's conservative fragmentation model that may be translated into the language of general branching processes. It gives a central limit theorem for the number of individuals born up to time t, but it requires that the offspring variable $N := \xi([0,\infty))$ be bounded and the additional assumption that $\int e^{-x}\,\xi(\mathrm{d}x) = 1$ almost surely, a rather restrictive assumption in the context of general branching processes. Another related result is the central limit theorem for Nerman's martingale [20].

In the present paper, we close the gap in the literature and present a central limit theorem for the general branching process counted with random characteristic. Our main result, Theorem 2.8, contains and extends all results for single-type processes summarized above. A non-exhaustive list of applications given in Section 3 contains Galton-Watson processes, Nerman's martingale and its complex-valued counterparts, epidemic models, binary homogeneous Crump-Mode-Jagers processes, and conservative fragmentation models.

2. Setup, preliminaries and main results

We continue with a formal description of the general branching process.

2.1. The general branching process counted with random characteristic.

We introduce the general (Crump-Mode-Jagers) branching process following Jagers [21, 22]. The process starts with a single individual, the ancestor, born at time 0. The ancestor produces offspring born at the points of a reproduction point process $\xi = \sum_{j=1}^{N} \delta_{X_j}$ on $[0,\infty)$ where $N = \xi([0,\infty))$ takes values in $\mathbb{N}_0 \cup \{\infty\}$ with $\mathbb{N}_0 := \{0,1,2,\ldots\}$ and $X_j := \inf\{t \geq 0 : \xi([0,t]) \geq j\}$. Here and throughout the paper, the infimum of the empty set is defined to be ∞. The ancestor has a random lifetime ζ, which may be dependent on ξ. Formally, ζ is a random variable assuming values in $[0,\infty]$.

Individuals are indexed by $u \in \mathcal{I} = \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n$ according to their genealogy. Here, $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}^0 := \{\varnothing\}$ is the singleton set containing only the empty tuple $\varnothing$. We use the usual Ulam-Harris notation. We abbreviate a tuple $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$ by $u_1 \ldots u_n$ and refer to n as the length or generation of u; we write $|u| = n$. In this context, any $u = u_1 \ldots u_n \in \mathcal{I}$ is called (potential) individual. Its ancestral line is encoded by

$$\varnothing \to u_1 \to u_1 u_2 \to \ldots \to u_1 \ldots u_n = u$$

where u_1 is the u_1th child of the ancestor, $u_1 u_2$ the u_2th child of u_1, etc. If $v = v_1 \ldots v_m \in \mathcal{I}$, then uv is short for $u_1 \ldots u_n v_1 \ldots v_m$. For $u \in \mathcal{I}$ and $i \in \mathbb{N}$, the individuals ui will be called children of u. Conversely, u will be called mother of ui. More generally, w will be called descendant of u (short: $u \preceq w$) iff $uv = w$ for some $v \in \mathcal{I}$. Conversely, u will be called an ancestor/progenitor of w. We write $u \prec w$ if $u \preceq w$ and $u \neq w$. Often, we shall refer to $\mathbb{N}^n$ as the (potential) nth generation ($n \in \mathbb{N}_0$). With these notations, we have

$$|u| = n \quad \text{iff} \quad u \in \mathbb{N}^n \quad \text{iff} \quad u \text{ is an } n\text{th generation (potential) individual.}$$

For $u = u_1 \ldots u_n \in \mathbb{N}^n$ and $k \in \mathbb{N}_0$, let $u|_k$ denote the ancestor of u in the kth generation. Formally, $u|_k$ is the restriction of the vector u to its first k components:

$$u|_k = \begin{cases} \varnothing & \text{if } k = 0, \\ u_1 \ldots u_k & \text{if } 1 \leq k \leq |u|, \\ u & \text{if } k > |u|. \end{cases} \tag{2.1}$$

For typographical reasons, we may sometimes write $v|k$ instead of $v|_k$. For $u \in \mathcal{I}$ let $u\mathcal{I}$ denote the subtree of $\mathcal{I}$ emanating from u, that is,

$$u\mathcal{I} := \{uv : v \in \mathcal{I}\} = \{w \in \mathcal{I} : w|_{|u|} = u\}.$$

For each $u \in \mathcal{I}$ there is an independent copy (ξ_u, ζ_u) of the pair (ξ, ζ) that determines the birth times of u's offspring relative to its time of birth, and the duration of its life. Quantities derived from (ξ_u, ζ_u) are indexed by u. For instance, N_u is the number of offspring of u and $X_{u,k}$ is the difference between the birth-time of the kth child of u and u itself, etc. The birth-times $S(u)$ for $u \in \mathcal{I}$ are defined recursively. We set $S(\varnothing) := 0$ and, for $n \in \mathbb{N}_0$,

$$S(uj) := S(u) + X_{u,j} \quad \text{for } u \in \mathbb{N}^n \text{ and } j \in \mathbb{N}.$$

The family tree of all individuals ever born is denoted by $\mathcal{T} := \{u \in \mathcal{I} : S(u) < \infty\}$. We call

$$\mathcal{S} := \bigcap_{n \in \mathbb{N}} \{\#(\mathcal{T} \cap \mathbb{N}^n) \geq 1\}$$

survival set and its complement $\mathcal{S}^{\mathsf{c}} = \cup_{n \in \mathbb{N}} \{\#(\mathcal{T} \cap \mathbb{N}^n) = 0\}$ extinction set. The time of death of individual u is $S(u) + \zeta_u$. An individual u is alive at time $t \geq 0$ if it is born, but not yet dead at time t, i.e., if

$$S(u) \leq t < S(u) + \zeta_u.$$

We now construct the canonical space for the general branching process. For $u \in \mathcal{I}$, let $(\Omega_u, \mathcal{A}_u, P_u)$ be a copy of a given probability space $(\Omega_\varnothing, \mathcal{A}_\varnothing, P_\varnothing)$, the *life space* of the ancestor. An element $\omega \in \Omega_u$ is a possible life career for individual u and any property of interest of u like its mass at some age or its life span is viewed as a measurable function on the life space. In particular, ξ and ζ, the reproduction point process and the life span, are measurable functions defined on $(\Omega_\varnothing, \mathcal{A}_\varnothing)$.

From the life space, we construct the population space:

$$(\Omega,\mathcal{F},\mathbb{P}) := \big(\times_{u\in\mathcal{I}}\Omega_u, \bigotimes_{u\in\mathcal{I}}\mathcal{A}_u, \bigotimes_{u\in\mathcal{I}}P_u\big).$$

For $u \in \mathcal{I}$, we write π_u for the projection $\pi_u : \times_{v\in\mathcal{I}}\Omega_v \to \Omega_u$ and θ_u for the shift $\theta_u((\omega_v)_{v\in\mathcal{I}}) = (\omega_{uv})_{v\in\mathcal{I}}$. To formally lift an entity χ defined on the life space, i.e. a function χ on Ω_u, to the population space, we define $\chi_u := \chi\circ\pi_u$. In particular, $\xi_u = \xi\circ\pi_u$ and $\zeta_u = \zeta\circ\pi_u$. In slight abuse of notation, if χ is defined on the life space, when working on the population space, we write χ instead of $\chi\circ\pi_\varnothing$. For instance, we sometimes write $\mathbb{P}(\zeta\le t)$ for $P_u(\zeta\le t) = \mathbb{P}(\zeta_\varnothing\le t)$.

We are interested in the general branching process counted with random characteristic. A random characteristic φ is a random process on $(\Omega_\varnothing,\mathcal{A}_\varnothing,P_\varnothing)$ taking values in the Skorokhod space of right-continuous functions $f:\mathbb{R}\to\mathbb{R}^d$ with existing left limits at every point in $\mathbb{R}$. The characteristic φ may also be viewed as a stochastic process $\varphi:\Omega_\varnothing\times\mathbb{R}\to\mathbb{R}^d$, $(\omega,t)\mapsto\varphi(\omega,t)$ with right-continuous paths and existing left limits. It is known that such a process is product-measurable. Define $\varphi_u = \varphi\circ\pi_u$. By product measurability, $\varphi_u(t-S(u))$ is a random variable. Note that, for given $u\in\mathcal{I}$, φ_u is independent of $S(u)$. However, φ_u and $S(v)$ can be dependent, when u is an ancestor of v. The general branching process counted with characteristic φ is $\mathcal{Z}^\varphi = (\mathcal{Z}^\varphi_t)_{t\in\mathbb{R}}$ where $\mathcal{Z}^\varphi_t$ is defined by

$$\mathcal{Z}^\varphi_t := \sum_{u\in\mathcal{I}}\varphi_u(t-S(u)), \quad t\in\mathbb{R}. \tag{2.2}$$

Here, we use the convention $\varphi(-\infty) := 0$ and so the above sum involves only terms associated with individuals that are eventually born. Conditions for the finiteness of the general branching process are given in [21, Section 6.2]. In the special case $\varphi = \mathbb{1}_{[0,\zeta)}$,

$$\mathcal{Z}^{\mathbb{1}_{[0,\zeta)}}_t = \sum_{u\in\mathcal{I}}\mathbb{1}_{[0,\zeta_u)}(t-S(u)) = \sum_{u\in\mathcal{I}}\mathbb{1}_{\{S(u)\le t<S(u)+\zeta_u\}}, \tag{2.3}$$

i.e., $\mathcal{Z}^{\mathbb{1}_{[0,\zeta)}}_t$ is the number of individuals alive at time t. Similarly,

$$N((t,t+a]) := \mathcal{Z}^{\mathbb{1}_{[0,a)}}_{t+a} = \sum_{u\in\mathcal{I}}\mathbb{1}_{[0,a)}(t+a-S(u)) = \sum_{u\in\mathcal{I}}\mathbb{1}_{\{t<S(u)\le t+a\}} \tag{2.4}$$

is the number of individuals born strictly after time t and up to and including time $t+a$, $a>0$. The setup covers a wide range of possible applications. Some special cases and specific examples are covered in Section 3.

2.2. **Assumptions.** We write $\mu(\cdot)$ for the intensity measure $\mathbb{E}[\xi(\cdot)]$ of the point process $\xi(\cdot)$ and $\mathcal{L}\mu$ for its Laplace transform, i.e.,

$$\mathcal{L}\mu(\lambda) := \int e^{-\lambda x}\,\mu(\mathrm{d}x) = \mathbb{E}\Big[\sum_{j=1}^N e^{-\lambda X_j}\Big] \tag{2.5}$$

for all $\lambda\in\mathbb{C}$ for which the above integral converges absolutely. The following assumptions are essential in the law of large numbers [14, 34] and, therefore, also for the central limit theorem studied here.

(A1) The process is *supercritical*, i.e., $\mathbb{E}[N] = \mu([0,\infty)) \in (1,\infty]$. (We do not exclude the case $\mathbb{P}(N=+\infty)>0$.)

(A2) There exists a *Malthusian parameter* $\alpha > 0$, i.e., an $\alpha > 0$ satisfying

$$\mathcal{L}\mu(\alpha) = \int e^{-\alpha x}\,\mu(\mathrm{d}x) = 1. \tag{2.6}$$

For the rest of the paper, we assume that (A1) and (A2) are satisfied. Then a nonnegative martingale, called *Nerman's martingale*, is given by

$$W_t = W_t(\alpha) = \sum_{u\in\mathcal{C}_t} e^{-\alpha S(u)}, \qquad t \geq 0 \tag{2.7}$$

where $\mathcal{C}_t := \{uj \in \mathcal{T} : S(u) \leq t < S(uj)\}$ is the coming generation at time t. We denote the almost sure limit of Nerman's martingale by W. It is known ([14, Theorem 3.3] and [30]) that $\mathbb{E}[W] = 1$ iff $(W_t)_{t\geq 0}$ is uniformly integrable. Sufficient conditions for this are that the (right) derivative of $\mathcal{L}\mu$ at α exists and satisfies $(\mathcal{L}\mu)'(\alpha) \in (-\infty, 0)$ and

$$\mathbb{E}[Z_1 \log_+ Z_1] < \infty \tag{$Z\log Z$}$$

holds where

$$Z_n = \sum_{|u|=n} e^{-\alpha S(u)}, \quad n \in \mathbb{N}_0. \tag{2.8}$$

The process $(Z_n)_{n\in\mathbb{N}_0}$ is also a nonnegative martingale, called Biggins' martingale, and it has the same almost sure limit W as Nerman's martingale $(W_t)_{t\geq 0}$ [14, Theorem 3.3].

If (A1), (A2) and the condition

$$\beta := -(\mathcal{L}\mu)'(\alpha) = \int x e^{-\alpha x}\,\mu(\mathrm{d}x) = \mathbb{E}\Big[\sum_{j=1}^{N} X_j e^{-\alpha X_j}\Big] \in (0,\infty) \tag{2.9}$$

hold, and some additional technical conditions regarding ξ and φ are satisfied, then, in the non-lattice case, the law of large numbers by Nerman states that

$$e^{-\alpha t}\mathcal{Z}_t^{\varphi} \to \beta^{-1}\mathcal{L}\mathbb{E}[\varphi](\alpha)W \quad \text{as } t\to\infty \tag{2.10}$$

where $\mathcal{L}\mathbb{E}[\varphi] := \int e^{-\alpha x}\mathbb{E}[\varphi](x)\,\mathrm{d}x$ is the bilateral Laplace transform of the function $\mathbb{E}[\varphi]$ (that maps $x\in\mathbb{R}$ to $\mathbb{E}[\varphi(x)]$) at α, a constant depending on ξ (via α) and φ only. Gatzouras's law of large numbers provides the analogous statement in the lattice case. The convergence in (2.10) holds in probability, in L^1 or in the almost sure sense depending on the conditions imposed on ξ and φ.

We further assume that the Laplace transform $\mathcal{L}\mu$ is finite on an open half-space $\mathrm{Re}(\lambda) > \vartheta$ for some $\vartheta < \frac{\alpha}{2}$:

(A3) There exists $\vartheta \in (0, \alpha/2)$ such that

$$\mathcal{L}\mu(\vartheta) = \mathbb{E}\Big[\sum_{j=1}^{N} e^{-\vartheta X_j}\Big] < \infty. \tag{2.11}$$

For the central limit theorem, we need a second moment assumption, namely:

(A4) There exists $\delta \in (0,1)$ such that with $h(t) := \big(1 + |t|\log_+^{1+\delta}|t|\big)^{1/2}$ we have

$$\mathbb{E}\Big[\Big(\sum_{j=1}^{N}\big(h(X_j) + X_j^{k-1}\big)e^{-\frac{\alpha}{2}X_j}\Big)^2\Big] < \infty \tag{2.12}$$

where k is the largest multiplicity of the roots of $\mathcal{L}\mu(z) = 1$ on $\mathrm{Re}(z) = \frac{\alpha}{2}$ (if there is no such root, we set $k := 1$).

Remark 2.1. Notice that Condition (A6) in [25], namely, the existence of a $\vartheta < \frac{\alpha}{2}$ such that

$$\mathbb{E}\bigg[\bigg(\sum_{j=1}^{N} e^{-\vartheta X_j}\bigg)^2\bigg] < \infty,$$

implies both our conditions (A3) and (A4). Janson's condition (A6) may be easier to check in cases where it holds.

For later use, we note that the function h defined above satisfies a useful inequality, namely,

$$h(s+t) \leq h(2\max\{|s|,|t|\}) \leq 2h(s)h(t) \tag{2.13}$$

for all $s, t \in \mathbb{R}$.

As a consequence of (A3), $\mathcal{L}\mu$ is holomorphic on the half-space $\mathrm{Re}(\lambda) > \vartheta$. In particular, $(\mathcal{L}\mu)'(\alpha)$ exists and is finite. Then (A1), (A2) and the strict convexity of $\mathcal{L}\mu$ on $[\vartheta, \infty)$ imply $(\mathcal{L}\mu)'(\alpha) < 0$, i.e., (2.9) holds. Since (A4) immediately implies ($Z \log Z$), we infer that validity of (A1) through (A4) implies that both martingales, $(W_t)_{t\geq 0}$ and $(Z_n)_{n\in\mathbb{N}_0}$, converge a. s. and in L^1 to the same limit $W \geq 0$. What is more, the holomorphy of $\mathcal{L}\mu$ implies that all higher derivatives of $\mathcal{L}\mu$ exist on $\mathrm{Re}(\lambda) > \vartheta$, in particular at $\lambda = \alpha$, which in turn implies (5.4) in [34] (for instance with $g(t) = 1 \wedge t^{-2}$ there). Hence, Condition 5.1 of [34] and Condition 3.2 of [14] are satisfied. This ensures that the convergence in (2.10) holds in the almost sure sense and in L^1 provided that φ satisfies appropriate assumptions, namely, Condition 5.2 of [34] in the non-lattice case, Condition 3.1 in [14] in the lattice case or Eq. (3.3) in [31] in the case where φ does not vanish on the negative halfline.

We continue with assumptions concerning the random characteristic φ. These assumptions are not made throughout the paper, but in certain results only. It will be explicitly stated, when this is the case.

We begin with some notation. For a function $f : \mathbb{R} \mapsto \mathbb{R}$ we define the total variation function $\mathrm{V}f$ by

$$\mathrm{V}f(x) := \sup\bigg\{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| : -\infty < x_0 < \ldots < x_n \leq x,\ n \in \mathbb{N}\bigg\}$$

for $x \in \mathbb{R}$. Further, throughout the paper, if φ is a nonnegative or integrable characteristic (meaning that $\mathbb{E}[\varphi(t)]$ is finite for every $t \in \mathbb{R}$), then we write $\mathbb{E}[\varphi]$ for the (measurable) function that maps $t \mapsto \mathbb{E}[\varphi](t) := \mathbb{E}[\varphi(t)]$. Similarly, we write $\mathrm{Var}[\varphi]$ for the variance function $\mathbb{E}[(\varphi - \mathbb{E}[\varphi])^2]$, so $\mathrm{Var}[\varphi](t) = \mathrm{Var}[\varphi(t)]$.

(A5) There is $\vartheta < \frac{\alpha}{2}$ such that

$$\int (\mathrm{V}\mathbb{E}[\varphi])(x)\big(e^{-\vartheta x} + e^{-\alpha x}\big)\,\mathrm{d}x < \infty.$$

(A6) The function $t \mapsto e^{-\alpha t}\,\mathrm{Var}[\varphi](t)$ is directly Riemann integrable.

We use the symbol ϑ both in (A3) and in (A5) to denote some parameter $< \frac{\alpha}{2}$ at which the corresponding condition is satisfied. If we make both assumptions at the same time, there is no harm in assuming that the ϑ's coincide, which is why we do not distinguish them by our notation.

Throughout this paper we distinguish between the lattice and the non-lattice case. Here, we say that ξ is *lattice* if $\mu([0,\infty)\setminus h\mathbb{N}_0)=0$ for some $h>0$, and we say that ξ is *non-lattice*, otherwise. In the lattice case, without loss of generality, we assume that the lattice span is 1, $\mu([0,\infty)\setminus \mathbb{N}_0)=0$ and $\mu([0,\infty)\setminus h\mathbb{N}_0)>0$ for all $h>1$. We set $\mathbb{G}:=\mathbb{R}$ in the non-lattice case and $\mathbb{G}:=\mathbb{Z}$ in the lattice case. We use the symbol ℓ to denote the Lebesgue measure in the non-lattice case and the counting measure on $\mathbb{Z}$ in the lattice case, respectively.

In the non-lattice case we additionally assume:

(A7) The intensity measure μ is absolutely continuous with respect to Lebesgue measure.

(A8) For any $t\in\mathbb{R}$ there is an $\varepsilon>0$ such that the family

$$(|\varphi(x)|^2)_{|x-t|\leq\varepsilon} \quad \text{is uniformly integrable.}$$

Notice that if φ is deterministic real-valued, then (A8) holds whenever φ is locally bounded.

In the lattice case, if $t\in\mathbb{Z}$, then $t-S(u)\in\mathbb{Z}$ for all individuals u with $S(u)<\infty$. Then $\mathcal{Z}_t^\varphi$ depends only on the values $\varphi_u(x)$ for $x\in\mathbb{Z}$ ($u\in\mathcal{I}$). In particular, the values of φ on $\mathbb{R}\setminus\mathbb{Z}$ are irrelevant for our purposes. Therefore, in the lattice case, we make the assumption that φ has paths that are constant on intervals of the form $[n,n+1)$, $n\in\mathbb{Z}$. With this assumption, condition (A8) is meaningful also in the lattice case, but reduces to the condition that $\varphi(x)\in L^2$ for all $x\in\mathbb{Z}$, a condition implicitly contained in (A6).

Remark 2.2. Notice that if the random characteristics φ and ψ satisfy condition (A8), then so does any linear combination of them. Further, by the dominated convergence theorem, both the expectation function and the variance function of any linear combination of φ and ψ have left and right limits at any point. This particularly implies that these functions are locally bounded and a.e. continuous. Consequently, if, in addition to (A8), also (A6) holds for φ and ψ, then (A6) also holds for any linear combination of φ and ψ. Indeed, for $\beta_1,\beta_2\in\mathbb{R}$,

$$e^{-\alpha t}\operatorname{Var}[\beta_1\varphi(t)+\beta_2\psi(t)]\leq 2\beta_1^2e^{-\alpha t}\operatorname{Var}[\varphi(t)]+2\beta_2^2e^{-\alpha t}\operatorname{Var}[\psi(t)],\quad t\in\mathbb{R}.$$

By [37, Remark 3.10.5 on p. 237], the function in focus is directly Riemann integrable as a locally Riemann integrable function which is dominated by a directly Riemann integrable function.

2.3. **Main results.** By Λ we denote the set of solutions to the equation

$$\mathcal{L}\mu(\lambda)=1 \tag{2.14}$$

such that $\mathrm{Re}(\lambda)>\frac{\alpha}{2}$ and by $\partial\Lambda$ we denote the set of roots on the *critical line* $\mathrm{Re}(\lambda)=\frac{\alpha}{2}$. In the lattice case, $\mathcal{L}\mu$ is $2\pi\mathrm{i}$-periodic. In this case, we define Λ to be the set of λ with $\mathrm{Re}(\lambda)>\frac{\alpha}{2}$ satisfying (2.14) and $\mathrm{Im}(\lambda)\in(-\pi,\pi)$. Analogously, in this case, $\partial\Lambda$ denotes the set of roots λ with $\mathrm{Re}(\lambda)=\frac{\alpha}{2}$ satisfying $\mathrm{Im}(\lambda)\in(-\pi,\pi)$. Finally, in both cases, we set $\Lambda_\geq:=\Lambda\cup\partial\Lambda$. Notice that $\alpha\in\Lambda$ and that every other element $\lambda\in\Lambda_\geq,\lambda\neq\alpha$ satisfies $\mathrm{Re}(\lambda)\in[\frac{\alpha}{2},\alpha)$ and $\mathrm{Im}(\lambda)\neq 0$. Further, $\lambda=\theta+\mathrm{i}\eta\in\Lambda_\geq$ implies that the complex conjugate $\overline{\lambda}=\theta-\mathrm{i}\eta\in\Lambda_\geq$.

For each root $\lambda \in \mathbb{C}$ of the function $\mathcal{L}\mu - 1$, we denote its multiplicity by $k(\lambda) \in \mathbb{N}$. Then, for any $j = 0, \ldots, k(\lambda)-1$, we can define a martingale

$$W_t^{(j)}(\lambda) := (-1)^j \sum_{u \in \mathcal{C}_t} S(u)^j e^{-\lambda S(u)}, \qquad t \in \mathbb{R} \tag{2.15}$$

with respect to the filtration $(\mathcal{F}_{\mathcal{C}_t})_{t\in\mathbb{R}}$ to be introduced below in Section 4.3. The Malthusian parameter $\alpha > 0$ is a root of multiplicity 1 and gives rise to one martingale, namely, Nerman's martingale $(W_t(\alpha))_{t\in\mathbb{R}} = (W_t)_{t\in\mathbb{R}}$, which is of great importance in the law of large numbers for the general branching process. On the other hand, the martingales corresponding to $\lambda \in \Lambda$ are relevant in the central limit theorem.

Theorem 2.3. *Suppose that (A1) through (A4) hold. Then, for any $\lambda \in \Lambda$ and $j = 0, \ldots, k(\lambda)-1$, there is a random variable $W^{(j)}(\lambda) \in L^2$ such that*

$$W_t^{(j)}(\lambda) \to W^{(j)}(\lambda) \quad \text{a. s. and in } L^2 \text{ as } t \to \infty.$$

There are more technicalities to deal with before the main result can be stated in its most general form but special cases are given now as illustration.

Theorem 2.4. *Suppose that (A1) through (A4) hold and that ξ is non-lattice. Further, suppose that there are no roots of the equation $\mathcal{L}\mu(z) = 1$ in the strip $\vartheta < \mathrm{Re}(z) < \alpha$. Then, for any characteristic φ satisfying (A5), (A6) and (A8), there exists $\sigma \geq 0$ such that, for $a_\alpha := \beta^{-1}\int \mathbb{E}[\varphi(x)]\mathrm{e}^{-\alpha x}\,\ell(\mathrm{d}x)$ and a standard normal random variable $\mathcal{N}$ independent of W,*

$$e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^\varphi - a_\alpha e^{\alpha t} W\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N} \quad \text{as } t \to \infty,\ t \in \mathbb{G}.$$

The constant σ can be explicitly computed, see the formula given in Theorem 2.8 below.

Theorem 2.5. *Suppose that (A1) through (A4) hold and that ξ is non-lattice. Then there are $b_{\lambda,l}$, $l = 0, \ldots, k(\lambda)-1$, $\lambda \in \Lambda_{\geq}$ satisfying $b_{\overline{\lambda},l} = \overline{b_{\lambda,l}}$ such that for any characteristic φ satisfying (A5), (A6) and (A8) there exists $\sigma \geq 0$ such that, for a standard normal random variable $\mathcal{N}$ independent of W, the following assertions hold.*

(i) If there are no roots of $\mathcal{L}\mu(z) = 1$ on the critical line $\mathrm{Re}(z) = \frac{\alpha}{2}$, then

$$e^{-\frac{\alpha}{2}t}\bigg(\mathcal{Z}_t^\varphi - \sum_{\lambda\in\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \sum_{j=0}^{l} \binom{l}{j} W^{(j)}(\lambda) \int (t-x)^{l-j}\mathbb{E}[\varphi(x)]e^{-\lambda x}\,\mathrm{d}x\bigg) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N}$$

as $t \to \infty$.

(ii) Otherwise, let $k \in \mathbb{N}$ be the maximal multiplicity $k(\lambda)$ of a root $\lambda \in \partial\Lambda$. Then

$$e^{-\frac{\alpha}{2}t}t^{-k+\frac{1}{2}}\bigg(\mathcal{Z}_t^\varphi - \sum_{\lambda\in\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \sum_{j=0}^{l} \binom{l}{j} W^{(j)}(\lambda) \int (t-x)^{l-j}\mathbb{E}[\varphi(x)]e^{-\lambda x}\,\mathrm{d}x\bigg) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N}$$

as $t \to \infty$.

There is an obvious lattice analogue to this theorem, but we refrain from stating it separately.

Remark 2.6. In the above theorem, we do not exclude the case $\sigma = 0$. There, a more precise limit theorem can be derived with the help of Theorem 2.8. In particular, the expression in parentheses in (i) is just a deterministic function of the order $O(e^{\theta t})$ for some $\theta < \frac{\alpha}{2}$. In case (ii), we need to additionally subtract a linear combination of $e^{\lambda t} t^l$, where λ runs over the roots on the critical line and $l < k(\lambda)$ with $k(\lambda)$ denoting the order of the root, and to use a different normalization in order to get a nontrivial limit.

Remark 2.7. The constants $b_{\lambda,l}$, $l = 0, \ldots, k(\lambda)-1$, $\lambda \in \Lambda_{\geq}$ can be computed using Proposition 6.8.

In order to derive the asymptotic expansion of $\mathcal{Z}_t^{\varphi}$, we first need an expansion for the mean $m_t^{\varphi} := \mathbb{E}[\mathcal{Z}_t^{\varphi}]$, $t \in \mathbb{R}$. It turns out that, under mild assumptions, the following holds:

$$m_t^{\varphi} = \mathbb{1}_{[0,\infty)}(t) \sum_{\lambda \in \Lambda_{\geq}} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l + r(t) \qquad \text{as } t \to \infty,\ t \in \mathbb{G} \tag{2.16}$$

for some constants $a_{\lambda,l} \in \mathbb{C}$ and a function r satisfying $|r(t)| \leq C e^{\alpha t/2}/h(t)$ where h is as defined in Assumption (A4). Condition (2.16) holds in many particular applications. Sufficient conditions ensuring (2.16) that can be effectively used in various situations are provided in Section 6. The conditions given are applied to several specific models in Section 3.

Although, one can consider cases where in expansion (2.16) there are infinitely many non-zero summands, for the ease of presentation, we restrict ourselves to the case where $\Lambda_{\geq}$ is finite.

From (2.16) we can derive an asymptotic expansion of $\mathcal{Z}_t^{\varphi}$ where the principal terms are of the form constant times $e^{\lambda t} t^j W^{(j)}(\lambda)$ for $\lambda \in \Lambda$ and $j = 0, \ldots, k(\lambda)-1$. More precisely, the principal terms are given by the expression

$$H_{\Lambda}(t) := \sum_{\lambda \in \Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} \sum_{j=0}^{l-1} a_{\lambda,l} \binom{l}{j} t^j W^{(l-j)}(\lambda).$$

If, additionally, there are roots $\lambda \in \partial\Lambda$, then the next terms in the expansion are given by the following deterministic sum

$$H_{\partial\Lambda}(t) := \sum_{\lambda \in \partial\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l, \qquad t \in \mathbb{R}.$$

(Of course, if $\Lambda_{\geq} = \partial\Lambda \cup \{\alpha\}$, the terms from $H_{\partial\Lambda}(t)$ are the subleading terms.) We set $H(t) := H_{\Lambda}(t) + H_{\partial\Lambda}(t)$, $t \in \mathbb{R}$. Further, for any $\lambda \in \partial\Lambda$ and $l = 0, \ldots, k(\lambda)-1$, we define a random variable $R_{\lambda,l}$ by

$$R_{\lambda,l} := \sum_{j=l}^{k(\lambda)-1} a_{\lambda,j} \binom{j}{l} \sum_{k=1}^{N} (-X_k)^{j-l} e^{-\lambda X_k}.$$

Assumption (A4) guarantees that $R_{\lambda,l} \in L^2$ for all $\lambda \in \partial\Lambda$ and $l = 0, \ldots, k(\lambda) - 1$. We may thus define

$$\rho_l^2 := \sum_{\substack{\lambda \in \partial\Lambda: \\ k(\lambda) > l}} \mathrm{Var}[R_{\lambda,l}]$$

where $\mathrm{Var}[R_{\lambda,l}] = \mathbb{E}[|R_{\lambda,l}|^2] - |\mathbb{E}[R_{\lambda,l}]|^2$. In general, throughout the paper, if Y is a complex-valued random variable with finite mean, we set $\mathrm{Var}[Y] := \mathbb{E}[|Y - \mathbb{E}[Y]|^2]$.

As a final preparation for our main result, we recall the fact that if a sequence of random variables (Y_n) defined on $(\Omega, \mathcal{F}, \mathbb{P})$ converges in distribution to some random variable Y, this convergence is said to be *stable* if for all continuity points y of the distribution function of Y and all $E \in \mathcal{F}$, the limit

$$\lim_{n\to\infty} \mathbb{P}(\{Y_n \leq y\} \cap E) = Q_y(E)$$

exists and if $\lim_{y\to\infty} Q_y(E) = \mathbb{P}(E)$. For a measurable function f we write $f * \xi$ for the Lebesgue-Stieltjes convolution of f and ξ, i.e.,

$$f * \xi(t) = \int f(t-x)\,\xi(\mathrm{d}x) = \sum_{j=1}^{N} f(t - X_j), \quad t \in \mathbb{R}.$$

Theorem 2.8. *Suppose that ξ satisfies (A1) through (A4) and that the real-valued characteristic φ satisfies (A6) and (A8). Further, assume that m_t^φ satisfies (2.16), that there are only finitely many $\rho_l \neq 0$, and let $n := \max\{l \in \mathbb{N}_0 : \rho_l > 0\}$ with $n = -1$ if the set is empty; in that case we set $\rho_{-1} := 0$. Then there exists a finite constant $\sigma \geq 0$ such that, with*

$$a_t^2 := \sigma^2 + \tfrac{\rho_n^2}{2n+1} t^{2n+1}, \qquad t > 0$$

it holds

(i) if $\sigma^2 = \rho_n^2 = 0$, then $\mathcal{Z}_t^\varphi = H(t) + r(t)$ for all $t \geq 0$,

(ii) if $\sigma^2 > 0$ or $\rho_n^2 > 0$, then

$$a_t^{-1} e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^\varphi - H(t)\big) \overset{\mathrm{d}}{\to} \sqrt{\tfrac{W}{\beta}}\mathcal{N} \tag{2.17}$$

as $t \to \infty$, $t \in \mathbb{G}$ where $\mathcal{N}$ is a standard normal random variable independent of W and β is as defined in (2.9). The above convergence is stable. In particular, it can be rewritten as

$$a_t^{-1} \sqrt{\frac{c_\alpha}{N(t)}}\big(\mathcal{Z}_t^\varphi - H(t)\big) \overset{\mathrm{d}}{\to} \mathcal{N}, \tag{2.18}$$

where $c_\alpha = (1 - e^\alpha)^{-1}$ in the lattice case, $c_\alpha = \alpha^{-1}$ in the non-lattice case, and $N(t)$ is the number of individuals born up to and including time t. If $n = -1$ the constant σ can be explicitly computed, namely,

$$\sigma^2 = \int \mathrm{Var}\big[\varphi(x) + h^\varphi * \xi(x)\big] e^{-\alpha x}\,\ell(\mathrm{d}x) \tag{2.19}$$

where

$$h^\varphi(t) := m_t^\varphi - \sum_{\lambda \in \Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l.$$

In the situation of Theorem 2.8, the following remarks are in order.

Remark 2.9. Observe that $N(t) = \mathcal{Z}_t^f$ for $f = \mathbb{1}_{[0,\infty)}$. Formula (2.18) is a consequence of the stable convergence in (2.17) and a particular case of (2.10) with $\varphi = f$, i.e.,

$$e^{-\alpha t} N(t) \to \frac{1}{\beta} \int\limits_{[0,\infty)} e^{-\alpha x}\, \ell(\mathrm{d}x)\, W = \frac{c_\alpha}{\beta} W \quad \text{a.s. as } t \to \infty,\ t \in \mathbb{G}.$$

Remark 2.10. (i) Notice that formula (2.19) is not well-defined in the case $n \geq 0$ as then the integral diverges. However, in this case, the exact value of σ is irrelevant and can be set to $\sigma := 0$.

(ii) For a function f satisfying a suitable integrability condition we shall write $\mathcal{L}f(z) = \int_{\mathbb{R}} e^{-zx} f(x)\,\mathrm{d}x$ for the bilateral Laplace transform. In the non-lattice case the variance σ^2 given by (2.19) can be calculated using the bilateral Laplace transform $\mathcal{L}h$ of $h = h^\varphi$. Indeed, by Plancherel's theorem,

$$\begin{aligned}
\sigma^2 &= \int \mathrm{Var}[\varphi(x) + h * \xi(x)] e^{-\alpha x}\,\mathrm{d}x \\
&= \mathbb{E}\Big[\int \big(\big(\varphi(x) - \mathbb{E}[\varphi](x) + (h * (\xi - \mu))(x)\big) e^{-\frac{\alpha}{2}x}\big)^2\,\mathrm{d}x\Big] \\
&= \frac{1}{2\pi}\mathbb{E}\Big[\int \big|\mathcal{L}((\varphi(\cdot) - \mathbb{E}[\varphi](\cdot)) e^{-\alpha\cdot/2})(\mathrm{i}\eta) + \mathcal{L}\big(((\xi-\mu) * h)(\cdot) e^{-\frac{\alpha}{2}\cdot}\big)(\mathrm{i}\eta)\big|^2\,\mathrm{d}\eta\Big] \\
&= \frac{1}{2\pi}\mathbb{E}\Big[\int \big|\mathcal{L}(\varphi - \mathbb{E}[\varphi])(\tfrac{\alpha}{2} + \mathrm{i}\eta) + \mathcal{L}\big(((\xi-\mu) * h)(\cdot)\big)(\tfrac{\alpha}{2} + \mathrm{i}\eta)\big|^2\,\mathrm{d}\eta\Big] \\
&= \frac{1}{2\pi} \int\limits_{\mathrm{Re}(z) = \frac{\alpha}{2}} \mathrm{Var}[\mathcal{L}\varphi(z) + \mathcal{L}\xi(z)\mathcal{L}h(z)]\,|\mathrm{d}z|.
\end{aligned} \tag{2.20}$$

An analogous formula holds in the lattice case, see [25]. We refrain from giving further details.

(iii) In the case where all the roots in Λ are simple and there are no roots on the critical line $\{\mathrm{Re}(z) = \frac{\alpha}{2}\}$, the bilateral Laplace transform $\mathcal{L}h$ of h coincides on a neighborhood of $\{\mathrm{Re}(z) = \frac{\alpha}{2}\}$ with the function

$$z \mapsto \frac{\mathcal{L}(\mathbb{E}[\varphi])(z)}{1 - \mathcal{L}\mu(z)}.$$

Indeed, let $h_1(t) := m_t^\varphi - \sum_{\lambda \in \Lambda} a_{\lambda,0} e^{\lambda t} \mathbb{1}_{[0,\infty)}(t)$ and $h_2(t) := h^\varphi(t) - h_1(t)$. Since $h_1(t) = O(e^{(\frac{\alpha}{2} - \epsilon)t})$ for some $\epsilon > 0$, we conclude that the bilateral Laplace transform $\mathcal{L}h_1$ is well-defined on $\{\mathrm{Re}(z) > \frac{\alpha}{2} - \epsilon\}$. Moreover, for $\mathrm{Re}(z) > \alpha$,

$$\mathcal{L}h_1(z) = \mathcal{L}m^\varphi(z) - \sum_{\lambda \in \Lambda} \frac{a_{\lambda,0}}{z - \lambda} = \frac{\mathcal{L}(\mathbb{E}[\varphi])(z)}{1 - \mathcal{L}\mu(z)} - \sum_{\lambda \in \Lambda} \frac{a_{\lambda,0}}{z - \lambda}.$$

The right-hand side, being holomorphic on $\{\mathrm{Re}(z) > \frac{\alpha}{2} - \epsilon\}$ (all the singularities are removable), coincides with $\mathcal{L}h_1$ on that domain. On the other hand, decreasing ε if needed we can and do assume that $\Lambda \subseteq \{\mathrm{Re}(z) > \frac{\alpha}{2} + \epsilon\}$. In particular, $\mathcal{L}h_2(z)$ is well-defined on $\{\mathrm{Re}(z) < \frac{\alpha}{2} + \epsilon\}$ and equal to $\sum_{\lambda \in \Lambda} \frac{a_{\lambda,0}}{z-\lambda}$. As a result, on the set $\{\frac{\alpha}{2} - \epsilon < \mathrm{Re}(z) < \frac{\alpha}{2} + \epsilon\}$ we have $\mathcal{L}h(z) = \mathcal{L}h_1(z) + \mathcal{L}h_2(z) = \frac{\mathcal{L}(\mathbb{E}[\varphi])(z)}{1 - \mathcal{L}\mu(z)}$.

Remark 2.11. Suppose that ξ satisfies (A1) through (A4) and that the real-valued characteristics $\varphi_1, \ldots, \varphi_d$ satisfy (A6) and (A8). Further, assume that each $m_t^{\varphi_j}$ satisfies (2.16) (with coefficients $a_{\lambda,l}^j$ and remainder r_j depending on j). Then Theorem 2.8 gives joint convergence in distribution of the vector $(\mathcal{Z}_t^{\varphi_1}, \ldots, \mathcal{Z}_t^{\varphi_d})$. Indeed, by the Cramér-Wold device, convergence in distribution of the vector is equivalent to convergence of all linear combinations of the form

$$\sum_{j=1}^d c_j \mathcal{Z}_t^{\varphi_j} = \mathcal{Z}_t^{\sum_{j=1}^d c_j \varphi_j}.$$

A routine verification shows that the characteristic $\sum_{j=1}^d c_j \varphi_j$ satisfies the assumptions of Theorem 2.8.

As a particular case of Remark 2.11 with $\varphi_j(\cdot) = \varphi(\cdot - s_j)$ for $-\infty < s_1 < \ldots < s_d < \infty$ and a given random characteristic φ, we get the following result for the finite-dimensional distributions:

Corollary 2.12. *In the situation of Theorem 2.8 suppose that $\sigma^2 \neq 0$ or $\rho_n \neq 0$. Then for $d := (2n+1) \vee 0$,*

$$t^{-\frac{d}{2}} e^{-\frac{\alpha}{2}t} \big(\mathcal{Z}_{t-s}^{\varphi} - H(t-s)\big)_{s \in \mathbb{R}} \stackrel{\text{f.d.}}{\to} \sqrt{\tfrac{W}{\beta}} (G_s)_{s \in \mathbb{R}}$$

where $(G_s)_{s \geq 0}$ is a centered Gaussian process with the covariance function

$$\mathbb{E}[G_s G_u] = \int \operatorname{Cov}\big[\varphi(x-s) - h^{\varphi} * \xi(x-s), \varphi(x-u) - h^{\varphi} * \xi(x-u)\big] e^{-\alpha x}\, \ell(\mathrm{d}x)$$

for any $s, u \in \mathbb{R}$ if $d = 0$, i.e., $n = -1$ and

$$\mathbb{E}[G_s G_u] = \frac{1}{d} \sum_{\substack{\lambda \in \partial\Lambda: \\ k(\lambda) \geq n+1}} \operatorname{Cov}\Bigg[\sum_{j=n}^{k(\lambda)-1} a_{\lambda,j} \binom{j}{n} \sum_{k=1}^N (-X_k - s)^{j-n} e^{-\lambda(X_k + s)},$$
$$\sum_{j=n}^{k(\lambda)-1} a_{\lambda,j} \binom{j}{n} \sum_{k=1}^N (-X_k - u)^{j-n} e^{-\lambda(X_k + u)}\Bigg]$$

if $d = 2n+1$ with $n \geq 0$.

3. Applications

3.1. **The Galton-Watson process.** Consider the situation of a supercritical Galton-Watson branching process, i.e., $\xi = \sum_{k=1}^N \delta_1 = N\delta_1$ where N is a random variable taking values in $\mathbb{N}_0$ with $m := \mathbb{E}[N] \in (1, \infty)$ and $\mathbb{E}[N^2] < \infty$. Then ξ is lattice with span 1 and (A1) holds. Further,

$$\mathcal{L}\mu(\lambda) = \mathbb{E}\Bigg[\sum_{k=1}^N e^{-\lambda}\Bigg] = m e^{-\lambda}, \quad \lambda \in \mathbb{C}.$$

The equation $\mathcal{L}\mu(z) = 1$ is equivalent to $e^z = m$ and has only one solution in the strip $\operatorname{Im}(z) \in (-\pi, \pi)$. In particular, (A2) holds, i.e., there is a Malthusian parameter $\alpha > 0$, namely, $\alpha = \log m$, and $\Lambda_{\geq} = \{\alpha\}$. In this case the parameter β defined by (2.9) is equal to 1. Then

$$\mathbb{E}\Big[\Big(\sum_{k=1}^{N} e^{-\theta}\Big)^2\Big] = e^{-2\theta}\mathbb{E}[N^2] < \infty \quad \text{for all } \theta \in \mathbb{R}.$$

By Remark 2.1, this implies that (A3) and (A4) hold.

Consider the characteristic $\phi_\alpha(t) := e^{-\alpha}\mathbb{1}_{[0,1)}(t)$. Then for any $n \in \mathbb{N}_0$, $\mathcal{Z}_n^{\phi_\alpha}$ is the number of individuals in the n^{th} generation and the corresponding Nerman's martingale (2.7) is the size of n^{th} generation normalized by its expectation m^n, i.e., $W_n = e^{-\alpha n}\mathcal{Z}_n^{\phi_\alpha}$. Clearly, ϕ_α satisfies (A5), (A6) and (A8). Therefore, we may apply the lattice counterpart of Theorem 2.4, which yields

$$m^{-\frac{n}{2}}\big(m^n W_n - a_\alpha m^n W\big) = e^{-\frac{\alpha}{2}n}\big(\mathcal{Z}_n^{\phi_\alpha} - a_\alpha e^{\alpha n} W\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{W}\mathcal{N} \quad \text{as } n \to \infty$$

where $\mathcal{N}$ is standard normal and independent of W. To calculate $a_\alpha := a_{\alpha,0}$, we use (2.16): $m^n = \mathbb{E}[\mathcal{Z}_n^{\phi_\alpha}] = a_\alpha m^n$, i.e., $a_\alpha = 1$. Further, $\sigma > 0$ is given by (2.19), i.e.,

$$\sigma^2 = \sum_{n\in\mathbb{Z}} \mathrm{Var}\,\big[\phi_\alpha(n) + h * \xi(n)\big]e^{-\alpha n} = \sum_{n\in\mathbb{Z}} \mathrm{Var}\,\big[h * \xi(n)\big]m^{-n},$$

where

$$h(n) = m_n^{\phi_\alpha} - a_\alpha e^{\alpha n} = m^n\mathbb{1}_{\{n\geq 0\}} - m^n = -m^n\mathbb{1}_{\{n<0\}}.$$

Now, since $h * \xi(n) = Nh(n-1)$ we infer

$$\sigma^2 = \sum_{n<1} \mathrm{Var}[N]m^{2n-2}m^{-n} = \frac{\mathrm{Var}[N]}{m^2 - m}.$$

Consequently,

$$m^{\frac{n}{2}}\big(W_n - W\big) \stackrel{\mathrm{d}}{\to} \Big(\frac{\mathrm{Var}[N]W}{m^2 - m}\Big)^{\frac{1}{2}}\mathcal{N} \quad \text{as } n \to \infty.$$

We have thus just rediscovered Heyde's classical central limit theorem for the martingale in the Galton-Watson process [17].

We can also deal with the total number of individuals in the generations $0, \ldots, n$. Indeed, this number is $\mathcal{Z}_n^f$ for $f(t) := \mathbb{1}_{[0,\infty)}(t)$, $t \in \mathbb{R}$, which satisfies (A5), (A6) and (A8). Thus, a lattice version of Theorem 2.4 applies and gives

$$e^{-\frac{\alpha}{2}n}\big(\mathcal{Z}_n^f - a_\alpha e^{\alpha n} W\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{W}\mathcal{N} \quad \text{as } n \to \infty.$$

This time a_α can be computed with the help of (2.16) as follows. We have the asymptotic expansion

$$m_n^f = \mathbb{E}[\mathcal{Z}_n^f] = \sum_{k=0}^{n} m^k = \frac{m^{n+1} - 1}{m - 1} = \frac{m}{m-1}e^{\alpha n} - \frac{1}{m-1},$$

for $n \geq 0$ and 0 otherwise. Consequently, $a_\alpha = \frac{m}{m-1}$ and thereupon

$$m^{-\frac{n}{2}}\Big(\mathcal{Z}_n^f - \frac{m^{n+1}}{m-1}W\Big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{W}\mathcal{N} \quad \text{as } n \to \infty.$$

This time $\sigma > 0$ is given by $\sigma^2 = \sum_{n\in\mathbb{Z}} \mathrm{Var}[N]|h(n-1)|^2 m^{-n}$ with

$$h(n) = m_n^f - \frac{m}{m-1}e^{\alpha n}.$$

Therefore,

$$\begin{aligned}\sigma^2 &= \mathrm{Var}[N]\Big(\sum_{n<1}\Big(\frac{m}{m-1}m^{n-1}\Big)^2 m^{-n} + \sum_{n\geq 1}\Big(\frac{1}{m-1}\Big)^2 m^{-n}\Big)\\ &= \frac{1}{(m-1)^2}\mathrm{Var}[N]\Bigg(\sum_{n\leq 0} m^n + \sum_{n>0} m^{-n}\Bigg) = \frac{m+1}{(m-1)^3}\mathrm{Var}[N].\end{aligned}$$

3.2. **Nerman's martingales.** Suppose that ξ is non-lattice and satisfies (A1) through (A4), and let $\lambda = \theta + \mathrm{i}\eta$ be a root to $\mathcal{L}\mu(z) = 1$ with $\mathrm{Re}(\lambda) = \theta < \frac{\alpha}{2}$. Further, suppose that

$$\mathbb{E}\Bigg[\Bigg(\sum_{k=1}^{N} e^{-\theta X_k}\Bigg)^2\Bigg] < \infty.$$

For simplicity let $Z_1(\lambda) := \sum_{k=1}^{N} e^{-\lambda X_k}$. We can view the complex variable $Z_1(\lambda)$ as a random variable taking value in $\mathbb{R}^2$. We denote by Σ^λ the corresponding covariance matrix. The aforementioned condition guarantees that Σ^λ is well-defined.

Let $(W_t(\lambda)_{t\geq 0}$ be the martingale defined by (2.15) for $j = 0$. Then, since $\mathrm{Re}(\lambda) < \frac{\alpha}{2}$, we cannot apply Theorem 2.3, but one can still wonder what the long-term behavior of the process is. To analyze this, we shall apply a special case of our main result, Theorem 2.8. Let

$$\phi_\lambda(t) := e^{\lambda t}\sum_{j=1}^{N} \mathbb{1}_{[0,X_j)}(t) e^{-\lambda X_j}, \quad t \in \mathbb{R}.$$

Then one can verify that

$$\mathcal{Z}_t^{\phi_\lambda} = e^{\lambda t} W_t(\lambda),$$

and $\mathbb{E}[\mathcal{Z}_t^{\phi_\lambda}] = e^{\lambda t}\mathbb{1}_{[0,\infty)}(t)$. Since

$$|\phi_\lambda(t)|^2 \leq e^{2\theta t}\mathbb{1}_{[0,\infty)}(t)\Bigg|\sum_{j=1}^{N} e^{-\theta X_j}\Bigg|^2,$$

both $\mathrm{Re}(\phi_\lambda)$ and $\mathrm{Im}(\phi_\lambda)$ fulfill (A6) and (A8). By Theorem 5.8, we deduce

$$e^{-\frac{\alpha}{2}t}(\mathcal{Z}_t^{\mathrm{Re}(\phi_\lambda)}, \mathcal{Z}_t^{\mathrm{Im}(\phi_\lambda)}) \xrightarrow{\mathrm{d}} \sqrt{\tfrac{W}{\beta}}\mathcal{N}$$

or equivalently

$$(\mathrm{Re}(e^{(\lambda-\frac{\alpha}{2})t}W_t(\lambda)), \mathrm{Im}(e^{(\lambda-\frac{\alpha}{2})t}W_t(\lambda))) \xrightarrow{\mathrm{d}} \sqrt{\tfrac{W}{\beta}}\mathcal{N}$$

where $\mathcal{N}$ is a 2-dimensional centered Gaussian vector with covariance matrix Σ, which can be explicitly computed. Indeed, we have

$$
\begin{aligned}
\Sigma_{11} &:= \int \operatorname{Var}\Big[\operatorname{Re}(\phi_\lambda(x)) + \sum_{j=1}^{N} \operatorname{Re}(e^{\lambda(x-X_j)})\mathbb{1}_{[0,\infty)}(x-X_j)\Big] e^{-\alpha x}\,\ell(\mathrm{d}x) \\
&= \int_0^\infty \operatorname{Var}\big[\operatorname{Re}(e^{\lambda x} Z_1(\lambda))\big] e^{-\alpha x}\,\mathrm{d}x \\
&= \int_0^\infty e^{2\theta x}\Big(\cos^2(\eta x)\Sigma_{11}^\lambda - 2\sin(\eta x)\cos(\eta x)\Sigma_{12}^\lambda + \sin^2(\eta x)\Sigma_{22}^\lambda\Big) e^{-\alpha x}\,\mathrm{d}x \\
&= \frac{\Sigma_{11}^\lambda}{\alpha-2\theta}\frac{2\eta^2+(\alpha-2\theta)^2}{4\eta^2+(\alpha-2\theta)^2} - \Sigma_{12}^\lambda \frac{2\eta}{4\eta^2+(\alpha-2\theta)^2} + \frac{\Sigma_{22}^\lambda}{\alpha-2\theta}\frac{2\eta^2}{4\eta^2+(\alpha-2\theta)^2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\Sigma_{22} &= \int_0^\infty \operatorname{Var}\big[\operatorname{Im}(e^{\lambda x} Z_1(\lambda))\big] e^{-\alpha x}\,\mathrm{d}x \\
&= \frac{\Sigma_{22}^\lambda}{\alpha-2\theta}\frac{2\eta^2+(\alpha-2\theta)^2}{4\eta^2+(\alpha-2\theta)^2} + \Sigma_{12}^\lambda \frac{2\eta}{4\eta^2+(\alpha-2\theta)^2} + \frac{\Sigma_{11}^\lambda}{\alpha-2\theta}\frac{2\eta^2}{4\eta^2+(\alpha-2\theta)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma_{12} &= \int_0^\infty \operatorname{Cov}\big[\operatorname{Re}(e^{\lambda x} Z_1(\lambda)), \operatorname{Im}(e^{\lambda x} Z_1(\lambda))\big] e^{-\alpha x}\,\mathrm{d}x \\
&= \frac{(\Sigma_{11}^\lambda - \Sigma_{22}^\lambda)\eta}{4\eta^2+(\alpha-2\theta)^2} + \Sigma_{12}^\lambda \frac{\alpha-2\theta}{4\eta^2+(\alpha-2\theta)^2}.
\end{aligned}
$$

3.3. **Epidemic models.** In this section, we consider the epidemic model discussed in [11]. In this model, the role of the ancestor is that of the first person in a community infected by an infectious disease. Birth events become infection events etc.

Suppose that ξ is a Poisson point process on $[0,\infty)$ with intensity measure $R_0 g(x)\,\mathrm{d}x$ where

$$
g(x) = \mathbb{1}_{(0,\infty)}(x)\frac{b^a x^{a-1}}{\Gamma(a)} e^{-bx}, \quad x \in \mathbb{R}
$$

is the density of the Gamma distribution with parameters $a, b > 0$ and $R_0 > 1$ is the basic reproduction mean. (No additional difficulties would occur if R_0 was replaced by a positive random variable N with mean R_0 and finite variance.) The function g is the infection rate scaled to become a probability density. It models the time delay between the infection of a person and a random person infected by that person. Characteristics of interest are $I(t) = R_0 g(t)$ and $f(t) = \mathbb{1}_{[0,\infty)}(t)$ with $\mathcal{Z}_t^I$ being the incidence at time t and $\mathcal{Z}_t^f$ the number of infections up to time t. In the given situation, the Laplace transform $\mathcal{L}\mu$ can be calculated explicitly in terms of a, b and R_0, namely,

$$
\mathcal{L}\mu(\lambda) = \int_0^\infty e^{-\lambda x}\,\mu(\mathrm{d}x) = R_0\Big(\frac{b}{b+\lambda}\Big)^a, \quad \operatorname{Re}(\lambda) > -b.
$$

Hence the equation $\mathcal{L}\mu(\lambda) = 1$ takes the form

$$R_0\Big(\frac{b}{b+\lambda}\Big)^a = 1.$$

Write $\frac{b}{b+\lambda} = re^{\mathrm{i}\varphi}$ with $r > 0$ and $|\varphi| < \pi$. Then $R_0(re^{\mathrm{i}\varphi})^a = 1$ is equivalent to

$$e^{-\mathrm{i}a\varphi} = r^a R_0.$$

This implies $r = R_0^{-1/a}$ and $\varphi \in (2\pi/a)\mathbb{Z} \cap (-\pi, \pi)$. Solving for λ yields

$$\lambda = b(R_0^{1/a} e^{-\mathrm{i}\varphi} - 1) \tag{3.1}$$

with $\varphi \in (2\pi/a)\mathbb{Z} \cap (-\pi, \pi)$. The Malthusian parameter is obtained by setting $\varphi = 0$, i.e.,

$$\alpha = b(R_0^{1/a} - 1).$$

The real part of a root λ as in (3.1) is given by

$$\mathrm{Re}(\lambda) = b(R_0^{1/a} \cos\varphi - 1). \tag{3.2}$$

A second root exists only if $a > 2$ (otherwise $(2\pi/a)\mathbb{Z} \cap (-\pi, \pi) = \{0\}$), in which case the root $\lambda \neq \alpha$ with largest real part is $\lambda = b(R_0^{1/a} e^{\mathrm{i}2\pi/a} - 1)$ with

$$\mathrm{Re}(\lambda) = b(R_0^{1/a} \cos(\tfrac{2\pi}{a}) - 1).$$

This is clearly negative unless $a > 4$ since for $a \in (2, 4]$, we have $\cos(\frac{2\pi}{a}) \leq 0$. In the case $a > 4$, we have

$$\mathrm{Re}(\lambda) = b(R_0^{1/a} \cos(\tfrac{2\pi}{a}) - 1) \geq \frac{\alpha}{2} = \frac{b}{2}(R_0^{1/a} - 1)$$

if and only if $a > 6$ and $R_0 \geq R_0(a) := (2\cos(\frac{2\pi}{a}) - 1)^{-a}$. Notice that $R_0(a) \to \infty$ for $a \downarrow 6$ and $R_0(a) \to 1$ for $a \to \infty$. If $R_0 < R_0(a)$, Theorem 2.4 applies and yields Gaussian fluctuations of $\mathcal{Z}_t^I$ and $\mathcal{Z}_t^f$. That is, for $\mathcal{Z}_t^f$ we have

$$e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^f - a_\alpha e^{\alpha t} W\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N} \quad \text{as } t \to \infty,$$

with $\beta := R_0 a b^a (b+\alpha)^{-a-1}$ and $a_\alpha := (\alpha\beta)^{-1}$. Left with calculating σ we obtain with the help of Remark 2.10

$$\begin{aligned}
\sigma^2 &= \frac{1}{2\pi} \int\limits_{\mathrm{Re}(z) = \frac{\alpha}{2}} \mathrm{Var}\Big[\mathcal{L}f(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}f(z)}{1 - \mathcal{L}\mu(z)}\Big]\,|\mathrm{d}z| \\
&= \frac{1}{2\pi} \int\limits_{\mathrm{Re}(z) = \frac{\alpha}{2}} \Big|\frac{1}{z(1 - \mathcal{L}\mu(z))}\Big|^2 \mathrm{Var}[\mathcal{L}\xi(z)]\,|\mathrm{d}z| \\
&= \frac{1}{2\pi} \int\limits_{\mathrm{Re}(z) = \frac{\alpha}{2}} \Big|\frac{1}{z(1 - \mathcal{L}\mu(z))}\Big|^2 \mathcal{L}\mu(\alpha)\,|\mathrm{d}z| \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{(\alpha^2 + 4t^2) g(t)}\,\mathrm{d}t,
\end{aligned}$$

with $g(t) := 1 + R_0^2\Big(\frac{4b^2}{(2b+\alpha)^2 + 4t^2}\Big)^a - 2R_0\Big(\frac{4b^2}{(2b+\alpha)^2+4t^2}\Big)^{\frac{a}{2}} \cos\Big(a \arctan\Big(\frac{2t}{2b+\alpha}\Big)\Big)$.

If $R_0 \geq R_0(a)$, the more general Theorem 2.8 applies and gives additional periodic fluctuations of greater magnitude than the Gaussian fluctuations. We refrain from providing further details.

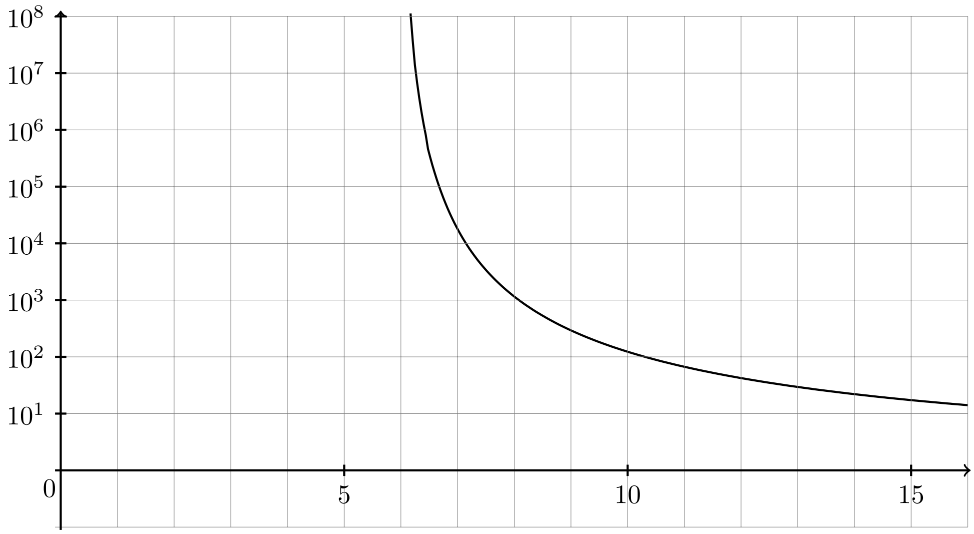

FIGURE 1. The figure shows the graph of the function $a \mapsto (2\cos(\frac{2\pi}{a}) - 1)^{-a}$ for $a > 6$ on a logarithmic scale.

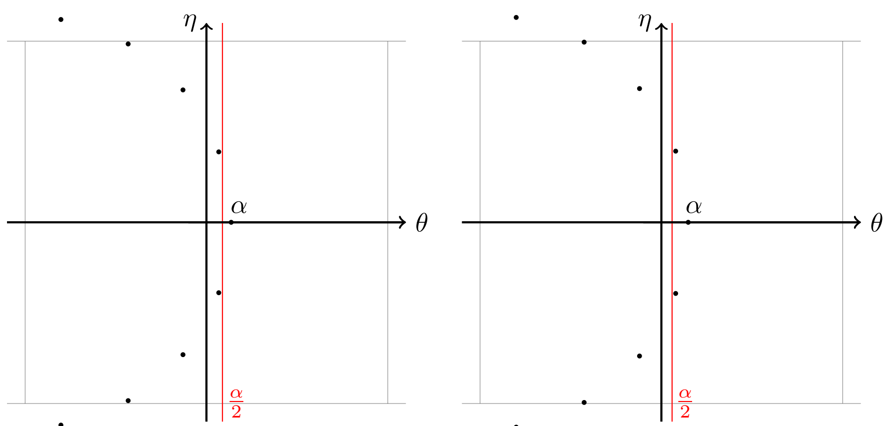

FIGURE 2. Solutions to $\mathcal{L}\mu(\lambda) = 1$ in the cases $a = 18$, $b = 1$, $R_0 = 10$ (left figure) and $a = 18$, $b = 1$, $R_0 = 12$. In the left figure, the root $\lambda \neq \alpha$ with largest real part has $\mathrm{Re}(\lambda) < \frac{\alpha}{2}$, in the right figure $\mathrm{Re}(\lambda) > \frac{\alpha}{2}$.

3.4. **A Poisson model.** In this section, we assume that

$$\xi := \sum_{i \geq 1} \mathbb{1}_{[0,\zeta)}(P_i)\delta_{P_i},$$

where $(P_i)_{i\geq 1}$ are the arrival times of a Poisson process with intensity $b > 0$, independent of a $[0,\infty]$-valued random variable ζ. We are interested in the limit theorem for $\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}}$ the number of individuals alive at time t, see (2.3). Thus, the corresponding characteristic ϕ is given by $\phi(t) := \mathbb{1}_{[0,\zeta)}(t)$ for $t \geq 0$.

We put $\mathcal{L}_\zeta(z) := \mathbb{E}[e^{-z\zeta}]$ for $\mathrm{Re}(z) \geq 0$ and start by noting that

$$\mathcal{L}\mu(z) = \mathbb{E}\sum_{i\geq 1} \mathbb{1}_{[0,\zeta)}(P_i)e^{-zP_i} = \mathbb{E}\int \mathbb{1}_{[0,\zeta)}(x)e^{-zx}b\mathrm{d}x = \int_0^\infty e^{-zx}b\mathbb{P}(\zeta > x)\mathrm{d}x$$
$$= \frac{b(1-\mathcal{L}_\zeta(z))}{z}, \quad \mathrm{Re}(z) > 0.$$

The Malthusian parameter α is the unique real number that satisfies

$$1 - \mathcal{L}_\zeta(\alpha) = \frac{\alpha}{b},$$

and the parameter β is given by

$$\beta = \frac{1}{\alpha}(1 + b\mathcal{L}_\zeta'(\alpha)) = \frac{1}{\alpha}(1 - b\mathbb{E}\zeta e^{-\alpha\zeta}).$$

Now we shall show that α is the only solution to $\mathcal{L}\mu(z) = 1$ with $\mathrm{Re}(z) > 0$. Indeed, for positive θ and η

$$\mathrm{Im}\big(\mathcal{L}\mu(\theta + \mathrm{i}\eta)\big) = \int_0^\infty \sin(\eta x)e^{-\theta x}b\mathbb{P}(\zeta > x)\mathrm{d}x$$
$$= b\sum_{l\geq 0} \int\limits_{2\pi l/\eta}^{\pi(2l+1)/\eta} \sin(\eta x)\Big(e^{-\theta x}\mathbb{P}(\zeta > x) - e^{-\theta(x+\pi/\eta)}\mathbb{P}(\zeta > x + \pi/\eta)\Big)\mathrm{d}x > 0$$

which, together with $\mathcal{L}\mu(\overline{\lambda}) = \overline{\mathcal{L}\mu(\lambda)}$, shows that $\Lambda_\geq \cap \{z : \mathrm{Re}(z) > 0\} = \{\alpha\}$. Since $\mathcal{L}(\mathbb{E}[\phi])(\alpha) = b^{-1}\mathcal{L}\mu(\alpha) = b^{-1}$ an application of Theorem 2.4 yields

$$e^{-\frac{\alpha}{2}t}\Big(\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}} - e^{\alpha t}\frac{W}{b\beta}\Big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N}.$$

Next, we express the variance σ^2 in terms of the parameters b, α and β. By Remark 2.10

$$\sigma^2 = \frac{1}{2\pi}\int\limits_{\mathrm{Re}(z)=\frac{\alpha}{2}} \mathrm{Var}\left[\mathcal{L}\phi(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}(\mathbb{E}[\phi])(z)}{1-\mathcal{L}\mu(z)}\right]|\mathrm{d}z|.$$

For $\mathrm{Re}(z) > 0$

$$\mathrm{Var}\left[\mathcal{L}\phi(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}(\mathbb{E}[\phi])(z)}{1-\mathcal{L}\mu(z)}\right] = \mathbb{E}\Big[\mathrm{Var}\left[\mathcal{L}\phi(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|\zeta\right]\Big]$$
$$+ \mathrm{Var}\left[\mathbb{E}\Big[\mathcal{L}\phi(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|\zeta\Big]\right] =: I + II$$

with

$$I = \mathbb{E}\Big[\mathrm{Var}\left[\mathcal{L}\xi(z)\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|\zeta\right]\Big] = \mathbb{E}\big[\mathrm{Var}\left[\mathcal{L}\xi(z)|\zeta\right]\big]\Big|\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|^2$$
$$= \mathbb{E}\Big[\int_0^\zeta e^{-2\mathrm{Re}(z)x}b\,\mathrm{d}x\Big]\Big|\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|^2$$

and

$$II = \mathrm{Var}\Big[\frac{1-e^{-z\zeta}}{z} + \frac{b(1-e^{-z\zeta})}{z}\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big]$$
$$= \Big|\frac{1}{z(1-\mathcal{L}\mu(z))}\Big|^2\mathrm{Var}[e^{-z\zeta}].$$

Assuming now that $\mathrm{Re}(z)=\frac{\alpha}{2}$ we arrive at

$$I=\Big|\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|^2$$

because

$$\mathbb{E}\Big[\int_0^V e^{-2\mathrm{Re}(z)x}b\,\mathrm{d}x\Big]=\mathbb{E}\Big[\int_0^V e^{-\alpha x}b\,\mathrm{d}x\Big]=1,$$

and

$$II=\Big|\frac{1}{z(1-\mathcal{L}\mu(z))}\Big|^2\big(\mathcal{L}_\zeta(\alpha)-|\mathcal{L}_\zeta(z)|^2\big).$$

Observing that $\overline{z}=\alpha-z$ and $\overline{\mathcal{L}\mu(z)}=\mathcal{L}\mu(\alpha-z)$ whenever $\mathrm{Re}(z)=\frac{\alpha}{2}$ we further infer

$$\begin{aligned}\Big|\frac{z\mathcal{L}\mu(z)}{b}\Big|^2+\mathcal{L}_\zeta(\alpha)-|\mathcal{L}_\zeta(z)|^2&=|1-\mathcal{L}_\zeta(z)|^2+1-\frac{\alpha}{b}-|\mathcal{L}_\zeta(z)|^2\\ =2-2\mathrm{Re}(\mathcal{L}_\zeta(z))-\tfrac{\alpha}{b}&=\big(z(\mathcal{L}\mu(z)-1)+(\alpha-z)(\mathcal{L}\mu(\alpha-z)-1)\big)/b\end{aligned}$$

which in turn gives

$$\begin{aligned}I+II&=\Big|\frac{1}{z(1-\mathcal{L}\mu(z))}\Big|^2\Big(\Big|\frac{z\mathcal{L}\mu(z)}{b}\Big|^2+\mathcal{L}_\zeta(\alpha)-|\mathcal{L}_\zeta(z)|^2\Big)\\&=\frac{z(\mathcal{L}\mu(z)-1)+(\alpha-z)(\mathcal{L}\mu(\alpha-z)-1)}{bz(\mathcal{L}\mu(z)-1)(\alpha-z)(\mathcal{L}\mu(\alpha-z)-1)}\\&=\frac{1}{b(\alpha-z)(\mathcal{L}\mu(\alpha-z)-1)}+\frac{1}{bz(\mathcal{L}\mu(z)-1)}\\&=\mathrm{Re}\Big(\frac{2}{bz(\mathcal{L}\mu(z)-1)}\Big)\end{aligned}$$

We can now compute the variance as follows

$$\sigma^2=\lim_{R\to\infty}\frac{1}{b\pi}\mathrm{Re}\Big(\int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R}\frac{1}{z(\mathcal{L}\mu(z)-1)}\,|\mathrm{d}z|\Big)=\lim_{R\to\infty}\frac{1}{b\pi}\mathrm{Im}\Big(\int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R}\frac{1}{z(\mathcal{L}\mu(z)-1)}\,\mathrm{d}z\Big).$$

To calculate the limit we use the residue theorem, for $R>\alpha$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{Re^{\mathrm{i}\theta}\mathrm{d}\theta}{(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})(\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})-1)}-\int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R}\frac{\mathrm{d}z}{z(\mathcal{L}\mu(z)-1)}=2\pi\mathrm{i}\,\mathrm{Res}_{z=\alpha}\,\frac{1}{z(\mathcal{L}\mu(z)-1)}.$$

It suffices to show that the imaginary part of the integrand of the first integral decays to zero uniformly in θ as R goes to infinity. In view of the inequality $|\mathcal{L}\mu(z)|\le 2b|z|^{-1}$ and its consequence $|\mathcal{L}\mu(z)-1|\ge 1-2b|z|^{-1}$ (both hold true for $\mathrm{Re}(z)\ge 0$) we conclude that

$$\frac{Re^{\mathrm{i}\theta}}{(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})(\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})-1)}=\frac{(\frac{\alpha}{2}+Re^{-\mathrm{i}\theta})Re^{\mathrm{i}\theta}}{\big|(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})(\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})-1)\big|^2}+O(R^{-1}),$$

whence

$$\mathrm{Im}\Big(\frac{Re^{\mathrm{i}\theta}}{(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})(\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})-1)}\Big)=O(R^{-1})$$

as $R \to \infty$ uniformly in θ. Finally,

$$\sigma^2 = -\frac{2}{\alpha b \mathcal{L}\mu'(\alpha)} = \frac{2}{\alpha b \beta}$$

and thereupon

$$e^{-\frac{\alpha}{2}t}\Big(\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}} - e^{\alpha t}\frac{W}{b\beta}\Big) \stackrel{\mathrm{d}}{\to} \sqrt{\frac{2W}{\alpha b \beta^2}}\mathcal{N}.$$

3.5. **The conservative fragmentation model.** In this section we consider the conservative fragmentation model as discussed in [26]. Let $b \geq 2$ be integer and $(V_1, V_2, \ldots, V_b)$ a vector of nonnegative random variables such that $\sum_{j=1}^{b} V_j = 1$ a.s. Starting with an object of mass $x \geq 1$, we break it into pieces with masses $(V_1 x, V_2 x, \ldots, V_b x)$. Continue recursively with each piece of mass ≥ 1, using new and independent copies of the random vector $(V_1, V_2, \ldots, V_b)$ each time. The process terminates a. s. after a finite number of steps, leaving a finite set of fragments of masses < 1.

Denote by $n(x)$ the random number of fragmentation events, i.e., the number of pieces of mass ≥ 1 that appear during the process. Further, let $n_e(x)$ be the final number of fragments, i.e., the number of pieces of mass < 1 that appear. A limit theorem for $n(x)$ has been proved in [26], where it was shown that the asymptotic behavior of $n(x)$ as x goes to infinity depends on the position of the roots of the function $z \mapsto \sum_{j\geq 1} \mathbb{E}[V_j^z]$.

Letting $\xi := \sum_{i\geq 1} \delta_{-\log V_i}$, we conclude that the corresponding Malthusian parameter is 1, i.e., $\alpha = 1$ and the limit of Nerman's martingale satisfies $W = 1$ a. s. Note also that $n(x) = N(\log x)$ corresponds to the number of individuals born up to and including time $\log x$ and similarly, we can represent $n_e(x)$ as a general branching process, namely, $n_e(x) = \mathcal{Z}^{\varphi}_{\log x}$, with $\varphi(t) := \#\{j \in \{1, \ldots, b\} : V_j > 0 \text{ and } -\log|V_j| > t\}$ for $t \geq 0$ and $\varphi(t) := 0$ for $t < 0$. Hence, our main result provides (precise) limit theorems for both n and n_e. For instance, in the case when all root from Λ are simple we get

$$x^{-1/2}\Big(n(x) - \sum_{\lambda\in\Lambda} c_\lambda W(\lambda) x^\lambda\Big) \stackrel{\mathrm{d}}{\to} \sigma\mathcal{N} \quad \text{if } \partial\Lambda \text{ is empty}$$

$$\text{and} \quad x^{-1/2}\big(\log x\big)^{-k+1/2}\Big(n(x) - \sum_{\lambda\in\Lambda} c_\lambda W_\lambda x^\lambda\Big) \stackrel{\mathrm{d}}{\to} \rho_{k-1}\mathcal{N} \quad \text{if } \partial\Lambda \neq \emptyset,$$

where k is the largest multiplicity in $\partial\Lambda$. Moreover, all the constants above can be explicitly computed.

4. Preliminaries for the proofs of the main results

In this section, we gather facts from the literature, introduce some notation used throughout the paper and perform some basic calculations.

4.1. **Change of measure and the connection to renewal theory.** Throughout the present subsection, we suppose that (A1) and (A2) hold. We define a (zero-delayed) random walk $(S_n)_{n\in\mathbb{N}_0}$ on some probability space with underlying probability measure P and increment distribution given by

$$\mathsf{P}(S_1 \in B) = \mathbb{E}\bigg[\sum_{|u|=1} e^{-\alpha S(u)} \mathbb{1}_B(S(u))\bigg], \qquad B \in \mathcal{B}(\mathbb{R}). \tag{4.1}$$

With this definition, the many-to-one formula (see, e.g., [40, Theorem 1.1]) holds:

$$\mathsf{E}[f(S_1, \ldots, S_n)] = \mathbb{E}\bigg[\sum_{|u|=n} e^{-\alpha S(u)} f(S(u|_1), \ldots, S(u))\bigg] \tag{4.2}$$

for all Borel measurable $f : \mathbb{R}^n \to \mathbb{R}$ such that the expectation on the left- or right-hand side of (4.2) is well-defined, possibly infinite. In particular, if (A3) holds, then

$$\mathsf{E}[S_1] = \mathbb{E}\bigg[\sum_{|u|=1} e^{-\alpha S(u)} S(u)\bigg] = -(\mathcal{L}\mu)'(\alpha) = \beta \in (0, \infty). \tag{4.3}$$

In other words, the increments of the random walk $(S_n)_{n\in\mathbb{N}_0}$ have positive, finite mean. As a consequence, the associated renewal measure

$$\mathsf{U}(\cdot) = \sum_{n\in\mathbb{N}_0} \mathsf{P}(S_n \in \cdot)$$

is uniformly locally finite in the sense that

$$\mathsf{U}([t, t+h]) \leq \mathsf{U}([0, h]) < \infty \qquad \text{for all } t, h \geq 0. \tag{4.4}$$

Again by the many-to-one formula, (A3) also guarantees that the increments of the associated random walk $(S_n)_{n\in\mathbb{N}_0}$ have a finite exponential moment of order $\alpha - \vartheta > \alpha/2$ since

$$\mathsf{E}[e^{(\alpha-\vartheta)S_1}] = \mathbb{E}\bigg[\sum_{|u|=1} e^{-\vartheta S(u)}\bigg] = \mathcal{L}\mu(\vartheta) < \infty. \tag{4.5}$$

4.2. **The expectation of the general branching process.** There is a connection between the renewal measure U and the expectation $m_t^\varphi = \mathbb{E}[\mathcal{Z}_t^\varphi]$ of the general branching process counted with characteristic φ provided that φ satisfies suitable assumptions. For instance, if $t \mapsto e^{-\alpha t}\mathbb{E}[\varphi(t)]$ is a directly Riemann integrable function, then we infer from the many-to-one formula

$$\begin{aligned}
e^{-\alpha t} m_t^\varphi := e^{-\alpha t}\mathbb{E}[\mathcal{Z}_t^\varphi] &= \sum_{n=0}^\infty \mathbb{E}\bigg[\sum_{|u|=n} e^{-\alpha S(u)} e^{-\alpha(t-S(u))}\varphi_u(t - S(u))\bigg] \\
&= \sum_{n=0}^\infty \mathbb{E}\bigg[\sum_{|u|=n} e^{-\alpha S(u)} e^{-\alpha(t-S(u))}\mathbb{E}[\varphi](t - S(u))\bigg] \\
&= \sum_{n=0}^\infty \mathsf{E}\big[e^{-\alpha(t-S_n)}\mathbb{E}[\varphi](t - S_n)\big] \\
&= \int e^{-\alpha(t-x)}\mathbb{E}[\varphi](t-x)\,\mathsf{U}(\mathrm{d}x).
\end{aligned} \tag{4.6}$$

By the direct Riemann integrability of $t \mapsto e^{-\alpha t}\mathbb{E}[\varphi](t)$ and (4.4), the function $t \mapsto e^{-\alpha t} m_t^\varphi$ is bounded and, moreover,

$$\lim_{\substack{t\to\infty,\\ t\in\mathbb{G}}} e^{-\alpha t} m_t^\varphi = \frac{1}{\beta}\int e^{-\alpha x}\mathbb{E}[\varphi](x)\,\ell(\mathrm{d}x) =: \frac{1}{\beta}(\mathcal{L}\mathbb{E}[\varphi])(\alpha) \tag{4.7}$$

by the key renewal theorem, see [3, Theorem 4.2] in the non-lattice case and [1, Theorem 2.5.3] in the lattice (and non-lattice) case. Here, $\mathcal{L}\mathbb{E}[\varphi]$ denotes the 'discrete' bilateral Laplace transform of $\mathbb{E}[\varphi]$ in the lattice case and the 'continuous' bilateral Laplace transform in the non-lattice case.

4.3. **Stopping times, optional lines and filtrations.** Regarding optional lines and filtrations, we follow [22, Sections 2 and 4]. For a subset $L \subseteq \mathcal{I}$ and $v \in \mathcal{I}$, we write $L \preceq v$ if $u \preceq v$ for some $u \in L$ and say that v stems from L. If $M \subseteq \mathcal{I}$, then we write $L \preceq M$ if $L \preceq v$ for every $v \in M$, i.e., every $v \in M$ stems from a $u \in L$. We define the pre-L-σ-algebra $\mathcal{F}_L$ via

$$\mathcal{F}_L = \sigma(\pi_u : u \not\succeq L).$$

Our true interest lies in the pre-$\mathcal{J}$-σ-algebra $\mathcal{F}_\mathcal{J}$ if $\mathcal{J}$ is an optional line. A line L is a random subset of $\mathcal{I}$ such that for all $u, v \in L$, it holds that $u \preceq v$ implies $u = v$, i.e., no individual stems from another. An optional line $\mathcal{J}$ is a random line such that $\{\mathcal{J} \preceq L\} \in \mathcal{F}_L$ for every line $L \subseteq \mathcal{I}$. The significance of optional lines stems from Jagers' strong Markov branching property [22, Theorem 4.14], namely,

$$\mathbb{E}\bigg[\prod_{u\in\mathcal{J}} \chi_u \circ \theta_u \,\bigg|\, \mathcal{F}_\mathcal{J}\bigg] = \prod_{u\in\mathcal{J}} \mathbb{E}[\chi_u] \quad \text{a.s.} \tag{4.8}$$

for every optional line $\mathcal{J}$ and all measurable $\chi_u : \Omega \to [0, 1]$.

If τ is a stopping time with respect to the canonical filtration of $(S_n)_{n\in\mathbb{N}_0}$, then it can be written in the form

$$\tau = \inf\{n \in \mathbb{N}_0 : (S_0, S_1, \ldots, S_n) \in B_n\}$$

for a sequence $B_n \in \mathcal{B}(\mathbb{R}^{n+1})$ of Borel sets. In this case, we may associate an optional line with τ, namely,

$$\mathcal{J} := \{v \in \mathcal{I} : (0, S(v|_1), \ldots, S(v)) \in B_{|v|} \text{ but } (0, S(u|_1), \ldots, S(u)) \notin B_{|u|} \text{ for } u \prec v\}.$$

It is not difficult to check that $\mathcal{J}$ is indeed optional as defined above.

It can further be checked that the many-to-one formula (4.2) carries over from S_n to S_τ as follows:

$$\mathsf{E}[f(S_\tau)] = \mathbb{E}\bigg[\sum_{u\in\mathcal{J}} e^{-\alpha S(u)} f(S(u))\bigg] \tag{4.9}$$

for every Borel measurable $f : \mathbb{R} \to \mathbb{R}$ such that at least one expectation (and hence also the other) is well-defined.

The most important special case for us is that of the optional line associated with the first-passage time $\tau_t := \inf\{n \in \mathbb{N}_0 : S_n > t\}$ of the associated random walk $(S_n)_{n\in\mathbb{N}_0}$. The resulting optional line is $\mathcal{C}_t = \{uj \in \mathcal{T} : S(u) \leq t < S(uj)\}$, the coming generation at time t.

4.4. **Matrix notation.** For any $s \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ we define the following lower triangular $k \times k$ matrix

$$\exp(\gamma, s, k) := e^{\gamma s} \times \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ s & 1 & 0 & \ldots & 0 \\ s^2 & 2s & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s^{k-1} & \binom{k-1}{1} s^{k-2} & \binom{k-1}{2} s^{k-3} & \ldots & 1 \end{pmatrix}. \tag{4.10}$$

The (i,j)th entry of the matrix is $e^{\gamma s}\binom{i-1}{j-1}s^{i-j}$, $i,j=1,\dots,k$. Matrices of this form will be very useful since they simplify the notation and allow us to deal with polynomial terms with relative ease. Elementary algebra gives

$$\exp(\gamma,s,k)\cdot\exp(\gamma,t,k)=\exp(\gamma,s+t,k).$$

With $\|\cdot\|$ denoting the operator norm and $\|\cdot\|_{\mathsf{HS}}$ denoting the Hilbert-Schmidt norm, the following (crude) bound holds for every $\delta>0$:

$$\|\exp(\gamma,s,k)\|\leq\|\exp(\gamma,s,k)\|_{\mathsf{HS}}\leq C'e^{\mathrm{Re}(\gamma)s}(1+|s|)^{k-1}\leq Ce^{\mathrm{Re}(\gamma)s+\delta|s|} \tag{4.11}$$

for some constant $C'>0$ depending on k only and another constant $C>0$ depending on k and $\delta>0$. We write $\mathsf{e}_1,\mathsf{e}_2,\dots$ for the canonical base vectors in Euclidean space. Here, for ease of notation, we are slightly sloppy as we do not specify the dimension of that space (formally, all Euclidean spaces may be embedded into an appropriate infinite-dimensional space such as ℓ^2). Then, for instance,

$$\exp(\gamma,s,k)\cdot\mathsf{e}_1=e^{\gamma s}\begin{pmatrix}1\\ s\\ s^2\\ \vdots\\ s^{k-1}\end{pmatrix}.$$

Throughout the paper, for $\mathrm{Re}(\lambda)\geq\vartheta$, $n\in\mathbb{N}_0$ and $k\in\mathbb{N}$, we denote by $Z_n(\lambda,k)$ the following random matrix

$$Z_n(\lambda,k):=\sum_{|u|=n}\exp(\lambda,-S(u),k). \tag{4.12}$$

We set $Z_n(\lambda):=Z_n(\lambda,1)$. In particular, $\mu(\theta)=\mathbb{E}[Z_1(\theta)]$ and hence (A3) reads $\mathbb{E}[Z_1(\vartheta)]<\infty$.

4.5. **Nerman's martingales.** The next examples are crucial for the paper as they constitute the building blocks for the asymptotic expansion of $\mathcal{Z}^{\varphi}$.

Example 4.1 (Nerman martingales with complex parameters)**.** Let $\lambda\in\mathbb{C}$ be a root of multiplicity $k=k(\lambda)\in\mathbb{N}$ of the mapping $z\mapsto\mathcal{L}\mu(z)-1$, i.e.,

$$\mathcal{L}\mu(\lambda)=\mathbb{E}\Big[\sum_{|u|=1}e^{-\lambda S(u)}\Big]=1, \tag{4.13}$$

$$(-1)^j(\mathcal{L}\mu)^{(j)}(\lambda)=\mathbb{E}\Big[\sum_{|u|=1}S(u)^je^{-\lambda S(u)}\Big]=0\quad\text{for } j=1,\dots,k(\lambda)-1, \tag{4.14}$$

$$(\mathcal{L}\mu)^{(k(\lambda))}(\lambda)\neq 0. \tag{4.15}$$

Conditions (4.13) and (4.14) are equivalent to

$$\mathbb{E}[Z_1(\lambda,k)]=\mathbb{E}\Big[\sum_{|u|=1}\exp(\lambda,-S(u),k)\Big]=I_k \tag{4.16}$$

where I_k is the $k\times k$ identity matrix. We define a matrix-valued characteristic

$$\phi_\lambda(t):=\sum_{j=1}^N\mathbb{1}_{[0,X_j)}(t)\exp(\lambda,t-X_j,k). \tag{4.17}$$

By definition, characteristics take values in $\mathbb{R}^d$ for some $d \in \mathbb{N}$, but here we use an obvious extension to $\mathbb{C}$ by splitting into real and imaginary part. Then the following process

$$W_t(\lambda,k) := \exp(\lambda,-t,k)\cdot \mathcal{Z}_t^{\phi_\lambda} = \sum_{u\in\mathcal{C}_t} \exp(\lambda,-S(u),k), \qquad t\in\mathbb{R}$$

is a (matrix-valued) martingale with respect to $(\mathcal{F}_{\mathcal{C}_t})_{t\in\mathbb{R}}$. Then $W_t(\lambda,k)$ has the following form

$$W_t(\lambda,k) = \begin{pmatrix} W_t^{(0)}(\lambda) & 0 & 0 & \dots & 0 \\ W_t^{(1)}(\lambda) & W_t^{(0)}(\lambda) & 0 & \dots & 0 \\ W_t^{(2)}(\lambda) & 2W_t^{(1)}(\lambda) & W_t^{(0)}(\lambda) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_t^{(k-1)}(\lambda) & \binom{k-1}{1}W_t^{(k-2)}(\lambda) & \binom{k-1}{2}W_t^{(k-3)}(\lambda) & \dots & W_t^{(0)}(\lambda) \end{pmatrix}, \tag{4.18}$$

where $W_t^{(j)}(\lambda) = \sum_{u\in\mathcal{C}_t}(-S(u))^j e^{-\lambda S(u)}$ as in (2.15).

Example 4.2 (Limits of Nerman's martingales at complex parameters)**.** Suppose now that the martingale $(W_t(\lambda,k))_{t\in\mathbb{R}}$ is uniformly integrable. Then it converges in L^1 as $t\to\infty$ to some random matrix $W(\lambda,k)$ of the form

$$W(\lambda,k) = \begin{pmatrix} W^{(0)}(\lambda) & 0 & 0 & \dots & 0 \\ W^{(1)}(\lambda) & W^{(0)}(\lambda) & 0 & \dots & 0 \\ W^{(2)}(\lambda) & 2W^{(1)}(\lambda) & W^{(0)}(\lambda) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W^{(k-1)}(\lambda) & \binom{k-1}{1}W^{(k-2)}(\lambda) & \binom{k-1}{2}W^{(k-3)}(\lambda) & \dots & W^{(0)}(\lambda) \end{pmatrix}. \tag{4.19}$$

By uniform integrability, $\mathbb{E}[W(\lambda,k)] = I_k$. Further, the random matrix $W(\lambda,k)$ satisfies the following fixed-point equation:

$$W(\lambda,k) = \sum_{|u|=1} \exp(\lambda,-S(u),k)\cdot(W(\lambda,k)\circ\theta_u) \qquad \text{a.s.} \tag{4.20}$$

The sum above can be taken over an optional line $\mathcal{C}_t$ for $t\geq 0$, i.e.,

$$W(\lambda,k) = \sum_{u\in\mathcal{C}_t} \exp(\lambda,-S(u),k)\cdot(W(\lambda,k)\circ\theta_u) \qquad \text{a.s.} \tag{4.21}$$

Equation (4.21) can be rewritten in the following form

$$\begin{aligned} W(\lambda,k) &= \sum_{u\in\mathcal{I}} \mathbb{1}_{[0,\infty)}(t-S(u)) \sum_{j=1}^{N_u} \mathbb{1}_{\{S(uj)>t\}} \exp(\lambda,-S(uj),k)\cdot(W(\lambda,k)\circ\theta_{uj}) \\ &= \exp(\lambda,-t,k)\cdot \sum_{u\in\mathcal{I}} \phi_{\lambda,u}^{(\infty)}(t-S(u)), \end{aligned}$$

with

$$\phi_{\lambda,u}^{(\infty)}(t) := \sum_{j=1}^{N_u} \mathbb{1}_{[0,X_{u,j})}(t)\exp(\lambda,t-X_{u,j},k)(W(\lambda,k)\circ\theta_u).$$

In particular,

$$\mathcal{Z}_t^{\phi_\lambda^{(\infty)}} = \exp(\lambda, t, k) \cdot W(\lambda, k) \tag{4.22}$$

and hence for this characteristic $\phi_\lambda^{(\infty)}$ we know the asymptotic behavior of $\mathcal{Z}_t^{\phi_\lambda^{(\infty)}}$ as $t \to \infty$.

Note that $\phi_{\lambda,u}^{(\infty)}$ and $\phi_{\lambda,v}^{(\infty)}$ in the above example are not functions of the life spaces of the individuals u and v, but of the population spaces with ancestor u and v, respectively. In particular, $\phi_{\lambda,u}^{(\infty)}$ and $\phi_{\lambda,v}^{(\infty)}$ are not independent if $u \preceq v$ and the limit behavior of general branching processes with random characteristics of this type is beyond the scope of the present paper. To overcome this problem, we introduce the following characteristics.

Example 4.3. Let

$$\chi_\lambda(t) := \mathbb{1}_{(-\infty,0)}(t) \exp(\lambda, t, k)(Z_1(\lambda, k) - I_k), \quad t \in \mathbb{R}. \tag{4.23}$$

Note that, for $\phi_\lambda^{(\infty)}$ from Example 4.2, ϕ_λ defined in formula (4.17) and $t \geq 0$, we have

$$\begin{aligned}
&\exp(\lambda, t, k) \cdot \big(W(\lambda, k) - W_t(\lambda, k)\big) = \mathcal{Z}_t^{\phi_\lambda^{(\infty)}} - \mathcal{Z}_t^{\phi_\lambda} \\
&= \sum_{u \in \mathcal{I}} \big(\phi_{\lambda,u}^{(\infty)}(t - S(u)) - \phi_{\lambda,u}(t - S(u))\big) \\
&= \sum_{u \in \mathcal{I}} \mathbb{1}_{\{S(u) \leq t\}} \sum_{j=1}^{N_u} \mathbb{1}_{\{S(uj) > t\}} \exp(\lambda, t - S(uj), k)\big(W(\lambda, k) \circ \theta_u - I_k\big) \\
&= \sum_{u \in \mathcal{I}} \mathbb{1}_{\{S(u) \leq t\}} \sum_{j=1}^{N_u} \mathbb{1}_{\{S(uj) > t\}} \exp(\lambda, t - S(uj), k) \\
&\qquad\qquad \cdot \sum_{w \in \mathcal{I}} \exp(\lambda, S(uj) - S(ujw), k)\big(Z_1(\lambda, k) \circ \theta_{ujw} - I_k\big) \\
&= \sum_{w \in \mathcal{I}} \exp(\lambda, t - S(w), k) \mathbb{1}_{\{S(w) > t\}}\big(Z_1(\lambda, k) \circ \theta_w - I_k\big) \\
&= \sum_{u \in \mathcal{I}} \chi_{\lambda,u}(t - S(u)) = \mathcal{Z}_t^{\chi_\lambda}.
\end{aligned}$$

In particular, for $t \geq 0$,

$$\mathcal{Z}_t^{\phi_\lambda + \chi_\lambda} = \exp(\lambda, t, k) \cdot W(\lambda, k). \tag{4.24}$$

Moreover, both ϕ_λ and χ_λ are $\sigma(\xi)$–measurable and, in particular, the tuples $(\xi_u, \phi_{\lambda,u}, \chi_{\lambda,u})$, $u \in \mathcal{I}$ are i. i. d.

5. Proofs of the main results

We investigate the asymptotic behavior of the general branching process $\mathcal{Z}_t^\varphi$ counted with characteristic φ as $t \to \infty$ in several steps. In the first step, we prove convergence of Nerman's martingales at complex parameters.

5.1. **Convergence of Nerman's martingales.** The following lemma implies Theorem 2.3.

Lemma 5.1. *Suppose that (A1) through (A4) hold and let λ be a solution to (2.14) with multiplicity k and $\mathrm{Re}(\lambda) > \alpha/2$. Then, for any $x, y \in \mathbb{R}^k$, the process $\langle x, W_t(\lambda, k)y\rangle$ is an L^2-bounded martingale.*

In particular, for every $0 \leq j \leq k-1$, the martingale $(W_t^{(j)}(\lambda))_{t\geq 0}$ converges a. s. and in L^2.

Proof. Obviously $(\langle x, W_t(\lambda, k)y\rangle)_{t\geq 0}$ is a martingale. It suffices to show the boundedness in L^2. Without loss of generality we assume that $|x|, |y| \leq 1$. Define the random matrix

$$Y_u := Z_{u,1}(\lambda, k) - I_k = \int \exp(\lambda, -x, k)\, \xi_u(\mathrm{d}x) - I_k.$$

Notice that $\mathbb{E}[Y_u]$ is the $k \times k$ zero matrix by (4.16). By the penultimate inequality in (4.11) and (A4), we have $\mathbb{E}[|Y_u y|^2] \leq C < \infty$ for some constant $C > 0$ that depends only on λ and k. For $0 < \delta < \mathrm{Re}(\lambda) - \alpha/2$ we can write

$$\begin{aligned}
\mathbb{E}\Big[\big|\big\langle x, W_t(\lambda,k)y\big\rangle - \big\langle x,y\big\rangle\big|^2\Big] &= \mathbb{E}\Bigg[\bigg|\sum_{u\in\mathcal{I}} \mathbb{1}_{\{u\in\mathcal{C}_t\}}\big\langle x, \exp(\lambda, -S(u), k)y\big\rangle - \big\langle x, y\big\rangle\bigg|^2\Bigg] \\
&= \mathbb{E}\Bigg[\bigg|\sum_{u\in\mathcal{I}} \mathbb{1}_{\{u\prec\mathcal{C}_t\}}\big\langle x, \exp(\lambda, -S(u), k)Y_u y\big\rangle\bigg|^2\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{u\in\mathcal{I}} \mathbb{1}_{\{u\prec\mathcal{C}_t\}}\big|\big\langle x, \exp(\lambda, -S(u), k)Y_u y\big\rangle\big|^2\Bigg] \\
&\leq \mathbb{E}\Bigg[\sum_{u\in\mathcal{I}} \|\exp(\lambda, -S(u), k)\|_{\mathsf{HS}}^2\, \mathbb{E}\big[|Y_u y|^2\big]\Bigg] \\
&\leq C\mathbb{E}\Bigg[\sum_{u\in\mathcal{I}} e^{(-2\mathrm{Re}(\lambda)+2\delta)S(u)}\Bigg] \\
&= C/(1-\mathcal{L}\mu(2\mathrm{Re}(\lambda)-2\delta)) < \infty,
\end{aligned}$$

where we have used (4.11) for the second inequality. This shows the boundedness in L^2. □

Lemma 5.2. *Suppose (A4) holds. Let $\lambda \in \Lambda$ be a solution to (2.14) with multiplicity k and $x, y \in \mathbb{R}^k$. Then both characteristics $\langle x, \mathrm{Re}(\phi_\lambda)y\rangle$ and $\langle x, \mathrm{Re}(\chi_\lambda)y\rangle$ fulfill (A6) and (A8).*

Proof. Clearly, both characteristics have càdlàg paths. Without loss of generality, we may assume that $|x|, |y| \leq 1$. From (4.11) and the assumption that $\mathrm{Re}(\lambda) > \frac{\alpha}{2}$ we infer the existence of a constant C that depends only on λ and k such that, for $t \leq 0$,

$$\big|\big\langle x, \mathrm{Re}(\exp(\lambda, t, k))y\big\rangle\big| \leq \big|\big\langle x, \exp(\lambda, t, k)y\big\rangle\big| \leq \|\exp(\lambda, t, k)\|_{\mathsf{HS}} \leq Ce^{\frac{\alpha}{2}t}/h(t)$$

where the function h is the one from (A4). Then, taking into account that h satisfies (2.13) and, thus, $h(t) \leq 2h(t-x)h(x)$ for all $t, x \geq 0$, we can write

$$\begin{aligned}\big|\langle x, \mathrm{Re}(\phi_\lambda(t))y\rangle\big| &\leq \sum_{j=1}^{N} \mathbb{1}_{[0,X_j)}(t)\big|\langle x, \mathrm{Re}(\exp(\lambda, t-X_j, k))y\rangle\big| \\ &\leq \sum_{j=1}^{N} \mathbb{1}_{[0,X_j)}(t) C e^{\frac{\alpha}{2}(t-X_j)}/h(t-X_j) \\ &\leq 2C\mathbb{1}_{[0,\infty)}(t)\frac{e^{\frac{\alpha}{2}t}}{h(t)}\sum_{j=1}^{N} h(X_j)e^{-\frac{\alpha}{2}X_j}.\end{aligned}$$

Since $\mathbb{1}_{[0,\infty)}(t)h(t)^{-2}$ is monotone and integrable with respect to Lebesgue measure, this implies the claim regarding the first characteristic (see also Remark 2.2). A similar argument applies to the second characteristic:

$$\begin{aligned}\big|\langle x, \chi_\lambda(t)y\rangle\big| &= \mathbb{1}_{(-\infty,0)}(t)\big|\langle x, \exp(\lambda, t, k)(Z_1(\lambda, k) - I_k)y\rangle\big| \\ &\leq \mathbb{1}_{(-\infty,0)}(t)\|\exp(\lambda, t, k)\|\|Z_1(\lambda, k) - I_k\| \\ &= C^2\mathbb{1}_{(-\infty,0)}(t)\frac{e^{\frac{\alpha}{2}t}}{h(t)}\bigg(1 + \sum_{j=1}^{N} e^{-\frac{\alpha}{2}X_j}\bigg),\end{aligned}$$

which completes the proof. □

5.2. **Centered characteristics.** In this section we study the fluctuations of $\mathcal{Z}_t^\chi$ as $t \to \infty$ for centered characteristics, that is, for characteristics χ satisfying $\mathbb{E}[\chi(t)] = 0$ for all $t \in \mathbb{R}$. Theorem 5.5 below plays a key role in the proof of our main result Theorem 2.8. Before we state it, we give a preparatory lemma.

Lemma 5.3. *Suppose that (A1) through (A4) hold. Let $\theta \geq 0$ and $f : \mathbb{R} \to [0, \infty)$ be a uniformly continuous function with $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| < \infty$ such that the limit*

$$\lim_{t\to\infty} \frac{1}{t^{\theta+1}} \int_0^t x^\theta f(x)\, \ell(\mathrm{d}x) =: c \in [0, \infty) \tag{5.1}$$

exists. Then, for $\varphi(t) := t^\theta e^{\alpha t} f(t)\mathbb{1}_{[0,\infty)}(t)$, the function $t \mapsto e^{-\alpha t}t^{-\theta-1}\mathbb{E}[\mathcal{Z}_t^\varphi]$ is bounded and

$$\frac{e^{-\alpha t}}{t^{\theta+1}}\mathcal{Z}_t^\varphi \to \frac{cW}{\beta} \quad \text{as } t \to \infty \text{ almost surely.} \tag{5.2}$$

Proof. By assumption $|f(t)| \leq \|f\|_\infty < \infty$ for all $t \in \mathbb{R}$. From [34, Corollary 3.3] in the non-lattice case and [14, Corollary 5.1] in the lattice case, we infer that $e^{-\alpha t}N((t-1, t])) = e^{-\alpha t}\mathcal{Z}_t^{\mathbb{1}_{[0,1)}}$ converges in L^1 as $t \to \infty$, $t \in \mathbb{G}$, so

$$e^{-\alpha t}t^{-\theta-1}\mathbb{E}[\mathcal{Z}_t^\varphi] \leq \|f\|_\infty\bigg(\frac{\mathbb{E}[N(\{0\})]}{t} + \frac{e^{-\alpha t}}{t^{\theta+1}}\sum_{n=0}^{\lfloor t\rfloor}(t-n)^\theta e^{\alpha(t-n)}\mathbb{E}[N((n, n+1])]\bigg)$$

is bounded. It remains to show (5.2). First notice that for any fixed $r > 0$

$$e^{-\alpha t}t^{-\theta-1}\sum_{\substack{u\in\mathcal{I}:\\ S(u)\leq r}}(t-S(u))^\theta e^{\alpha(t-S(u))}f(t-S(u)) \leq t^{-1}\cdot\|f\|_\infty\cdot N([0, r]) \to 0$$

almost surely as $t \to \infty$. Hence, almost surely, the limiting behavior of $e^{-\alpha t} t^{-\theta-1} \mathcal{Z}_t^{\varphi}$ as $t \to \infty$, $t \in \mathbb{G}$, is the same as that of

$$e^{-\alpha t} t^{-\theta-1} \sum_{\substack{u \in \mathcal{I}:\\ r < S(u) \leq t}} (t - S(u))^{\theta} e^{\alpha(t - S(u))} f(t - S(u)). \tag{5.3}$$

Now first consider the lattice case and notice that by [31, Corollary 3.1(b)] for given $\varepsilon > 0$ we may choose $r \in \mathbb{N}$ so large that

$$(1-\varepsilon) e^{\alpha k} \frac{W}{\beta} \leq N(\{k\}) \leq (1+\varepsilon) e^{\alpha k} \frac{W}{\beta}$$

for all $k \in \mathbb{N}$, $k \geq r$. Then, for $t \in \mathbb{N}$ with $t > r$,

$$\begin{aligned} e^{-\alpha t} t^{-\theta-1} &\sum_{\substack{u \in \mathcal{I}:\\ r \leq S(u) \leq t}} (t - S(u))^{\theta} e^{\alpha(t - S(u))} f(t - S(u)) \\ &= t^{-\theta-1} \sum_{k=r}^{t} (t-k)^{\theta} f(t-k) e^{-\alpha k} N(\{k\}) \\ &\leq (1+\varepsilon) \frac{W}{\beta} \frac{1}{t^{\theta+1}} \sum_{k=0}^{t-r} k^{\theta} f(k) \to (1+\varepsilon) \frac{cW}{\beta} \quad \text{as } t \to \infty \end{aligned}$$

by (5.1). The corresponding lower bound can be obtained analogously. Now (5.2) follows by letting $\varepsilon \to 0$.

Next, we turn to the non-lattice case and fix small $\varepsilon, \delta > 0$. By [31, Corollary 3.1(a)], we may choose $r \in \delta\mathbb{N}$ so large that

$$(1-\varepsilon) e^{\alpha t} \frac{e^{\alpha\delta} - 1}{\alpha} \frac{W}{\beta} \leq N((t, t+\delta]) \leq (1+\varepsilon) e^{\alpha t} \frac{e^{\alpha\delta} - 1}{\alpha} \frac{W}{\beta}$$

for all $t \geq r - \delta$ almost surely. For $t \geq r$, define $I_k^{\delta} := [k\delta, (k+1)\delta)$ for $k = 0, \ldots, t_{\delta} - 1$ where $t_{\delta} := \lfloor \frac{t-r}{\delta} \rfloor$, and $I_{t_{\delta}}^{\delta} := [t_{\delta}\delta, t - r)$. Notice that $t - t_{\delta}\delta \geq r$. Hence, almost surely,

$$\begin{aligned} \limsup_{t \to \infty} \frac{e^{-\alpha t}}{t^{\theta+1}} &\sum_{\substack{u \in \mathcal{I}:\\ r < S(u) \leq t}} (t - S(u))^{\theta} e^{\alpha(t - S(u))} f(t - S(u)) \\ &\leq \limsup_{t \to \infty} \frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{k=0}^{t_{\delta}} \sum_{\substack{u \in \mathcal{I}:\\ t - S(u) \in I_k^{\delta}}} (t - S(u))^{\theta} e^{\alpha k\delta} f(t - S(u)) \\ &\leq \limsup_{t \to \infty} \frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{k=0}^{t_{\delta}} e^{\alpha k\delta} N(t - I_k^{\delta}) \sup_{x \in I_k^{\delta}} \big(x^{\theta} f(x)\big) \\ &\leq (1+\varepsilon) \frac{e^{\alpha\delta} - 1}{\alpha\delta} \frac{W}{\beta} e^{-\alpha\delta} \limsup_{t \to \infty} \frac{1}{t^{\theta+1}} \sum_{k=0}^{t_{\delta}} \delta \sup_{x \in I_k^{\delta}} \big(x^{\theta} f(x)\big). \end{aligned} \tag{5.4}$$

Write $w_f(\delta) := \sup_{|x-y| \leq \delta} |f(x) - f(y)|$ for the modulus of continuity of f. Since f is uniformly continuous by assumption, $w_f(\delta) \to 0$ as $\delta \to 0$. We now estimate the lim sup in (5.4):

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{\theta+1}} \sum_{k=0}^{t_\delta} \delta \sup_{x\in I_k^\delta} \big(x^\theta f(x)\big) \\
&\le \limsup_{t\to\infty} \Bigg(\frac{1}{t^{\theta+1}} \int_0^{t-r} x^\theta f(x)\,\mathrm{d}x + \frac{1}{t^{\theta+1}} \int_0^{t-r} (x+\delta)^\theta (f(x)+w_f(\delta)) - x^\theta f(x)\,\mathrm{d}x \Bigg) \\
&= c + \limsup_{t\to\infty} \Bigg(\frac{\|f\|_\infty}{t^{\theta+1}} \int_0^{t-r} \big((x+\delta)^\theta - x^\theta\big)\,\mathrm{d}x + \frac{w_f(\delta)}{t^{\theta+1}} \int_0^{t-r} (x+\delta)^\theta \mathrm{d}x \Bigg) \\
&\le c + \frac{w_f(\delta)}{\theta+1}.
\end{aligned}$$

Using this in (5.4) gives

$$\begin{aligned}
\limsup_{t\to\infty} \frac{e^{-\alpha t}}{t^{\theta+1}} &\sum_{\substack{u\in\mathcal{I}:\\ r\le S(u)\le t}} (t-S(u))^\theta e^{\alpha(t-S(u))} f(t-S(u)) \\
&\le (1+\varepsilon) \frac{e^{\alpha\delta}-1}{\alpha\delta} \frac{W}{\beta} e^{-\alpha\delta} \Big(c + \frac{w_f(\delta)}{\theta+1} \Big)
\end{aligned}$$

almost surely. Letting $\varepsilon, \delta \to 0$ yields the upper bound of (5.2). The lower bound can be obtained analogously. □

Remark 5.4. Notice that in the proof of Lemma 5.3 we actually do not use the full power of assumptions (A3) and (A4). Indeed, we only need the assumptions regarding ξ that allow us to apply [34, Corollary 3.3] in the non-lattice case and [14, Corollary 5.1] in the lattice case, respectively.

The following theorem gives the central limit theorem in the case of a centered characteristic χ.

Theorem 5.5. *Let χ be a real-valued, centered characteristic, and let $\mathcal{N}$ be a standard normal random variable independent of W.*

(i) Suppose that (A6) holds. Then

$$e^{-\frac{\alpha}{2}t} \mathcal{Z}_t^\chi \xrightarrow{\mathrm{d}} \Big(\frac{W}{\beta} \int \mathbb{E}[\chi^2](x) e^{-\alpha x}\,\ell(\mathrm{d}x) \Big)^{1/2} \mathcal{N} \quad \text{as } t\to\infty,\ t\in\mathbb{G}. \tag{5.5}$$

(ii) Suppose that there are $\theta \ge 0$ and a function f satisfying the conditions of Lemma 5.3, $\mathbb{E}[\chi^2(t)] = t^\theta e^{\alpha t} f(t) \mathbb{1}_{[0,\infty)}(t)$ and

$$\mathbb{E}\big[\chi^2(t) \mathbb{1}_{\{\chi^2(t) > \varepsilon e^{\alpha t} t^{\theta+1}\}}\big] = o(t^\theta e^{\alpha t}) \quad \text{as } t\to\infty \tag{5.6}$$

for every $\varepsilon > 0$. Then

$$\Big(e^{\alpha t} \int_0^t \mathbb{E}[\chi^2(x)] e^{-\alpha x}\,\ell(\mathrm{d}x) \Big)^{-1/2} \mathcal{Z}_t^\chi \xrightarrow{\mathrm{d}} \big(\tfrac{W}{\beta}\big)^{1/2} \mathcal{N} \text{ as } t\to\infty,\ t\in\mathbb{G}. \tag{5.7}$$

Moreover, the convergences in (5.5) and (5.7) are stable.

Proof of Theorem 5.5. Consider an increasing sequence $(\mathcal{I}_n)_{n\in\mathbb{N}_0}$ of subsets of $\mathcal{I}$ with the following property: for any $n \in \mathbb{N}_0$, $|\mathcal{I}_n| = n$ and if $v \in \mathcal{I}_n$, then $u \in \mathcal{I}_n$ for all $u \preceq v$. By v_n we denote the unique vertex in $\mathcal{I}_n \setminus \mathcal{I}_{n-1}$. It is not hard to see that such a sequence exists. Indeed, we can construct $(v_n)_{n\in\mathbb{N}}$ recursively. First, let $v_1 = \varnothing$. If we have constructed v_i for $i = 1, \ldots, 2^k$ where $k \in \mathbb{N}_0$, then, for any $2^k < i \leq 2^{k+1}$, we set $v_i := v_{i-2^k}j$ with the smallest $j \in \mathbb{N}$ such that $v_{i-2^k}j \notin \{v_1, \ldots, v_{2^k}\}$. Now we set

$$a_t := \begin{cases} \frac{1}{\beta} \int \mathbb{E}[\chi^2](x) e^{-\alpha x}\, \ell(\mathrm{d}x) \cdot e^{\alpha t} & \text{in case (i)}, \\ \frac{1}{\beta} \int_0^t \mathbb{E}[\chi^2](x) e^{-\alpha x}\, \ell(\mathrm{d}x) \cdot e^{\alpha t} & \text{in case (ii)} \end{cases}$$

for all $t \in \mathbb{G}$. If $\|\chi\|_{L^2(\mathrm{d}\mathbb{P}\otimes e^{-\alpha t}\ell(\mathrm{d}t))} = \int \mathbb{E}[\chi^2(x)] e^{-\alpha x}\, \ell(\mathrm{d}x) = 0$ in case (i) or if $\int_0^t \mathbb{E}[\chi^2(x)] e^{-\alpha x}\, \ell(\mathrm{d}x) = 0$ for all $t \geq 0$, $t \in \mathbb{G}$, then the assertion is trivial. Hence, we exclude these cases and may thus assume that $a_t > 0$ for all sufficiently large $t \in \mathbb{G}$. For those t, we define

$$M_n(t) := a_t^{-1/2} \sum_{u\in\mathcal{I}_n} \chi_u(t - S(u)).$$

Then $(M_n(t))_{n\in\mathbb{N}_0}$ is a martingale with respect to $(\mathcal{F}_{\mathcal{I}_n})_{n\in\mathbb{N}_0}$. Indeed, for any $u \in \mathcal{I}_n$ both $S(u)$ and χ_u are $\mathcal{F}_{\mathcal{I}_n}$–measurable and hence so is M_n. The martingale property then follows since $S(v_{n+1})$ is $\mathcal{F}_{\mathcal{I}_n}$–measurable whereas $\chi_{v_{n+1}}$ is independent of $\mathcal{F}_{\mathcal{I}_n}$, hence

$$\mathbb{E}[\chi_{v_{n+1}}(t - S(u)) | \mathcal{F}_{\mathcal{I}_n}] = \mathbb{E}[\chi](t - S(u)) = 0 \quad \text{almost surely}$$

since $\mathbb{E}[\chi](x) = \mathbb{E}[\chi(x)] = 0$ for all $x \in \mathbb{R}$.

Next, we observe that

$$\begin{aligned} \mathbb{E}[M_n^2(t)] &= a_t^{-1} \mathbb{E}\Big[\sum_{u\in\mathcal{I}_n} \chi_u^2(t - S(u))\Big] = a_t^{-1} \mathbb{E}\Big[\sum_{u\in\mathcal{I}_n} \mathbb{E}[\chi^2](t - S(u))\Big] \\ &\leq a_t^{-1} \mathbb{E}\Big[\sum_{u\in\mathcal{I}} \mathbb{E}[\chi^2](t - S(u))\Big] = a_t^{-1} \mathbb{E}[\mathcal{Z}_t^{\mathbb{E}[\chi^2]}] \\ &= a_t^{-1} e^{\alpha t} \int e^{-\alpha(t-x)} \mathbb{E}[\chi^2](t - x)\, \mathsf{U}(\mathrm{d}x) \leq C \end{aligned} \tag{5.8}$$

for some $C \geq 0$ independent of n and t by (4.6) where U is the renewal measure of the associated random walk $(S_n)_{n\in\mathbb{N}_0}$. Indeed, this follows from the uniform local finiteness of U (see (4.4)) and the direct Riemann integrability of $e^{-\alpha t}\,\mathrm{Var}[\chi(t)] = e^{-\alpha t}\mathbb{E}[\chi^2](t)$ in case (i) and Lemma 5.3 in case (ii). In particular, $M_n(t)$ converges almost surely and in L^2 to $M(t) = a_t^{-1/2} \mathcal{Z}_t^{\chi}$.

Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{G}$ that diverges to infinity. Then there exists an increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}[(M(t_n) - M_{k_n}(t_n))^2] \leq 2^{-n}$ for every $n \in \mathbb{N}$ and, therefore, $M(t_n) - M_{k_n}(t_n)$ converges to 0 almost surely as $n \to \infty$. In view of Slutsky's theorem [38, Theorem 8.6.1] and the martingale central limit theorem [16, Corollary 3.1 on p. 58] it suffices to verify that

$$a_{t_n}^{-1}\sum_{j=1}^{k_n}\mathbb{E}\Big[\chi_{v_j}^2(t_n-S(v_j))\Big|\mathcal{F}_{\mathcal{I}_{j-1}}\Big] \stackrel{\mathbb{P}}{\to} W \quad \text{as } n\to\infty \tag{5.9}$$

$$a_{t_n}^{-1}\sum_{j=1}^{k_n}\mathbb{E}\Big[\chi_{v_j}^2(t_n-S(v_j))\mathbb{1}_{\{|\chi_{v_j}(t_n-S(v_j))|>\varepsilon a_{t_n}^{1/2}\}}\Big|\mathcal{F}_{\mathcal{I}_{j-1}}\Big] \stackrel{\mathbb{P}}{\to} 0 \quad \text{as } n\to\infty \tag{5.10}$$

for every $\varepsilon>0$. To prove (5.9) observe that

$$a_{t_n}^{-1}\mathbb{E}\bigg[\sum_{j=k_n+1}^{\infty}\mathbb{E}\Big[\chi_{v_j}^2(t_n-S(v_j))\Big|\mathcal{F}_{\mathcal{I}_{j-1}}\Big]\bigg] = \mathbb{E}\big[(M(t_n)-M_{k_n}(t_n))^2\big] \leq 2^{-n}$$

and hence (5.9) is equivalent to

$$a_{t_n}^{-1}\sum_{j=1}^{\infty}\mathbb{E}\Big[\chi_{v_j}^2(t_n-S(v_j))\Big|\mathcal{F}_{\mathcal{I}_{j-1}}\Big] = a_{t_n}^{-1}\sum_{u\in\mathcal{I}}\mathbb{E}[\chi^2](t_n-S(u)) \stackrel{\mathbb{P}}{\to} W. \tag{5.11}$$

In case (i), (5.11) is equivalent to

$$e^{-\alpha t_n}\sum_{u\in\mathcal{I}}\mathbb{E}[\chi^2](t_n-S(u)) = e^{-\alpha t_n}\mathcal{Z}_{t_n}^{\mathbb{E}[\chi^2]} \stackrel{\mathbb{P}}{\to} \frac{W}{-\mathcal{L}\mu(\alpha)}\int e^{-\alpha x}\mathbb{E}[\chi^2](x)\,\ell(\mathrm{d}x),$$

which follows from [23, Theorem 6.1] in the non-lattice case. The lattice case is analogous. Lemma 5.3 gives (5.11) in case (ii).

Now we show (5.10). Let $v_2(t,s) := \mathbb{E}[\chi^2(t)\mathbb{1}_{\{|\chi(t)|>s\}}]$ for $t\in\mathbb{R}$ and $s\geq 0$. In case (i), for any $\varepsilon>0$,

$$\begin{aligned}
\limsup_{n\to\infty} e^{-\alpha t_n}&\sum_{j=1}^{k_n}\mathbb{E}\big[\chi_{v_j}^2(t_n-S(v_j))\mathbb{1}_{\{|\chi_{v_j}(t_n-S(v_j))|>\varepsilon e^{\alpha t_n/2}\}}\,\big|\,\mathcal{F}_{\mathcal{I}_{j-1}}\big]\\
&\leq \limsup_{n\to\infty} e^{-\alpha t_n}\sum_{j=1}^{\infty} v_2(t_n-S(v_j),\varepsilon e^{\alpha t_n/2})\\
&\leq \liminf_{s\to\infty}\limsup_{n\to\infty} e^{-\alpha t_n}\mathcal{Z}_{t_n}^{v_2(\cdot,s)}\\
&= \liminf_{s\to\infty}\frac{W}{\beta}\int v_2(x,s)\,\ell(\mathrm{d}x) = 0 \qquad \text{a.s.}
\end{aligned}$$

by [23, Theorem 6.1] in the non-lattice case and the dominated convergence theorem. The lattice case is analogous. This shows (5.10).
We turn to case (ii) and fix $\varepsilon>0$. We infer from (5.6) that for any $\varepsilon,\delta>0$ there is a $T\geq 0$ such that, for all $t\geq T$,

$$v_2(t,\varepsilon e^{\alpha t/2}t^{\frac{\theta+1}{2}}) = \mathbb{E}\Big[\chi(t)^2\mathbb{1}_{\{|\chi(t)|>\varepsilon e^{\alpha t/2}t^{\frac{\theta+1}{2}}\}}\Big] \leq \delta e^{\alpha t}t^{\theta}.$$

Therefore, with $\|\mathbb{E}[\chi^2]\|_{[0,T]} := \sup_{x\in[0,T]} \mathbb{E}[\chi^2](x)$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \sum_{j=1}^{k_n} \mathbb{E}\Big[\chi_{v_j}^2(t_n - S(v_j))\mathbb{1}_{\{|\chi_{v_j}(t_n-S(v_i))|>\varepsilon e^{\alpha t_n/2} t_n^{\frac{\theta+1}{2}}\}} \,\Big|\, \mathcal{F}_{\mathcal{I}_j}\Big] \\
&\leq \limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \sum_{u\in\mathcal{I}} v_2(t_n - S(u), \varepsilon e^{\alpha t_n/2} t_n^{\frac{\theta+1}{2}}) \\
&= \limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \Bigg(\sum_{\substack{u\in\mathcal{I}:\\ S(u)\leq t_n-T}} v_2(t_n - S(u), \varepsilon e^{\alpha t_n/2} t_n^{\frac{\theta+1}{2}}) + \|\mathbb{E}[\chi^2]\|_{[0,T]} \cdot N((t_n - T, t_n]) \Bigg) \\
&\leq \limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \Bigg(\delta \sum_{\substack{u\in\mathcal{I}:\\ S(u)\leq t_n-T}} e^{\alpha(t_n - S(u))}(t_n - S(u))^\theta + \|\mathbb{E}[\chi^2]\|_{[0,T]} \cdot N((t_n - T, t_n]) \Bigg) \\
&\leq \frac{\delta W}{\beta(\theta+1)} \qquad \text{a. s.}
\end{aligned}$$

by Lemma 5.3 with $f = 1$. Since $\delta > 0$ was arbitrary, we conclude that the limit is zero and, therefore, (5.10) is fulfilled in both cases. □

5.3. **Deterministic characteristics.** Let f be a deterministic characteristic, i.e., a càdlàg function $f : \mathbb{R} \to \mathbb{R}$. We investigate the behavior of $\mathcal{Z}_t^f$ as $t \to \infty$ by means of an auxiliary centered random characteristic χ_f defined by

$$\chi_f(t) := f * \xi * V(t) - f * \mu * V(t) = \xi * m_t^f - \mu * m_t^f, \tag{5.12}$$

where $V(\cdot) = \sum_{n=0}^{\infty} \mu^{*n}(\cdot) = \mathbb{E}[\sum_{u\in\mathcal{I}} \delta_{S(u)}(\cdot)]$ and $*$ denotes convolution. For instance, for every $t \in \mathbb{R}$,

$$f * V(t) = \int f(t-x)\, V(\mathrm{d}x) = \mathbb{E}\Big[\sum_{u\in\mathcal{I}} f(t - S(u))\Big] = m_t^f$$

if the integrals are well-defined. However, the latter is not guaranteed a priori. The following lemma provides a sufficient condition along with an important connection between $\mathcal{Z}_t^f$ and $\mathcal{Z}_t^{\chi_f}$.

Lemma 5.6. *Let f be a deterministic, càdlàg characteristic such that $t \mapsto e^{-\alpha t}|f(t)|$ is directly Riemann integrable.*

(a) The characteristic χ_f given by (5.12) is well-defined and has almost surely càdlàg paths. Moreover, for any $t \in \mathbb{R}$, $\mathbb{E}[\chi_f(t)] = 0$ and

$$\mathcal{Z}_t^f - m_t^f = \mathcal{Z}_t^{\chi_f} \qquad \text{for all } t \in \mathbb{R}. \tag{5.13}$$

(b) If (A4) holds and the function $t \mapsto e^{-\frac{\alpha}{2}t} h(t) m_t^f$ is bounded, then the characteristic $\chi_f(t)$ satisfies (A6) and (A8).

Proof. In order to see that χ_f is well-defined, it suffices to check that $|f| * \mu * V(t)$ is finite for all $t \in \mathbb{R}$. To this end, define $f^*(t) := \sup_{|x-t|\leq 1} |f(x)|$. Direct Riemann integrability of $t \mapsto e^{-\alpha t}|f(t)|$ entails local boundedness and continuity a.e. of $|f|$, hence of f^*. Furthermore, for every $t \in \mathbb{R}$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \sup_{n\le t<n+1} f^*(t) &= \sum_{n\in\mathbb{Z}} \sup_{n-1\le t<n+2} |f(t)| \\ &\le \sum_{n\in\mathbb{Z}} \Big(\sup_{n-1\le t<n} |f(t)| + \sup_{n\le t<n+1} |f(t)| + \sup_{n+1\le t<n+2} |f(t)| \Big) \\ &= 3 \sum_{n\in\mathbb{Z}} \sup_{n\le t<n+1} |f(t)| < \infty.\end{aligned}$$

Thus, by [37, Remark 3.10.4 on p. 236], f^* is directly Riemann integrable. We conclude that $f^* * \mu * V$ is everywhere finite as $\mu * V \le \delta_0 + \mu * V = V$ and hence

$$\begin{aligned}f^* * \mu * V(t) \le f^* * V(t) &= \sum_{n=0}^{\infty} \mathbb{E}\Big[\sum_{|u|=n} f^*(t - S(u)) \Big] \\ &= e^{\alpha t} \sum_{n=0}^{\infty} \mathbb{E}\Big[\sum_{|u|=n} e^{-\alpha S(u)} e^{-\alpha(t-S(u))} f^*(t - S(u)) \Big].\end{aligned}$$

The series is finite by (4.6). The finiteness of $f^* * \mu * V(t)$ for all t together with the dominated convergence theorem yield that $f * \mu * V$ is càdlàg. Similarly, as $f^* * \mu * V(t)$ is finite for all $t \in \mathbb{R}$, $f^* * \xi * V(t)$ is finite for all $t \in \mathbb{R}$ almost surely and hence $f * \xi * V$ has càdlàg paths almost surely. Consequently, χ_f has càdlàg paths. Next, due to the fact that f is deterministic, χ_f is ξ-measurable. Further, χ_f is centered since $\mu(\cdot) = \mathbb{E}[\xi(\cdot)]$. For $u \in \mathcal{I}$, we have $\chi_{f,u}(t) = f * \xi_u * V(t) - f * \mu * V(t)$. Using this and $V = \delta_0 + \mu * V$, we infer

$$\begin{aligned}&\sum_{0\le|u|\le n} \chi_f(t - S(u)) \\ &= \sum_{1\le|u|\le n+1} f * V(t - S(u)) - \sum_{0\le|u|\le n} f * \mu * V(t - S(u)) \\ &= \sum_{1\le|u|\le n} f(t - S(u)) + \sum_{|u|=n+1} f * V(t - S(u)) - f * \mu * V(t) \\ &= \sum_{0\le|u|\le n} f(t - S(u)) - f * V(t) + \sum_{|u|=n+1} f * V(t - S(u)).\end{aligned}$$

Now let $n \to \infty$ to obtain (5.13).

For the second part let $C := \sup_{t\in\mathbb{R}} |e^{-\frac{\alpha}{2}t} h(t) m_t^f| < \infty$. Then $C < \infty$ by assumption. Thus, for any $t \in \mathbb{R}$,

$$|\xi * m_t^f| = \Big| \sum_{|u|=1} m_{t-S(u)}^f \Big| \le C \sum_{|u|=1} \frac{e^{\frac{\alpha}{2}(t-S(u))}}{h(t - S(u))}.$$

From (2.13) we infer that

$$|\xi * m_t^f| \le 2C \frac{e^{\frac{\alpha}{2}t}}{h(t)} \sum_{|u|=1} h(S(u)) e^{-\frac{\alpha}{2}S(u)}.$$

Thus, $|\mu * m_t^f| = |\mathbb{E}[\xi * m_t^f]| \leq 2C\frac{e^{\frac{\alpha}{2}t}}{h(t)}\mathbb{E}\big[\sum_{|u|=1} h(S(u))e^{-\frac{\alpha}{2}S(u)}\big]$. Consequently,

$$|\chi_f(t)| \leq 2C\frac{e^{\frac{\alpha}{2}t}}{h(t)}\bigg(\sum_{|u|=1} h(S(u))e^{-\frac{\alpha}{2}S(u)} + \mathbb{E}\bigg[\sum_{|u|=1} h(S(u))e^{-\frac{\alpha}{2}S(u)}\bigg]\bigg).$$

Since $h(t)^{-2}$ is monotone and Lebesgue integrable, this proves that χ_f satisfies both, (A6) and (A8). □

The above lemma has the following immediate corollary.

Corollary 5.7. *If the assumptions of Lemma 5.6(a) or (b) are satisfied, then, for every $t \in \mathbb{R}$,*

$$\mathrm{Var}[\mathcal{Z}_t^f] = m_t^{\chi_f^2} < \infty.$$

5.4. **Slowly growing mean process with signed characteristics.** We now cover the case where m_t^φ grows relatively slowly as $|t| \to \infty$. Later, we shall reduce the general case to this one.

Theorem 5.8. *Suppose that (A1), (A2) and (A4) hold, that the random characteristic φ satisfies (A6) and that the function $t \mapsto e^{-\frac{\alpha}{2}t}h(t)m_t^\varphi$ is bounded. If $\mathbb{G} = \mathbb{R}$, then additionally assume that (A8) holds. Then, with $\mathcal{N}$ denoting a standard normal random variable independent of W,*

$$e^{-\frac{\alpha}{2}t}\mathcal{Z}_t^\varphi \stackrel{\mathrm{d}}{\to} \sigma_\varphi\sqrt{\tfrac{W}{\beta}}\mathcal{N},$$

with

$$\sigma_\varphi^2 := \int \mathrm{Var}[\varphi(x) + \xi * m_x^\varphi]e^{-\alpha x}\,\ell(\mathrm{d}x).$$

Proof. Clearly, $m^{\mathbb{E}[\varphi]} = m^\varphi$ where $\mathbb{E}[\varphi]$ denotes the function $t \mapsto \mathbb{E}[\varphi(t)]$. In view of Lemma 5.6, we can write

$$\mathcal{Z}_t^\varphi = \mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]} + \mathcal{Z}_t^{\mathbb{E}[\varphi]} = \mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]} + \mathcal{Z}_t^{\chi_{\mathbb{E}[\varphi]}} + m_t^\varphi = \mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]+\chi_{\mathbb{E}[\varphi]}} + m_t^\varphi.$$

By assumption, $e^{-\alpha t/2}m_t^\varphi \to 0$ as $t \to \infty$. Therefore, it suffices to show that $e^{-\frac{\alpha}{2}t}\mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]+\chi_{\mathbb{E}[\varphi]}}$ converges in distribution to the claimed normal distribution. Since the characteristic

$$t \mapsto \varphi(t) - \mathbb{E}[\varphi(t)] + \chi_{\mathbb{E}[\varphi]}(t) \tag{5.14}$$

is centered, it is reasonable to apply Theorem 5.5(i). To this end, we need to check that (A6) holds for the characteristic in (5.14), i.e., that the function

$$\begin{aligned} t &\mapsto e^{-\alpha t}\,\mathrm{Var}[\varphi(t) - \mathbb{E}[\varphi(t)] + \chi_{\mathbb{E}[\varphi]}(t)] \\ &= e^{-\alpha t}\,\mathrm{Var}[\varphi(t) + \chi_{\mathbb{E}[\varphi]}(t)] \text{ is directly Riemann integrable.} \end{aligned} \tag{5.15}$$

Invoking Lemma 5.6(b) we conclude that $\chi_{\mathbb{E}[\varphi]}(t)$ satisfies (A6) and (A8). This is also true for φ. Hence, (5.15) holds by Remark 2.2. □

5.5. **Proof of Theorem 2.8.** In the proof of Theorem 2.8, we use the following fact.

Lemma 5.9. *Let $\eta_1, \ldots, \eta_m$ be different real numbers and let $(Y_{j,l})_{1\leq j\leq m, 0\leq l\leq n}$ be a collection of centered, square-integrable random variables with $\sum_{j=1}^m \operatorname{Var}[Y_{j,l}] > 0$ for each integer $l \in \{0, \ldots, n\}$. Then for*

$$\chi(t) := e^{\frac{\alpha}{2}t} \sum_{l=0}^{n} \sum_{j=1}^{m} t^l e^{\mathrm{i}\eta_j t} Y_{j,l}$$

it holds that

$$\frac{1}{t^{2n+1}} \int_{[0,t]} \operatorname{Var}\left[\chi(x)\right] e^{-\alpha x}\, \ell(\mathrm{d}x) \to \frac{1}{2n+1} \sum_{j=1}^{m} \operatorname{Var}[Y_{j,n}] \quad \text{as } t \to \infty,\ t \in \mathbb{G} \tag{5.16}$$

and, for any $\varepsilon > 0$,

$$\mathbb{E}\left[|\chi(t)|^2 \mathbb{1}_{\{|\chi(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}\}}\right] = o(t^{2n} e^{\alpha t}) \quad \text{as } t \to \infty,\ t \in \mathbb{G}. \tag{5.17}$$

In other words, $\chi(t)$ fulfills the assumption of Theorem 5.5(ii) with $\theta = 2n$.

Proof. As $x \to \infty$, expanding the variance gives

$$\operatorname{Var}\left[\sum_{l=0}^{n} \sum_{j=1}^{m} x^l e^{\mathrm{i}\eta_j x} Y_{j,l}\right] = x^{2n} \sum_{j=1}^{m} \operatorname{Var}[Y_{j,n}] + x^{2n} \sum_{j\neq k} e^{\mathrm{i}(\eta_j - \eta_k)x} \operatorname{Cov}[Y_{j,n}, Y_{k,n}] + O(x^{2n-1}). \tag{5.18}$$

Further notice that, for $\eta \in \mathbb{R}$,

$$\frac{1}{t^{2n+1}} \int_{[0,t]} x^{2n} e^{\mathrm{i}\eta x}\, \ell(\mathrm{d}x) \to \begin{cases} \frac{1}{2n+1}, & \text{if } \eta = 0, \\ 0, & \text{if } \eta \neq 0. \end{cases} \tag{5.19}$$

This follows from the fundamental theorem of calculus in the non-lattice case and integration by parts if $\eta \neq 0$, whereas in the lattice case, it follows from Faulhaber's formula if $\eta = 0$ and from summation by parts if $\eta \neq 0$. Relation (5.16) now follows from (5.18) and (5.19).

In order to prove that (5.17) holds we set $\chi_j(t) := e^{\frac{\alpha}{2}t} \sum_{l=0}^{n} t^l e^{\mathrm{i}\eta_j t} Y_{j,l}$. Since for any complex numbers $c_1, \ldots, c_m$ and $y > 0$

$$|c_1 + \cdots + c_m|^2 \mathbb{1}_{\{|c_1+\cdots+c_m|>y\}} \leq m^2\big(|c_1|^2 \mathbb{1}_{\{|c_1|>y/m\}} + \cdots + |c_m|^2 \mathbb{1}_{\{|c_m|>y/m\}}\big),$$

it suffices to prove (5.17) for χ_j instead of χ. By Markov's inequality, we have

$$\mathbb{P}(|\chi_j(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}) \leq \frac{(n+1)^2}{\varepsilon t^{2n+1}} \sum_{l=0}^{n} t^{2l} \mathbb{E}[|Y_{j,l}|^2] \to 0 \quad \text{as } t \to \infty$$

as the sum is of the order t^{2n} as $t \to \infty$. Consequently,

$$\begin{aligned} &\mathbb{E}\left[|\chi_j(t)|^2 \mathbb{1}_{\{|\chi_j(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}\}}\right] \\ &\quad \leq 2t^{2n} e^{\alpha t} \mathbb{E}\left[|Y_{j,n}|^2 \mathbb{1}_{\{|\chi_j(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}\}}\right] + O(t^{2n-1} e^{\alpha t}) = o(t^{2n} e^{\alpha t}) \end{aligned}$$

as $t \to \infty$. This proves (5.17). □

We now turn to the proof of Theorem 2.8.

Proof of Theorem 2.8. Suppose that φ is a random characteristic satisfying

$$m_t^{\varphi} = \mathbb{1}_{[0,\infty)}(t) \sum_{\lambda\in\Lambda_{\geq}} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l e^{\lambda t} + r(t), \qquad t \in \mathbb{G} \tag{2.16}$$

for some constants $a_{\lambda,l} \in \mathbb{R}$ and a function r such that $|r(t)| \leq Ce^{\frac{\alpha}{2}t}/h(t)$ for a finite constant $C > 0$. For any $\lambda \in \Lambda_{\geq}$, we put

$$\vec{a}_\lambda := \sum_{l=1}^{k(\lambda)} a_{\lambda,l-1}\mathsf{e}_l$$

and consider the following characteristic

$$\psi_\Lambda(t) = \sum_{\lambda\in\Lambda} \big\langle \vec{a}_\lambda, (\phi_\lambda(t) + \chi_\lambda(t))\mathsf{e}_1 \big\rangle$$

for ϕ_λ and χ_λ defined in (4.17) and (4.23), respectively. Then, by (4.24),

$$\mathcal{Z}_t^{\psi_\Lambda} = \sum_{\lambda\in\Lambda} \big\langle \vec{a}_\lambda, \mathcal{Z}_t^{\phi_\lambda+\chi_\lambda}\mathsf{e}_1 \big\rangle = \sum_{\lambda\in\Lambda} \big\langle \vec{a}_\lambda, \exp(\lambda, t, k(\lambda))W(\lambda, k)\mathsf{e}_1 \big\rangle = H_\Lambda(t) \tag{5.20}$$

Similarly, putting

$$\psi_{\partial\Lambda}(t) = \sum_{\lambda\in\partial\Lambda} \big\langle \vec{a}_\lambda, \mathbb{E}[\phi_\lambda(t)]\mathsf{e}_1 \big\rangle,$$

we obtain, for any $t \in \mathbb{R}$,

$$\begin{aligned} m_t^{\psi_{\partial\Lambda}} = \mathbb{E}[\mathcal{Z}_t^{\psi_{\partial\Lambda}}] &= \sum_{\lambda\in\partial\Lambda} \big\langle \vec{a}_\lambda, \mathbb{E}[\mathcal{Z}_t^{\phi_\lambda}]\mathsf{e}_1 \big\rangle \\ &= \sum_{\lambda\in\partial\Lambda} \big\langle \vec{a}_\lambda, \exp(\lambda, t, k(\lambda))\mathsf{e}_1 \big\rangle \mathbb{1}_{[0,\infty)}(t) = H_{\partial\Lambda}(t)\mathbb{1}_{[0,\infty)}(t). \end{aligned} \tag{5.21}$$

We write

$$\mathcal{Z}_t^{\varphi} = \mathcal{Z}_t^{\psi_\Lambda} + \mathcal{Z}_t^{\psi_{\partial\Lambda}} + \mathcal{Z}_t^{\varrho} = H_\Lambda(t) + \mathcal{Z}_t^{\psi_{\partial\Lambda}} + \mathcal{Z}_t^{\varrho},$$

where $\varrho := \varphi - \psi_\Lambda - \psi_{\partial\Lambda}$. Next, since $\psi_{\partial\Lambda}$ is deterministic, we may consider the associated centered characteristic $\chi_{\psi_{\partial\Lambda}}$ defined in (5.12), namely,

$$\begin{aligned} \chi_{\psi_{\partial\Lambda}}(t) &= \xi * m_t^{\psi_{\partial\Lambda}} - \mu * m_t^{\psi_{\partial\Lambda}} \\ &= \sum_{\lambda\in\partial\Lambda} \sum_{i=1}^{N} \big\langle \vec{a}_\lambda, \exp(\lambda, t - X_i, k(\lambda))\mathsf{e}_1 \big\rangle \mathbb{1}_{[0,\infty)}(t - X_i) - \mu * m_t^{\psi_{\partial\Lambda}} \\ &= \sum_{\lambda\in\partial\Lambda} \psi_\lambda(t) - \phi_{\partial\Lambda}(t) =: \chi(t) - \phi_{\partial\Lambda}(t), \end{aligned}$$

where

$$\begin{aligned} \psi_\lambda(t) := \mathbb{1}_{[0,\infty)}(t) &\sum_{i=1}^{N} \big\langle \vec{a}_\lambda, \exp(\lambda, t - X_i, k(\lambda))\mathsf{e}_1 \big\rangle \\ &- \mathbb{1}_{[0,\infty)}(t)\mathbb{E}\Big[\sum_{i=1}^{N} \big\langle \vec{a}_\lambda, \exp(\lambda, t - X_i, k(\lambda))\mathsf{e}_1 \big\rangle\Big] \end{aligned}$$

and

$$\begin{aligned}\phi_{\partial\Lambda}(t) &:= \sum_{\lambda\in\partial\Lambda}\sum_{i=1}^{N}\big\langle\vec{a}_\lambda, \exp(\lambda, t-X_i, k(\lambda))\mathsf{e}_1\big\rangle\mathbb{1}_{[0,X_i)}(t)\\ &\quad-\sum_{\lambda\in\partial\Lambda}\mathbb{E}\Big[\sum_{i=1}^{N}\big\langle\vec{a}_\lambda, \exp(\lambda, t-X_i, k(\lambda))\mathsf{e}_1\big\rangle\mathbb{1}_{[0,X_i)}(t)\Big]\\ &= \sum_{\lambda\in\partial\Lambda}\big\langle\vec{a}_\lambda, \phi_\lambda(t)\mathsf{e}_1\big\rangle-\sum_{\lambda\in\partial\Lambda}\mathbb{E}\big[\big\langle\vec{a}_\lambda, \phi_\lambda(t)\mathsf{e}_1\big\rangle\big].\end{aligned}$$

Note also that, for every $\lambda\in\partial\Lambda$,

$$\begin{aligned}\sum_{i=1}^{N}\big\langle\vec{a}_\lambda, \exp(\lambda, t-X_i, k(\lambda))\mathsf{e}_1\big\rangle &= \sum_{i=1}^{N}e^{\lambda(t-X_i)}\sum_{l=0}^{k(\lambda)-1}a_{\lambda,l}(t-X_i)^l\\ &= \sum_{i=1}^{N}e^{\lambda(t-X_i)}\sum_{l=0}^{k(\lambda)-1}a_{\lambda,l}\sum_{j=0}^{l}\binom{l}{j}t^j(-X_i)^{l-j}\\ &= \sum_{j=0}^{k(\lambda)-1}\sum_{l=j}^{k(\lambda)-1}\sum_{i=1}^{N}e^{-\lambda X_i}a_{\lambda,l}\binom{l}{j}(-X_i)^{l-j}t^je^{\lambda t}\\ &=: \sum_{j=0}^{k(\lambda)-1}R_{\lambda,j}t^je^{\lambda t}.\end{aligned}$$

With this at hand, we infer

$$\psi_\lambda(t) = \mathbb{1}_{[0,\infty)}(t)\sum_{l=0}^{k(\lambda)-1}(R_{\lambda,l}-\mathbb{E}[R_{\lambda,l}])t^le^{\lambda t}, \quad t\in\mathbb{R}. \tag{5.22}$$

Therefore, for $t\geq 0$, using (5.13) for $f=\psi_{\partial\Lambda}$, namely, that $\mathcal{Z}_t^{\psi_{\partial\Lambda}}-m_t^{\psi_{\partial\Lambda}}=Z_t^{\chi_{\psi_{\partial\Lambda}}}$ and, hence, $\mathcal{Z}_t^{\psi_{\partial\Lambda}}=Z_t^{\chi_{\psi_{\partial\Lambda}}}+m_t^{\psi_{\partial\Lambda}}=Z_t^{\chi_{\psi_{\partial\Lambda}}}+H_{\partial\Lambda}(t)$, we write

$$\mathcal{Z}_t^{\varphi}=H_\Lambda(t)+H_{\partial\Lambda}(t)+\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}+\mathcal{Z}_t^{\chi}.$$

It suffices to prove the limit theorem for $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}$ and $\mathcal{Z}_t^{\chi}$. To this end, we invoke Theorem 5.8 for the first process and Theorem 5.5(ii) for the second. We begin with $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}$ and first notice that, in view of Lemma 5.2 and Remark 2.2, the characteristic $\varrho-\phi_{\partial\Lambda}$ has càdlàg paths and satisfies (A6) and (A8). Moreover, since $m_t^{\varrho-\phi_{\partial\Lambda}}=m_t^{\varrho}=r(t)$, we may apply Theorem 5.8 to conclude that

$$e^{-\frac{\alpha}{2}t}\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}\stackrel{\mathrm{d}}{\to}\sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N},$$

where

$$\sigma^2 := \int v(x)e^{-\alpha x}\,\ell(\mathrm{d}x), \tag{5.23}$$

and $v(t) := \mathrm{Var}\left[\varrho(t) - \phi_{\partial\Lambda}(t) + r * \xi(t)\right]$. The function v can be further simplified in the following way

$$\begin{aligned}
v(t) &= \mathrm{Var}\left[\varphi(t) - \psi_\Lambda(t) - \psi_{\partial\Lambda}(t) - \phi_{\partial\Lambda}(t) + r * \xi(t)\right] \\
&= \mathrm{Var}\left[\varphi(t) - \sum_{\lambda\in\Lambda} \left\langle \vec{a}_\lambda, (\phi_\lambda(t) + \chi_\lambda(t))\mathsf{e}_1\right\rangle - \sum_{\lambda\in\partial\Lambda} \left\langle \vec{a}_\lambda, \phi_\lambda(t)\mathsf{e}_1\right\rangle + r * \xi(t)\right] \\
&= \mathrm{Var}\Bigg[\varphi(t) - \sum_{j=1}^{N}\Bigg(\sum_{\lambda\in\Lambda} \left\langle \vec{a}_\lambda, \mathbb{1}_{(-\infty,X_j)}(t)\exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1\right\rangle \\
&\qquad - \sum_{\lambda\in\partial\Lambda} \left\langle \vec{a}_\lambda, \mathbb{1}_{[0,X_j)}(t)\exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1\right\rangle + r(t - X_j)\Bigg)\Bigg] \\
&= \mathrm{Var}\Bigg[\varphi(t) - \sum_{j=1}^{N}\Bigg(\sum_{\lambda\in\Lambda} \left\langle \vec{a}_\lambda, \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1\right\rangle \\
&\qquad - \mathbb{1}_{[0,\infty)}(t)\sum_{\lambda\in\partial\Lambda} \left\langle \vec{a}_\lambda, \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1\right\rangle + m(t - X_j)\Bigg)\Bigg].
\end{aligned}$$

This proves the theorem under the assumption that $\rho_l = 0$ for all $l \geq 0$. Indeed, in this case $\mathcal{Z}_t^\chi = 0$ and $\left\langle \vec{a}_\lambda, \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1\right\rangle$ is deterministic for any $\lambda \in \partial\Lambda$, and (2.19) follows. If, additionally, $\sigma^2 = 0$, then $\mathcal{Z}_t^{\varrho - \phi_{\partial\Lambda}} = r(t)$.

It remains to prove the theorem in the case where $\rho_l > 0$ for some $l \geq 0$. First notice that by the already established central limit theorem for $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}$, we have $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}} = o(t^{\frac{1}{2}} e^{\frac{\alpha}{2}t})$ as $t \to \infty$ in probability.

Let $n \in \mathbb{N}_0$ be maximal with $\rho_n > 0$. We show that the characteristic χ satisfies the assumptions of Theorem 5.5(ii) with $\theta = 2n$. First note that, for any $\lambda \in \partial\Lambda$ and $l \leq k(\lambda) - 1$,

$$|R_{\lambda,l}| \leq C_{\lambda,\vec{a}_\lambda} \sum_{i=1}^{N} (1 + X_i^{k(\lambda)-1}) e^{-\frac{\alpha}{2}X_i}$$

for some constant $C_{\lambda,\vec{a}_\lambda}$ depending on λ, l and $\vec{a}_\lambda$. In view of assumption (A4) the random variable $R_{\lambda,l}$ is square integrable. Setting $R_{\lambda,l} := 0$ for $l \geq k(\lambda)$, we write

$$\chi(t) = \mathbb{1}_{[0,\infty)}(t) e^{\frac{\alpha}{2}t} \cdot \sum_{\lambda\in\partial\Lambda} \sum_{l=0}^{n} (R_{\lambda,l} - \mathbb{E}[R_{\lambda,l}]) t^l e^{\mathrm{i}\,\mathrm{Im}\lambda t}.$$

An application on Lemma 5.9 gives

$$\frac{1}{t^{2n+1}} \int_0^t \mathbb{E}[\chi^2(x)] e^{-\alpha x}\, \ell(\mathrm{d}x) \to \frac{\sum_{\lambda\in\partial\Lambda} \mathrm{Var}[R_{\lambda,n}]}{2n+1} = \frac{\rho_n^2}{2n+1}$$

and

$$\mathbb{E}\left[|\chi(t)|^2 \mathbb{1}_{\{|\chi(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}\}}\right] = o(t^{2n} e^{\alpha t}) \quad \text{as } t \to \infty,\ t \in \mathbb{G}.$$

Finally, by Theorem 5.5(ii),

$$\Big(\frac{\rho_n^2 t^{2n+1}}{2n+1} e^{\alpha t}\Big)^{-\frac{1}{2}} \mathcal{Z}_t^\chi \xrightarrow{\mathrm{d}} \sqrt{\tfrac{W}{\beta}}\mathcal{N},$$

which finishes the proof. ∎

Proof of Corollary 2.12. First observe that linear combinations as well as the translations $\varphi(\cdot) \mapsto \varphi(\cdot - s)$ preserve the conditions (A6) and (A8). Moreover, for the characteristic $\psi(t) := \varphi(t-s)$ the mean function m_t^ψ has the expansion (2.16) with coefficients given by vectors $\big(\exp(\lambda, -s, k(\lambda))^\mathsf{T} \vec{a}_\lambda\big)$. According to the Cramér–Wold device the convergence in distribution of

$$t^{-\frac{d}{2}} e^{-\frac{\alpha}{2} t} \big(\mathcal{Z}_{t-s_1}^\varphi - H(t-s_1), \ldots, \mathcal{Z}_{t-s_n}^\varphi - H(t-s_n)\big)$$

is equivalent to the convergence in distribution of

$$t^{-\frac{d}{2}} e^{-\frac{\alpha}{2} t} \sum_{j=1}^{n} c_j \big(\mathcal{Z}_{t-s_j}^\varphi - H(t-s_j)\big),$$

for all choices $c_1, \ldots, c_n \in \mathbb{R}$, and the latter convergence follows from Theorem 2.8. The covariance can be obtained by the polarization identity applied to the variance and the fact that $m_t^\psi = m_{t-s}^\varphi$, $h^\psi(t) = h^\varphi(t-s)$. □

6. Asymptotic expansion of the mean

In this section we are concerned with the asymptotic expansion of the mean $m_t^\phi = \mathbb{E}[\mathcal{Z}_t^\phi]$ of a supercritical general branching process $(\mathcal{Z}_t^\phi)_{t\geq 0}$ as $t \to \infty$. Throughout the section, we assume that (A1), (A2) and (A3) hold. In the non-lattice case we work with the corresponding bilateral Laplace transforms whereas in the lattice case, we use generating functions.

6.1. The lattice case.

In the present subsection, we assume that μ is concentrated on the lattice $\mathbb{Z}$ (and no smaller lattice). In this case it is convenient to work with generating functions. Throughout this section we set

$$\mathcal{G}\mu(z) := \sum_{k=0}^{\infty} \mu(\{k\}) z^k = \int z^x \, \mu(\mathrm{d}x) \tag{6.1}$$

for all $z \in \mathbb{C}$ for which the series is absolutely convergent. In particular, $\mathcal{L}\mu(z) = \mathcal{G}\mu(e^{-z})$. Note that, due to assumption (A3), $\mathcal{G}\mu(e^{-\vartheta}) < \infty$ and hence the power series (6.1) defines a holomorphic function on $\{|z| < e^{-\vartheta}\}$. Further, by slightly increasing the value of ϑ if necessary, we may assume without loss of generality that there are only finitely many solutions of the equation $\mathcal{G}\mu(z) = 1$ in the disc $\{|z| < e^{-\vartheta}\}$. We set $\gamma := \min\{\mathrm{Re}(\lambda) : \lambda \in \Lambda\} > \frac{\alpha}{2}$.

Lemma 6.1. *Let $\theta \in (\vartheta, \frac{\alpha}{2})$ be such that there are no solutions to $\mathcal{G}\mu(z) = 1$ in $\{z : e^{-\alpha/2} < |z| \leq e^{-\theta}\}$. Then there are constants $b_{\lambda,l}$, $\lambda \in \Lambda_\geq$, $l = 0, \ldots, k(\lambda) - 1$ such that, for any characteristic φ with*

$$\sum_{n\in\mathbb{Z}} |\mathbb{E}[\varphi(n)]| (e^{-\theta n} + e^{-\alpha n}) < \infty,$$

it holds that, for $t \in \mathbb{Z}$,

$$m_t^\varphi = \begin{cases} \displaystyle\sum_{\lambda\in\Lambda_\geq} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \sum_{n\in\mathbb{Z}} \mathbb{E}[\varphi(n)](t-n)^l e^{\lambda(t-n)} + O(e^{\theta t}) & \text{as } t \to \infty \\ \displaystyle\sum_{\lambda\in\Lambda} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \sum_{n\in\mathbb{Z}} \mathbb{E}[\varphi(n)](t-n)^l e^{\lambda(t-n)} + O(e^{\gamma t}) & \text{as } t \to -\infty \end{cases}. \tag{6.2}$$

Remark 6.2. Defining $\vec{b}_\lambda := (b_{\lambda,l-1})_{l=1,\ldots,k(\lambda)}$, we may rewrite (6.2) more compactly in the form

$$m_t^\varphi = \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{t\geq 0 \text{ or } \lambda\in\Lambda\}} \sum_{n\in\mathbb{Z}} \mathbb{E}[\varphi(n)]\big\langle \vec{b}_\lambda, \exp(\lambda, t-n, k(\lambda))\mathsf{e}_1\big\rangle + O(e^{\theta t}\wedge e^{\gamma t})$$

as $t\to\pm\infty$, $t\in\mathbb{Z}$.

Proof of Lemma 6.1. For $r>0$, let $B_r = \{|z|<r\}$ and $\partial B_r = \{|z|=r\}$. Now fix $r<e^{-\alpha}$. As $\mathcal{G}(\mu^{*l}) = (\mathcal{G}\mu)^l$, for any $l\in\mathbb{N}$ and since $\mathcal{G}\mu$ is holomorphic on $B_{e^{-\vartheta}}$, we infer from Cauchy's integral formula that

$$\mu^{*l}(\{n\}) = \frac{1}{2\pi\mathrm{i}}\int_{\partial B_r} \frac{(\mathcal{G}\mu)^l(z)}{z^{n+1}}\,\mathrm{d}z.$$

In particular,

$$\mathbb{E}[N(\{n\})] = \sum_{l=0}^\infty \mu^{*l}(\{n\}) = \sum_{l=0}^\infty \frac{1}{2\pi\mathrm{i}}\int_{\partial B_r}\frac{(\mathcal{G}\mu)^l(z)}{z^{n+1}}\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_{\partial B_r}\frac{\mathrm{d}z}{(1-\mathcal{G}\mu(z))z^{n+1}}$$

where the last equality follows by Fubini's theorem. For $\lambda\in\Lambda_\geq$, let

$$\sum_{j=-k(\lambda)}^{-1} b_j(\lambda)(z-e^{-\lambda})^j$$

be the principle part of the Laurent expansion of the meromorphic function $(1-\mathcal{G}\mu(z))^{-1}$ around $e^{-\lambda}$. Then the function

$$H(z) := \frac{1}{1-\mathcal{G}\mu(z)} - \sum_{\lambda\in\Lambda_\geq}\sum_{j=-k(\lambda)}^{-1} b_j(\lambda)(z-e^{-\lambda})^j$$

is holomorphic on $B_{e^{-\vartheta}}$. On the other hand, for any $d\in\mathbb{N}_0$,

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial B_r}\frac{(z-e^{-\lambda})^{-d}}{z^{n+1}}\,\mathrm{d}z = (-e^{\lambda})^d e^{\lambda n}\binom{n+d-1}{d-1}$$

by the residue theorem. Therefore,

$$G(z) := \sum_{\lambda\in\Lambda_\geq}\sum_{j=-k(\lambda)}^{-1} b_j(\lambda)(z-e^{-\lambda})^j$$

satisfies

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial B_r}\frac{G(z)}{z^{n+1}}\,\mathrm{d}z = \sum_{\lambda\in\Lambda_\geq} p_\lambda(n)e^{\lambda n}$$

where p_λ, for $\lambda\in\Lambda_\geq$, is a polynomial with complex coefficients of degree $k(\lambda)-1$. From the analyticity of H, we infer

$$\Bigg|\int_{\partial B_{e^{-\theta}}}\frac{H(z)}{z^{n+1}}\,\mathrm{d}z\Bigg| = O(e^{\theta n})\quad\text{as } n\to\infty,$$

which in turn gives

$$\begin{aligned}\mathbb{E}[N(\{n\})] &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{G(z)+H(z)}{z^{n+1}}\,\mathrm{d}z \\ &= \sum_{\lambda\in\Lambda_{\geq}} \big\langle \vec{b}_\lambda, \exp(\lambda, n, k(\lambda))\mathsf{e}_1\big\rangle + O(e^{\theta n}) \quad \text{as } n\to\infty\end{aligned}$$

for some $\vec{b}_\lambda = \sum_{l=1}^{k(\lambda)} b_{\lambda,l-1}\mathsf{e}_l \in \mathbb{R}^{k(\lambda)}$. In other words, there exists a constant $C>0$ such that, for any $n\in\mathbb{Z}$,

$$\Big|\mathbb{E}[N(\{n\})] - \sum_{\lambda\in\Lambda_{\geq}} \big\langle \vec{b}_\lambda, \exp(\lambda, n, k(\lambda))\mathsf{e}_1\big\rangle\Big| \leq C(e^{\theta n}\wedge e^{\gamma n}). \tag{6.3}$$

Now we are ready to investigate the asymptotic behavior of m_t^φ as $t\to\pm\infty$, $t\in\mathbb{Z}$. Since $m_t^\varphi = m_t^{\mathbb{E}[\varphi]}$, we assume without loss of generality that $\varphi = f$ is a deterministic function satisfying

$$\sum_{n\in\mathbb{Z}} |f(n)|(e^{-\theta n}+e^{-\alpha n}) < \infty,$$

Then, for $t\in\mathbb{Z}$, we have $m_t^f = \sum_{n\in\mathbb{Z}} f(n)\mathbb{E}[N(\{t-n\})]$. We write

$$\begin{aligned}&\Big|m_t^f - \sum_{\lambda\in\Lambda_{\geq}} \mathbb{1}_{\{t\geq 0 \text{ or } \lambda\in\Lambda\}} \sum_{n\in\mathbb{Z}} f(n)\big\langle \vec{b}_\lambda, \exp(\lambda, t-n, k(\lambda))\mathsf{e}_1\big\rangle\Big| \\ &\leq \sum_{n\in\mathbb{Z}} |f(n)| \Big|\mathbb{E}[N(\{t-n\})] - \sum_{\lambda\in\Lambda_{\geq}} \mathbb{1}_{\{t\geq n \text{ or } \lambda\in\Lambda\}}\big\langle \vec{b}_\lambda, \exp(\lambda, t-n, k(\lambda))\mathsf{e}_1\big\rangle\Big| \\ &\quad + \sum_{n\in\mathbb{Z}} |f(n)| \Big|\sum_{\lambda\in\partial\Lambda} |\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t\geq n\}}| \big\langle \vec{b}_\lambda, \exp(\lambda, t-n, k(\lambda))\mathsf{e}_1\big\rangle\Big|.\end{aligned} \tag{6.4}$$

We use (6.3) to estimate the first sum on the right-hand side of (6.4) by

$$\begin{aligned}C\sum_{n\in\mathbb{Z}} |f(n)|(e^{\theta(t-n)}\wedge e^{\gamma(t-n)}) &\leq C\Big(\sum_{n\in\mathbb{Z}} |f(n)|e^{\theta(t-n)}\Big) \wedge \Big(\sum_{n\in\mathbb{Z}} |f(n)|e^{\gamma(t-n)}\Big) \\ &\leq (e^{\theta t}\wedge e^{\gamma t})\, C\sum_{n\in\mathbb{Z}} |f(n)|(e^{-\theta n}+e^{-\alpha n}).\end{aligned}$$

On the other hand, we use (4.11) to conclude that for any $0<\epsilon<\alpha/2-\theta$ and $\lambda\in\partial\Lambda$ there is a constant $C_\epsilon\geq 0$ such that $\|\exp(\lambda, n, k(\lambda))\| \leq C_\epsilon e^{\frac{\alpha}{2}n+\epsilon|n|}$. Hence the second sum on the right-hand side of (6.4) is bounded by

$$C_\epsilon \sum_{\lambda\in\partial\Lambda} |\vec{b}_\lambda| \sum_{n\in\mathbb{Z}} |f(n)||\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t\geq n\}}| e^{\frac{\alpha}{2}(t-n)+\epsilon|t-n|}.$$

The latter sum can be estimated as follows

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} |f(n)||\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t\geq n\}}| e^{\frac{\alpha}{2}(t-n)+\epsilon|t-n|} \\ &= \mathbb{1}_{\mathbb{N}_0}(t)\sum_{n>t} |f(n)|e^{(\frac{\alpha}{2}-\epsilon)(t-n)} + \mathbb{1}_{\mathbb{Z}\setminus\mathbb{N}_0}(t)\sum_{n\leq t} |f(n)|e^{(\frac{\alpha}{2}+\epsilon)(t-n)} \\ &\leq \mathbb{1}_{\mathbb{N}_0}(t)\sum_{n\in\mathbb{Z}} |f(n)|e^{\theta(t-n)} + \mathbb{1}_{\mathbb{Z}\setminus\mathbb{N}_0}(t)\sum_{n\leq t} |f(n)|e^{\alpha(t-n)} = O(e^{\theta t}\wedge e^{\alpha t}),\end{aligned}$$

as $t \to \pm\infty$. □

6.2. **The non-lattice case.** We again work under the conditions (A1), (A2) and (A3) as in Section 6.1, but now we assume that μ is non-lattice.

Similar to the lattice case, first we study the behavior of $\mathbb{E}[N(t)]$. This was already done in [26, Theorem 3.1] in the special case where $\mathcal{L}\mu(z) - 1$ has only simple roots. However, the proof given in the cited source can be adapted to the more general setting here. In order to make this paper self-contained and for the reader's convenience, we include the proof.

Lemma 6.3. *Suppose that, besides (A1) through (A3), the following condition holds:*

$$\limsup_{\eta\to\infty} |\mathcal{L}\mu(\tfrac{\alpha}{2} - \delta + \mathrm{i}\eta)| < 1 \tag{6.5}$$

for some $\delta \in (0, \frac{\alpha}{2} - \vartheta]$. Then Λ is finite. In fact, the function $\mathcal{L}\mu$ takes the value 1 only at finitely many points in the strip $\frac{\alpha}{2} - \delta < \mathrm{Re}(z) < \alpha$. Then, for any root $\lambda \in \Lambda_{\geq}$ of multiplicity $k(\lambda) \in \mathbb{N}$, there exist constants $c_{\lambda,l}$, $l = 0, \ldots, k(\lambda) - 1$ such that, for any $\theta \in (\frac{\alpha}{2} - \delta, \frac{\alpha}{2})$,

$$\mathbb{E}[N(t)] = \sum_{\lambda\in\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} c_{\lambda,l} t^l + O(e^{\theta t}) \quad \text{as } t \to \infty. \tag{6.6}$$

Remark 6.4. Note that (6.6) can be rewritten in the form

$$\mathbb{E}[N(t)] = \sum_{\lambda} \langle \vec{c}_\lambda, \exp(\lambda, t, k(\lambda))\mathsf{e}_1 \rangle + O(e^{\theta t})$$

as $t \to \infty$ with $\vec{c}_\lambda := \sum_{l=1}^{k(\lambda)} c_{\lambda,l-1}\mathsf{e}_l$.

Remark 6.5. Suppose that (A3) and (A7) hold. Then one can check using the Riemann-Lebesgue lemma that (6.5) holds. Hence, in this case, Lemma 6.3 applies.

Proof of Lemma 6.3. First, condition (6.5) implies that

$$\limsup_{\eta\to\infty} |\mathcal{L}\mu(\theta + \mathrm{i}\eta)| < 1$$

for all $\theta \geq \frac{\alpha}{2} - \delta$ and that there are only finitely many roots of the equation $\mathcal{L}\mu(z) = 1$ in the strip $\frac{\alpha}{2} - \delta \leq \mathrm{Re}(z) < \alpha$, see Lemmas 2.1 and 2.3 in [26].

Now let $f = \mathbb{1}_{[0,\infty)}$ and recall that $N(t) = \mathcal{Z}_t^f$, hence $V(t) := \mathbb{E}[N(t)] = m_t^f$ for $t \in \mathbb{R}$. In analogy to the derivation of [26, Eq. (3.11)], we use the recursive structure of $\mathcal{Z}_t^f$ to obtain a renewal equation for $V(t)$ as follows. We start with

$$\mathcal{Z}_t^f = f(t) + \sum_{j=1}^{N} \mathcal{Z}_{t-X_j}^f.$$

Taking expectations, then conditioning with respect to ξ, the reproduction point process of the ancestor, we infer

$$m_t^f = f(t) + \int m_{t-x}^f \,\mu(\mathrm{d}x) = f(t) + \mu * m_t^f, \quad t \in \mathbb{R}. \tag{6.7}$$

Our subsequent proof relies on a smoothing technique. So let $\rho := \mathbb{1}_{[0,1]}$. For any $\varepsilon > 0$, we set

$$\rho_\varepsilon(t) := \tfrac{1}{\varepsilon}\rho(\tfrac{t}{\varepsilon}) = \tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}(t), \quad t \in \mathbb{R}.$$

Then for $f_\varepsilon := f * \rho_\varepsilon$ (Lebesgue convolution), we have

$$f_\varepsilon(t) \leq f(t) \leq f_\varepsilon(t+\varepsilon)$$

for all $t \in \mathbb{R}$, which in turn gives

$$m_t^{f_\varepsilon} \leq m_t^f \leq m_{t+\varepsilon}^{f_\varepsilon}.$$

Also, one can check that $t \mapsto m_t^{f_\varepsilon}$ is a continuous function. First, we find the asymptotic expansion of this function and then, we let ε tend to 0 in a controlled way while letting $t \to \infty$ to deduce the asymptotic behavior of $V(t) = m_t^f$ from that of $m_t^{f_\varepsilon}$. From the renewal equation (6.7) we conclude that for $\mathrm{Re}(z) > \alpha$ it holds

$$\mathcal{L}m^{f_\varepsilon}(z) = \mathcal{L}(\rho_\varepsilon * m_t^f)(z) = \mathcal{L}f_\varepsilon(z) + \mathcal{L}\mu(z)\mathcal{L}m^{f_\varepsilon}(z),$$

hence,

$$\mathcal{L}m^{f_\varepsilon}(z) = \frac{\mathcal{L}f_\varepsilon(z)}{1-\mathcal{L}\mu(z)} \quad \text{for } \mathrm{Re}(z) > \alpha.$$

The function

$$\frac{\mathcal{L}f_\varepsilon(z)}{1-\mathcal{L}\mu(z)} = \frac{\mathcal{L}\rho_\varepsilon(z)\mathcal{L}f(z)}{1-\mathcal{L}\mu(z)} = \frac{1-e^{-\varepsilon z}}{\varepsilon z^2(1-\mathcal{L}\mu(z))}$$

defines a meromorphic extension of $\mathcal{L}m^{f_\varepsilon}$ on $\mathrm{Re}(z) > \vartheta$. This function decays as $|\mathrm{Im}(z)|^{-2}$ as $\mathrm{Im}(z) \to \pm\infty$ and $\mathrm{Re}(z)$ is constant, hence, it is integrable along vertical lines. Thus, for any $\theta > \alpha$, the Laplace inversion formula (see, for instance, [42, Theorem 7.3 on p. 66]) gives

$$m_t^{f_\varepsilon} = \frac{m_{t+}^{f_\varepsilon} + m_{t-}^{f_\varepsilon}}{2} = \frac{1}{2\pi \mathrm{i}} \int_{\theta-\mathrm{i}\infty}^{\theta+\mathrm{i}\infty} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z, \quad t > 0.$$

To simplify notation, we assume without loss of generality that $\vartheta = \frac{\alpha}{2} - \delta$ and that $\mathcal{L}\mu$ is holomorphic on a neighborhood of $\mathrm{Re}(z) \geq \vartheta$. Then, for large enough R, an application of the residue theorem gives

$$\int_{\theta-\mathrm{i}R}^{\theta+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{\lambda\in\Lambda_\geq} \mathrm{Res}_{z=\lambda}\left(e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\right) + \int_{\vartheta-\mathrm{i}R}^{\vartheta+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z$$
$$+ \int_{\vartheta+\mathrm{i}R}^{\theta+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z - \int_{\vartheta-\mathrm{i}R}^{\theta-\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z.$$

Here,

$$\Big|\int_{\vartheta+\mathrm{i}R}^{\theta+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z\Big| \leq e^{t\theta}\int_{\vartheta}^{\theta}\Big|\frac{1-e^{-\varepsilon(x+\mathrm{i}R)}}{\varepsilon(x+\mathrm{i}R)^2(1-\mathcal{L}\mu(x+\mathrm{i}R))}\Big|\,\mathrm{d}x$$
$$\leq Ce^{t\theta}\int_{\vartheta}^{\theta}\Big|\frac{1}{\varepsilon(x+\mathrm{i}R)^2}\Big|\,\mathrm{d}x \underset{R\to\infty}{\longrightarrow} 0$$

for some constant C that depends only on μ. Here we used that $\mathcal{L}\mu$ is uniformly bounded away from 1 independent of R, which is true by Lemma 2.1 of [26]. The same bound holds for the second horizontal integral. Therefore, by letting R tend to infinity we conclude

$$m^{f_\varepsilon}(t) = \sum_{\lambda\in\Lambda_{\geq}} \operatorname{Res}_{z=\lambda}\left(e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\right) + \frac{1}{2\pi\mathrm{i}}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z. \tag{6.8}$$

Next, denoting by $\{b_j(\lambda)\}_{j\in\mathbb{Z}}$ the coefficients in the Laurent expansion of the function $(1-\mathcal{L}\mu(z))^{-1}$ at $z=\lambda\in\Lambda_{\geq}$ (hence, in particular, $b_j(\lambda)=0$ for $j<-k(\lambda)$), we have

$$\begin{aligned}
\operatorname{Res}_{z=\lambda}\left(e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\right) &= \operatorname{Res}_{z=\lambda}\left(e^{tz}\frac{\mathcal{L}f_\varepsilon(z)}{1-\mathcal{L}\mu(z)}\right)\\
&= e^{\lambda t}\sum_{\substack{n,l\geq 0\\ n+l<k(\lambda)}} \frac{t^l}{l!}\frac{(\mathcal{L}f_\varepsilon)^{(n)}(\lambda)}{n!} b_{-1-n-l}(\lambda)\\
&= e^{\lambda t}\sum_{\substack{n,l\geq 0\\ n+l<k(\lambda)}} \frac{t^l}{l!}\frac{\int(-x)^n f_\varepsilon(x)e^{-\lambda x}\,\mathrm{d}x}{n!} b_{-1-n-l}(\lambda)\\
&= e^{\lambda t}\sum_{\substack{n,l\geq 0\\ n+l<k(\lambda)}} \frac{t^l}{l!}\frac{\int(-x)^n f(x)e^{-\lambda x}\mathrm{d}x}{n!} b_{-1-n-l}(\lambda) + \varepsilon O(e^{\mathrm{Re}(\lambda)t})\\
&=: e^{\lambda t}\sum_{0\leq l<k(\lambda)} c_{\lambda,l}t^l + \varepsilon O(e^{\mathrm{Re}(\lambda)t}),
\end{aligned} \tag{6.9}$$

where the implicit constant depends only on λ, not on ε. It remains to estimate the second term in (6.8). For $\varepsilon\leq\vartheta^{-1}$, using that $|1-e^{-z}|\leq|z|\wedge 2$ for $\mathrm{Re}(z)\geq 0$, we infer

$$\begin{aligned}
\left|\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z\right| &\leq e^{\vartheta t}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty}\left|\frac{1-e^{-\varepsilon z}}{\varepsilon z^2(1-\mathcal{L}\mu(z))}\right||\mathrm{d}z|\\
&\leq Ce^{\vartheta t}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty}\frac{|\varepsilon z|^{-1}\wedge 1}{|z|}\,|\mathrm{d}z|\\
&= Ce^{\vartheta t}\int_{\varepsilon\vartheta-\mathrm{i}\infty}^{\varepsilon\vartheta+\mathrm{i}\infty}(|z|^{-1}\wedge|z|^{-2})\,|\mathrm{d}z|\\
&\leq Ce^{\vartheta t}\int_{-\infty}^{\infty} x^{-1}\wedge x^{-2}\wedge(\varepsilon\vartheta)^{-1}\,\mathrm{d}x\\
&\leq C'e^{\vartheta t}(|\log\varepsilon|+1)
\end{aligned}$$

for some constant C' that depends neither on t nor on ε. Using (6.9) with $t+\varepsilon$ instead of t, we conclude that

$$\left|\operatorname{Res}_{z=\lambda}\left(e^{(t+\varepsilon)z}\mathcal{L}m^{f_\varepsilon}(z)\right) - \operatorname{Res}_{z=\lambda}\left(e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\right)\right| = \varepsilon O(e^{\alpha t}),$$

where we used $k(\lambda) = 1$ for $\lambda = \alpha$, and thereupon

$$m_{t+\varepsilon}^{f_\varepsilon} - m_t^{f_\varepsilon} \leq O(\varepsilon e^{\alpha t} + |\log \varepsilon| e^{\vartheta t}).$$

Setting now $\varepsilon := e^{-\alpha t}$, we infer

$$m_t^f = \sum_{\lambda \in \Lambda} e^{\lambda t} \sum_{0 \leq l \leq k(\lambda)-1} c_{\lambda,l} t^l + O(t e^{\vartheta t}),$$

which completes the proof of the lemma. □

Now we are ready to provide the asymptotic expansion for the expectation function of a general branching process counted with random characteristic φ.

Lemma 6.6. *Suppose that, besides (A1) through (A3), condition (6.5) holds and that the random characteristic φ satisfies (A5). Then there are constants $b_{\lambda,l}$, $\lambda \in \Lambda$, $0 \leq l < k(\lambda)$ such that, for any $\vartheta < \theta < \frac{\alpha}{2}$, we have*

$$m_t^\varphi = \begin{cases} \sum\limits_{\lambda \in \Lambda_\geq} \sum\limits_{l=0}^{k(\lambda)-1} b_{\lambda,l} \int (t-x)^l e^{\lambda(t-x)} \mathbb{E}[\varphi(x)] \, \mathrm{d}x + O(e^{\theta t}) & \text{as } t \to \infty \\ \sum\limits_{\lambda \in \Lambda} \sum\limits_{l=0}^{k(\lambda)-1} b_{\lambda,l} \int (t-x)^l e^{\lambda(t-x)} \mathbb{E}[\varphi(x)] \, \mathrm{d}x + O(e^{\gamma t}) & \text{as } t \to -\infty \end{cases}. \tag{6.10}$$

Remark 6.7. If we set $\vec{b}_\lambda := (b_{\lambda,l})_{l=0,\ldots,k(\lambda)-1}$, then formula (6.10) can be rewritten in the form

$$m_t^\varphi = \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t \geq 0 \text{ or } \lambda \in \Lambda\}} \int \langle \vec{b}_\lambda, \exp(\lambda, t-x, k(\lambda)) \mathsf{e}_1 \rangle \mathbb{E}[\varphi(x)] \, \mathrm{d}x + O(e^{\theta t} \wedge e^{\gamma t})$$

as $t \to \infty$.

Proof. Without loss of generality we assume that the characteristic $\varphi = f$ is a deterministic function. By Lemma 6.3 there are $\theta \in (\vartheta, \frac{\alpha}{2})$ and a constant C such that, for any $t \in \mathbb{R}$,

$$\Big| \mathbb{E}[N(t)] - \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t \geq 0 \text{ or } \lambda \in \Lambda\}} \langle \vec{c}_\lambda, \exp(\lambda, t, k(\lambda)) \mathsf{e}_1 \rangle \Big| \leq C(e^{\theta t} \wedge e^{\gamma t})$$

and hence for the characteristic $f(t) = \mathbb{1}_{[x,\infty)}(t) = \mathbb{1}_{[0,\infty)}(t-x)$, we find

$$\Big| m_t^f - \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t-x \geq 0 \text{ or } \lambda \in \Lambda\}} \langle \vec{c_\lambda}, \exp(\lambda, t-x, k(\lambda)) \mathsf{e}_1 \rangle \Big| \leq C(e^{\theta(t-x)} \wedge e^{\gamma(t-x)}).$$

Suppose now that $f \geq 0$ is a càdlàg, nondecreasing function with

$$\int f(x)(e^{-\alpha x} + e^{-\vartheta x}) \, \mathrm{d}x < \infty. \tag{6.11}$$

Then f is the measure-generating function of a locally finite measure ν on the Borel sets of $\mathbb{R}$, namely, for any $y \in \mathbb{R}$,

$$f(y) = \nu((-\infty, y]) = \int \mathbb{1}_{[x,\infty)}(y) \, \nu(\mathrm{d}x).$$

For any $t \in \mathbb{R}$, by an application of Fubini's theorem, we infer

$$m_t^f = \int \mathbb{E}[N(t-x)] \, \nu(\mathrm{d}x).$$

Using the functional equation of the matrix $\exp(\lambda, x, k)$, one can verify that $\frac{\mathrm{d}}{\mathrm{d}x}\exp(\lambda, x, k) = J_{\lambda,k}\exp(\lambda, x, k)$ where the matrix $J_{\lambda,k}$ is defined by

$$J_{\lambda,k} := \begin{pmatrix} \lambda & & & \\ 1 & \ddots & 0 & \\ & \ddots & \ddots & \\ 0 & & k-1 & \lambda \end{pmatrix}.$$

We show that (6.10) holds with $\vec{b}_\lambda := J_{\lambda,k}^\top \vec{c}_\lambda$, $\lambda \in \Lambda_\geq$. To this end, first notice that, as f fulfills (6.11), another application of Fubini's theorem yields

$$\int \exp(\lambda, -x, k)\,\nu(\mathrm{d}x) = \int J_{\lambda,k}\exp(\lambda, -x, k) f(x)\,\mathrm{d}x.$$

We now write

$$\begin{aligned}
&\Big| m_t^f - \int \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{t\geq 0 \text{ or } \lambda\in\Lambda\}} \big\langle \vec{b}_\lambda, J_{\lambda,k(\lambda)}\exp(\lambda, t-x, k(\lambda)) f(x)\mathsf{e}_1\big\rangle\,\mathrm{d}x \Big| \\
&= \Big| \int \mathbb{E}[N(t-x)] - \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{t\geq 0 \text{ or } \lambda\in\Lambda\}} \big\langle \vec{b}_\lambda, \exp(\lambda, t-x, k(\lambda))\mathsf{e}_1\big\rangle\,\nu(\mathrm{d}x) \Big| \\
&\leq \int \Big| \mathbb{E}[N(t-x)] - \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{t-x\geq 0 \text{ or } \lambda\in\Lambda\}} \big\langle \vec{b}_\lambda, \exp(\lambda, t-x, k(\lambda))\mathsf{e}_1\big\rangle \Big|\,\nu(\mathrm{d}x) \\
&\quad + \int \sum_{\lambda\in\partial\Lambda} \int \big|\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t-x\geq 0\}}\big| \Big| \big\langle \vec{b}_\lambda, \exp(\lambda, t-x, k(\lambda))\mathsf{e}_1\big\rangle \Big|\,\nu(\mathrm{d}x).
\end{aligned}$$

For the first term we have

$$\begin{aligned}
&\int \Big| \mathbb{E}[N(t-x)] - \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{t-x\geq 0 \text{ or } \lambda\in\Lambda\}} \big\langle \vec{c}_\lambda, \exp(\lambda, t-x, k(\lambda))\mathsf{e}_1\big\rangle \Big|\,\nu(\mathrm{d}x) \\
&\leq C \int (e^{\theta(t-x)} \wedge e^{\gamma(t-x)})\,\nu(\mathrm{d}x) \\
&\leq C\Big(\int e^{\theta(t-x)}\nu(\mathrm{d}x) \wedge \int e^{\gamma(t-x)}\,\nu(\mathrm{d}x) \Big) \\
&\leq C(e^{\theta t} \wedge e^{\gamma t})\Big(\theta \int f(x) e^{-\theta x}\,\mathrm{d}x + \gamma \int f(x) e^{-\gamma x}\,\mathrm{d}x \Big).
\end{aligned}$$

Next, for $\lambda \in \partial\Lambda$, we estimate

$$\begin{aligned}
&\int \big|\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t-x\geq 0\}}\big| \big| \big\langle \vec{b}_\lambda, \exp(\lambda, t-x, k(\lambda))\mathsf{e}_1\big\rangle \big|\,\nu(\mathrm{d}x) \\
&= \mathbb{1}_{(-\infty,0)}(t) \int_{(-\infty,t]} \big| \big\langle \vec{b}_\lambda, \exp(\lambda, t-x, k(\lambda))\mathsf{e}_1\big\rangle \big|\,\nu(\mathrm{d}x) \\
&\qquad + \mathbb{1}_{[0,\infty)}(t) \int_{(t,\infty)} \big| \big\langle \vec{b}_\lambda, \exp(\lambda, t-x, k(\lambda))\mathsf{e}_1\big\rangle \big|\,\nu(\mathrm{d}x) \\
&\leq C\mathbb{1}_{(-\infty,0)}(t) \int e^{\alpha(t-x)}\,\nu(\mathrm{d}x) + C\mathbb{1}_{[0,\infty)}(t) \int e^{\vartheta(t-x)}\,\nu(\mathrm{d}x) \\
&\leq C'\big(e^{\alpha t} \wedge e^{\vartheta t}\big),
\end{aligned}$$

where we have used (4.11) in the penultimate step. This completes the proof of the theorem for non-decreasing $f \geq 0$.

Now let f be an arbitrary càdlàg function satisfying the integrability condition (A5) (with f in place of $\mathbb{E}[\varphi]$). Define

$$f_{\pm}(x) := \sup\bigg\{ \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))^{\pm} : -\infty < x_0 < \ldots < x_n \leq x,\ n \in \mathbb{N} \bigg\}$$

for $x \in \mathbb{R}$. Clearly, $f_+, f_- : \mathbb{R} \to \mathbb{R}$ are nondecreasing with $f_{\pm} \geq 0$. It is known that $f = f_+ - f_-$. (This is the Jordan decomposition of f on $\mathbb{R}$.) It is further known that f_+ and f_- are càdlàg since f is. Further, $\mathrm{V}f(x) = f_+(x) + f_-(x)$ and hence (A5) implies that both, f_+ and f_- satisfy (6.11). The previous part of the proof thereby applies to f_+ and f_- and, by linearity, extends to f. □

6.3. **Determining the coefficients.** Note that although the constants c_l^λ c_l^λ and b_l^λ are not given explicitly it is not hard to follow the proofs and provide explicit expressions for them. However, even for small $k(\lambda)$, this approach may lead to tedious calculations, not to mention that there can also be several roots in the relevant strip. It seems that a more efficient way to determine the constants b_l^λ is an application of Lemma 6.1 or 6.6, respectively, to a characteristic for which we explicitly know the asymptotic behavior of the expectation of the associated general branching process.

Proposition 6.8. *Let λ be a root of $\mathcal{L}\mu(z) = 1$ of multiplicity k (in the lattice case we also assume that $\mathrm{Im}(\lambda) \in (-\pi, \pi)$). Then the vector $\vec{b}_\lambda$ appearing in Lemma 6.6 is given by $M^\lambda \vec{b}_\lambda = \mathsf{e}_1$, where M^λ is the k by k upper triangular matrix such that for $j \geq i$*

$$(M^\lambda)_{i,j} := \frac{(-1)^k (j-1)!(k-1)!}{(i-1)!(j-i+k)!} (\mathcal{L}\mu)^{(k+j-i)}(\lambda)$$

in the non-lattice case. In contrast, in the lattice case,

$$(M^\lambda)_{i,j} := \binom{j-1}{i-1} P_{k,j-i}\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big) \mathcal{L}\mu(z)_{|z=\lambda},$$

where the polynomials $P_{k,l}$ are given by

$$P_{k,l}(y) := (-1)^l \sum_{m=1}^{k} \binom{k-1}{m-1} (-y)^{k-m} \frac{B_{l+m}(y) - B_{l+m}(0)}{l+m},$$

and B_n is the n^{th} Bernoulli polynomial. In particular, in both cases, as $(\mathcal{L}\mu)^{(j)}(\lambda) = 0$ for $j < k$,

$$\det(M^\lambda) = \bigg(\frac{-m^{(k)}(\lambda)}{k} \bigg)^k \neq 0$$

and the matrix M^λ is invertible.

Proof. An application of Lemma 6.6 (resp. Lemma 6.1) to the characteristic

$$\phi(x) = \big\langle e_k, \mathbb{E}\phi_\lambda^{(1)}(x) e_1 \big\rangle = \mathbb{1}_{[0,\infty)}(x) \mathbb{E}\bigg[\sum_{|u|=1} \mathbb{1}_{\{S(u)>x\}} (x - S(u))^{k-1} e^{\lambda(x - S(u))} \bigg]$$

yields

$$\begin{aligned} t^{k-1}e^{\lambda t} &= \Big\langle \vec{b}_\lambda, \int \exp(\lambda, t-x, k)\phi(x)\ell(\mathrm{d}x)e_1 \Big\rangle \\ &= \Big\langle \Big(\int \exp(\lambda, -x, k)\phi(x)\ell(\mathrm{d}x)\Big)^\top \vec{b}_\lambda, \exp(\lambda, t, k)e_1 \Big\rangle, \end{aligned}$$

whence

$$\Big(\int \exp(\lambda, -x, k)\phi(x)\ell(\mathrm{d}x)\Big)^\top \vec{b}_\lambda = \mathsf{e}_k.$$

It suffices now to evaluate coefficients of the matrix $\int \exp(\lambda, -x, k)\phi(x)\ell(\mathrm{d}x)$.

First we deal with the non-lattice case. For this purpose, recalling basic properties of beta function B, we infer

$$\begin{aligned} \int (-x)^l e^{-\lambda x}\phi(x)\mathrm{d}x &= (-1)^l \mathbb{E}\Big[\sum_{|u|=1} \int_0^{S(u)} x^l (x-S(u))^{k-1} e^{-\lambda S(u)}\,\mathrm{d}x \Big] \\ &= (-1)^l \int \int_0^y x^l (x-y)^{k-1} \mathrm{d}x e^{-\lambda y}\,\mu(\mathrm{d}y) \\ &= (-1)^l \int B(l+1,k) y^{k+l} e^{-\lambda y}\,\mu(\mathrm{d}y) \\ &= \frac{(-1)^k l!(k-1)!}{(l+k)!} m^{(k+l)}(\lambda), \end{aligned}$$

and therefore

$$\begin{aligned} (M^\lambda)_{i,j} &= \Big(\int \exp(\lambda, -x, k)\phi(x)\mathrm{d}x \Big)_{j,i} \\ &= \mathbb{1}_{\{j\ge i\}} m^{(k+j-i)}(\lambda) \cdot \binom{j-1}{i-1} \frac{(-1)^k (j-i)!(k-1)!}{(j-i+k)!} \\ &= \mathbb{1}_{\{j\ge i\}} \frac{(-1)^k (j-1)!(k-1)!}{(i-1)!(j-i+k)!} m^{(k+j-i)}(\lambda). \end{aligned}$$

In the non-lattice case, invoking Faulhaber's formula, we have

$$\begin{aligned} \sum_{x\in\mathbb{Z}} (-x)^l e^{-\lambda x}\phi(x) &= (-1)^l \mathbb{E}\Big[\sum_{|u|=1} \sum_{0\le x<S(u)} x^l (x-S(u))^{k-1} e^{-\lambda S(u)} \Big] \\ &= (-1)^l \int \sum_{0\le x<y} x^l (x-y)^{k-1} e^{-\lambda y} \mu(\mathrm{d}y) \\ &= (-1)^l \int \sum_{m=1}^k \binom{k-1}{m-1} (-y)^{k-m} \sum_{0\le x<y} x^{l+m-1} e^{-\lambda y} \mu(\mathrm{d}y) \\ &= (-1)^l \int \sum_{m=1}^k \binom{k-1}{m-1} (-y)^{k-m} \frac{B_{l+m}(y)-B_{l+m}(0)}{l+m} e^{-\lambda y} \mu(\mathrm{d}y) \\ &= P_{k,l}\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big) m(z)_{|z=\lambda}, \end{aligned}$$

which gives

$$\begin{aligned}(M^{\lambda})_{i,j} &= \Big(\int \exp(\lambda, -x, k)\phi(x)\mathrm{d}x\Big)_{j,i} \\ &= \mathbb{1}_{\{j\geq i\}}\binom{j-1}{i-1}P_{k,j-i}\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)m(z)_{|z=\lambda}.\end{aligned}$$

□

7. Discussion and open problems

In this section we formulate several open problems which are closely related to the present framework.

Open problem 1. Prove a corresponding limit theorem for the multitype CMJ process.

Open problem 2. Provide a functional version of the theorems proved in this paper.

A drawback of our method in the non-lattice case is that, in order to find the asymptotic of the mean m_t^{ϕ} we need to assume that the measure μ is absolutely continuous with respect to the Lebesgue measure.

Open problem 3. In the non-lattice case, work out a proof that does not require the absolute continuity of the intensity measure μ.

The Gaussian fluctuations appearing in our theorems are caused by the finiteness of the second moment (2.12). In case when the condition is not satisfied one still may ask for a generalization.

Open problem 4. Prove a stable version of the limit theorems.

One of the basic ingredient of the CMJ process is the underlying branching random walk $(S(u))_{u\in\mathcal{T}}$ with positive increments. However, the process $\mathcal{Z}^{\phi}$ might also be defined for a branching random walk with two-sided increments and suitable ϕ.

Open problem 5. Investigate the behavior of $\mathcal{Z}_t^{\phi}$ for a branching random walk $(S(u))_{u\in\mathcal{T}}$ with two-sided increments.

A central limit theorem is usually complemented by a law of the iterated logarithm (see, for instance, [20]). This motivates the following.

Open problem 6. Prove a corresponding law of the iterated logarithm for $\mathcal{Z}_t^{\phi}$.

The martingale limits $W_{\lambda}^{(j)}$ play an important role in the asymptotic behavior of the general branching process. It is important to obtain more information about their distributions. In particular, the following problem seems to be quite relevant.

Open problem 7. Derive the first-order asymptotic behavior of the tail probabilities $\mathbb{P}(|W_{\lambda}^{(j)}| > t)$ as $t \to \infty$ for $j = 0, \dots, k(\lambda) - 1$.

We also believe that the approach developed in the paper might be useful for settling the following.

Open problem 8. Find large deviation estimates for Z_t^{ϕ}.

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Alexander Iksanov, Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, Ukraine
Email address: iksan@univ.kiev.ua

Konrad Kolesko, Mathematisches Institut, University of Giessen, Germany, and Mathematical Institute, University of Wrocław, Poland
Email address: konrad.kolesko@math.uni-giessen.de

Matthias Meiners, Mathematisches Institut, University of Giessen, Germany
Email address: matthias.meiners@math.uni-giessen.de