STABLE-LIKE FLUCTUATIONS OF BIGGINS' MARTINGALES

ALEXANDER IKSANOV, KONRAD KOLESKO, AND MATTHIAS MEINERS

Abstract. Let $(W_n(\theta))_{n\in\mathbb{N}_0}$ be Biggins' martingale associated with a supercritical branching random walk, and let $W(\theta)$ be its almost sure limit. Under a natural condition for the offspring point process in the branching random walk, we show that if the law of $W_1(\theta)$ belongs to the domain of normal attraction of an α-stable distribution for some $\alpha \in (1, 2)$, then, as $n \to \infty$, there is weak convergence of the tail process $(W(\theta) - W_{n-k}(\theta))_{k\in\mathbb{N}_0}$, properly normalized, to a random scale multiple of a stationary autoregressive process of order one with α-stable marginals.

1. Introduction and main result

1.1. **Introduction.** The branching random walk on the real line is a model for the evolution of a population with a spatial component. It has connections to classical objects of statistical physics such as directed polymers on disordered trees [8] to give just one example; we refer to the recent lecture notes [23] for further examples and references.

Certain nonnegative martingales, the *additive martingales*, are key tools in the description and analysis of the asymptotic behavior of the branching random walk such as the spread of particles at typical positions, see e.g. [2]. These martingales are sometimes called *Biggins' martingales* in honor of Biggins' seminal contribution [1], in which conditions for the convergence of these martingales to nondegenerate limits were found. It is then natural to ask for the speed of convergence.

In the present paper, we are interested in the rate of convergence of Biggins' martingale in the case where the martingale at time 1 has a power tail. Requiring only minimal assumptions, we prove convergence of the finite-dimensional distributions of the tail of Biggins' martingale, suitably normalized, to a randomly scaled stationary autoregressive process of order one with stable marginals.

1.2. **Model description.** A (one-dimensional) branching random walk is a particle system on the real line. At time $n = 0$ it consists of one particle, the ancestor, located at the origin. At time $n = 1$ the ancestor produces offspring (the first generation) the positions of which are given by the points of a point process $\mathcal{Z} = \sum_{j=1}^{N} \delta_{X_j}$ on $\mathbb{R}$. The number of offspring, $N = \mathcal{Z}(\mathbb{R})$, is a random variable taking values in $\mathbb{N}_0 \cup \{+\infty\} = \{0, 1, 2, \ldots\} \cup \{+\infty\}$. At time $n = 2$, the individuals of the first generation produce offspring, the second generation, with displacements with respect to their mothers' positions given by independent copies of the point process $\mathcal{Z}$. The further generations are formed analogously.

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More formally, let $\mathcal{I} = \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n$ be the set of all possible individuals. The ancestor label is the empty word $\varnothing$, its position is $S(\varnothing) = 0$. On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $(\mathcal{Z}(u))_{u\in\mathcal{I}}$ be a family of independent and identically distributed (i.i.d.) copies of the point process $\mathcal{Z}$. An individual of the nth generation with label $u = u_1 \dots u_n$ and position $S(u)$ produces a random number $N(u)$ of offspring at time $n + 1$. The offspring of the individual u are placed at random locations on $\mathbb{R}$ given by the positions of the point process

$$\delta_{S(u)} * \mathcal{Z}(u) = \sum_{j=1}^{N(u)} \delta_{S(u)+X_j(u)}$$

where $\mathcal{Z}(u) = \sum_{j=1}^{N(u)} \delta_{X_j(u)}$ and $N(u)$ is the number of points in $\mathcal{Z}(u)$. The offspring of the individual u are enumerated by $uj = u_1 \dots u_n j$, where $j = 1, \dots, N(u)$ (if $N(u) < \infty$) or $j = 1, 2, \dots$ (if $N(u) = \infty$), and the positions of the offspring are denoted by $S(uj)$. No assumptions are imposed on the dependence structure of the random variables $N(u), X_1(u), X_2(u), \dots$ for fixed $u \in \mathcal{I}$. The point process of the positions of the nth generation individuals will be denoted by $\mathcal{Z}_n$ so that $\mathcal{Z}_0 = \delta_0$ and

$$\mathcal{Z}_{n+1} = \sum_{|u|=n} \sum_{j=1}^{N(u)} \delta_{S(u)+X_j(u)},$$

where here and hereafter, $|u| = n$ means that the sum is taken over all individuals of the nth generation rather than over all $u \in \mathbb{N}^n$. The sequence of point processes $(\mathcal{Z}_n)_{n\in\mathbb{N}_0}$ is then called a *branching random walk* (BRW).

We assume throughout that $(\mathcal{Z}_n)_{n\in\mathbb{N}_0}$ is *supercritical*, i.e., $\mathbb{E}[N] > 1$. This implies $\mathbb{P}(\mathcal{S}) > 0$ where $\mathcal{S} = \{\mathcal{Z}_n(\mathbb{R}) > 0 \text{ for every } n \in \mathbb{N}_0\}$. The sequence of generation sizes in the BRW, $(\mathcal{Z}_n(\mathbb{R}))_{n\in\mathbb{N}_0}$, forms a Galton–Watson process if $\mathbb{P}(N < \infty) = 1$.

Consider the Laplace transform of the intensity measure $\mu(\cdot) := \mathbb{E}[\mathcal{Z}(\cdot)]$ of $\mathcal{Z}$,

$$m : \mathbb{R} \to [0, \infty], \qquad \theta \mapsto \int_{\mathbb{R}} \mathrm{e}^{-\theta x} \, \mu(\mathrm{d}x) = \mathbb{E}\Big[\int_{\mathbb{R}} \mathrm{e}^{-\theta x} \, \mathcal{Z}(\mathrm{d}x)\Big].$$

We assume that $m(\theta) < \infty$ for some $\theta \in \mathbb{R}$. For each such θ, let

$$W_n(\theta) := \frac{1}{m(\theta)^n} \int_{\mathbb{R}} \mathrm{e}^{-\theta x} \, \mathcal{Z}_n(\mathrm{d}x) = \frac{1}{m(\theta)^n} \sum_{|u|=n} e^{-\theta S(u)}, \quad n \in \mathbb{N}_0.$$

We write $|u| < n$ if $u \in \mathbb{N}^k$ for some $k < n$ and set $\mathcal{F}_n = \sigma(\mathcal{Z}(u) \colon |u| < n)$, the σ-algebra generated by the first n generations. It is well-known that, for every θ with $m(\theta) < \infty$, $(W_n(\theta))_{n\in\mathbb{N}_0}$ forms a nonnegative martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ and thus converges almost surely to a random variable $W(\theta)$ satisfying $\mathbb{E}[W(\theta)] \leq 1$. This martingale is called *additive* or *Biggins' martingale*.

1.3. **The main result.** Next, we introduce an object that appears in our main result. Let $(U_k)_{k\in\mathbb{N}_0}$ denote a stationary autoregressive process of order 1 with parameter $\varphi \in (0, 1)$ defined by

$$U_k = \varphi U_{k-1} + Q_k, \quad k \in \mathbb{N} \tag{1.1}$$

where U_0 is independent of the sequence $Q_1, Q_2, \dots$ of i.i.d. random variables which have characteristic function

$$\mathbb{E}\big[e^{\mathrm{i}tQ_k}\big] = \exp\bigg(\frac{\Gamma(2-\alpha)}{\alpha-1}c|t|^\alpha\Big(\cos\Big(\frac{\pi\alpha}{2}\Big) - \mathrm{i}\sin\Big(\frac{\pi\alpha}{2}\Big)\operatorname{sign}(t)\Big)\bigg),\ t \in \mathbb{R} \tag{1.2}$$

for some $c > 0$, where $\Gamma(\cdot)$ is the gamma function. Notice that the Q_k have spectrally positive α-stable laws. Observe that, for $t \in \mathbb{R}$,

$$\mathbb{E}\big[e^{\mathrm{i}tU_0}\big] = \prod_{j\geq 0}\mathbb{E}\big[e^{\mathrm{i}\varphi^j tQ_1}\big] = \exp\bigg(\frac{\Gamma(2-\alpha)}{\alpha-1}\frac{c|t|^\alpha}{1-\varphi^\alpha}\Big(\cos\Big(\frac{\pi\alpha}{2}\Big) - \mathrm{i}\sin\Big(\frac{\pi\alpha}{2}\Big)\operatorname{sign}(t)\Big)\bigg). \tag{1.3}$$

Our main result is the following theorem.

Theorem 1.1. *Suppose there exist $\alpha \in (1,2)$ and $c > 0$ such that*

$$\kappa := \tfrac{m(\alpha\theta)}{m(\theta)^\alpha} < 1 \tag{1.4}$$

and

$$\mathbb{P}(W_1(\theta) > x) \sim cx^{-\alpha} \quad \text{as } x \to \infty. \tag{1.5}$$

Further, let $(U_r)_{r\in\mathbb{N}_0}$ be independent of $W(\theta)$ and defined as in (1.1) with $\varphi = \kappa^{1/\alpha}$. Let c in (1.2) be the same as in (1.5). Then, with $W_j(\theta) = 1$ for $j < 0$, we have

$$\big(\kappa^{-(n-r)/\alpha}(W(\theta) - W_{n-r}(\theta))\big)_{r\in\mathbb{N}_0} \overset{\text{f.d.d.}}{\longrightarrow} W(\alpha\theta)^{1/\alpha}(U_r)_{r\in\mathbb{N}_0} \quad \text{as } n \to \infty \tag{1.6}$$

where $\overset{\text{f.d.d.}}{\longrightarrow}$ denotes convergence of the finite-dimensional distributions.

Remark 1.2. Without further assumptions, the martingale convergence theorem implies that $W(\theta) := \lim_{n\to\infty} W_n(\theta)$ exists almost surely, but $\mathbb{P}(W(\theta) = 0) = 1$ may hold. However, the assumptions of Theorem 1.1 guarantee $\mathbb{E}[W(\theta)] = 1$. More precisely, notice that $p \mapsto m_\theta(p) := m(p\theta)/m(\theta)^p$ is convex with $m_\theta(1) = 1$ and $m_\theta(\alpha) = \kappa < 1$. Thus $m_\theta'(1) < 0$, which gives $\theta m'(\theta)/m(\theta) - \log(m(\theta)) \in [-\infty, 0)$. Further, $\mathbb{E}[W_1(\theta)\log^+(W_1(\theta))] < \infty$ is a consequence of (1.5). Therefore, the main result of [17] together with the subsequent remark give $\mathbb{E}[W(\theta)] = 1$.

On the other hand, the assumptions of our main result do not rule out the case where $\mathbb{P}(W(\alpha\theta) = 0) = 1$. In this situation, Theorem 1.1 remains valid, but the limit process in (1.6) is trivial.

Specializing Theorem 1.1 for $r = 0$, we obtain the following one-dimensional result.

Corollary 1.3. *Under the assumptions of Theorem 1.1,*

$$\kappa^{-n/\alpha}(W(\theta) - W_n(\theta)) \overset{\text{d}}{\to} W(\alpha\theta)^{1/\alpha}U_0 \quad \text{as } n \to \infty$$

where, for $t \in \mathbb{R}$,

$$\mathbb{E}\big[e^{\mathrm{i}tW(\alpha\theta)^{\frac{1}{\alpha}}U_0}\big] = \mathbb{E}\bigg[\exp\bigg(\frac{\Gamma(2-\alpha)}{\alpha-1}\frac{cW(\alpha\theta)}{1-\kappa}|t|^\alpha\Big(\cos\Big(\frac{\pi\alpha}{2}\Big) - \mathrm{i}\sin\Big(\frac{\pi\alpha}{2}\Big)\operatorname{sign}(t)\Big)\bigg)\bigg].$$

The limit distribution in Corollary 1.3 is a scale mixture of α-stable laws.

1.4. **Related literature.** Rate of convergence results in the form of a central limit theorem and a law of the iterated logarithm are given in [14], see also [11] for a recent interesting contribution in the setting of branching Brownian motion. There are various earlier results, but here we confine ourselves to referring to [14, p. 1182] for a thorough account of the literature.

The counterpart of our Corollary 1.3 for the Galton–Watson process was proved in [12]. In the setting of weighted branching processes, which includes the branching random walk as a special case, an analogue of our Corollary 1.3 was obtained in [21] (since [21] is not easily available we also refer to the conference paper [22], which is an abridged version of [21]) under the assumption $m((\alpha+\varepsilon)\theta) < \infty$ for some $\varepsilon > 0$. This assumption is not required here.

1.5. **Heuristics.** We continue with an informal discussion of why Theorem 1.1 should be true. From the representation of $W_{n+j}(\theta) - W_{n+j-1}(\theta)$ as a random weighted sum of i.i.d. copies of $W_1(\theta) - 1$ and the limit theory for independent, infinitesimal triangular arrays it is plausible that

$$\big(\kappa^{-n/\alpha}(W_{n+j}(\theta) - W_{n+j-1}(\theta))\big)_{j\in\mathbb{N}} = \Big(\kappa^{(j-1)/\alpha}\frac{W_{n+j}(\theta) - W_{n+j-1}(\theta)}{\kappa^{(n+j-1)/\alpha}}\Big)_{j\in\mathbb{N}} \stackrel{\text{f.d.d.}}{\longrightarrow} W(\alpha\theta)^{1/\alpha}(\kappa^{(j-1)/\alpha}Q_j)_{j\in\mathbb{N}}.$$

In view of this one may expect that, for fixed $r \in \mathbb{N}$ as $n \to \infty$,

$$\kappa^{-(n-r)/\alpha}(W(\theta) - W_{n-r}(\theta)) = \kappa^{-(n-r)/\alpha}\sum_{j\geq 1}(W_{n-r+j}(\theta) - W_{n-r+j-1}(\theta)) \stackrel{\text{d}}{\to} W(\alpha\theta)^{1/\alpha}\sum_{j\geq 1}\kappa^{(j-1)/\alpha}Q_j \stackrel{\text{law}}{=} W(\alpha\theta)^{1/\alpha}\,U_0.$$

Similarly, for $r_1, r_2 \in \mathbb{N}_0$, $r_1 < r_2$ one would expect that

$$\begin{aligned}
&\Big(\frac{W(\theta) - W_{n-r_1}(\theta)}{\kappa^{(n-r_1)/\alpha}}, \frac{W(\theta) - W_{n-r_2}(\theta)}{\kappa^{(n-r_2)/\alpha}}\Big)\\
&= \Big(\frac{W(\theta) - W_{n-r_1}(\theta)}{\kappa^{(n-r_1)/\alpha}}, \kappa^{(r_2-r_1)/\alpha}\frac{W(\theta) - W_{n-r_1}(\theta)}{\kappa^{(n-r_1)/\alpha}}\\
&\qquad + \kappa^{(r_2-r_1-1)/\alpha}\frac{W_{n-r_1}(\theta) - W_{n-r_1-1}(\theta)}{\kappa^{(n-r_1-1)/\alpha}} + \ldots + \frac{W_{n-r_2+1}(\theta) - W_{n-r_2}(\theta)}{\kappa^{(n-r_2)/\alpha}}\Big)\\
&\stackrel{\text{d}}{\to} W(\alpha\theta)^{1/\alpha}(U_0, \kappa^{(r_2-r_1)/\alpha}U_0 + \kappa^{(r_2-r_1-1)/\alpha}Q_1 + \ldots + Q_{r_2-r_1})\\
&= W(\alpha\theta)^{1/\alpha}(U_0, U_{r_2-r_1}) \stackrel{\text{law}}{=} W(\alpha\theta)^{1/\alpha}(U_{r_1}, U_{r_2})
\end{aligned}$$

having utilized the stationarity of $(U_r)_{r\in\mathbb{N}_0}$ for the last distributional equality.

1.6. **Examples.** Typically, our main result applies when the number of offspring N has a heavy tail while the displacements X_j are 'tame'. For instance, if

$$\mathbb{P}(N > x) \sim dx^{-\alpha} \quad \text{as } x \to \infty \tag{1.7}$$

for some $\alpha \in (1,2)$ and $d > 0$, and if $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables independent of N such that condition (1.4) holds, that is,

$$\mathbb{E}[e^{-\alpha\theta X_1}] < (\mathbb{E}[N])^{\alpha-1}(\mathbb{E}[e^{-\theta X_1}])^{\alpha} < \infty, \tag{1.8}$$

then (1.5) holds according to Proposition 4.3 in [9]. In particular, condition (1.8) is satisfied for all sufficiently small $\theta > 0$ if the X_j have a standard normal law.

On the other hand, one may wonder whether there are point processes $\mathcal{Z}$ with infinitely many points satisfying the assumptions (1.4) and (1.5) of Theorem 1.1. In [13] it is demonstrated that (1.4) and (1.5) are incompatible if $N = \mathcal{Z}(\mathbb{R}) = \infty$ almost surely and $\mathcal{Z}$ is either an inhomogeneous Poisson process or a point process with independent points. Now we show that a slight modification of the example given in the first paragraph of the section leads to a point process $\mathcal{Z}$ with $\mathbb{P}(N = \infty) = 1$ which satisfies the assumptions of Theorem 1.1. Let K be a random variable taking positive integer values with the same tail behavior as in (1.7). Further, let $Y_1, Y_2, \ldots$ be independent copies of a positive random variable Y such that the sequence $(Y_k)_{k\in\mathbb{N}}$ is independent of K. For some $a > 0$ to be specified below, set

$$X_k := Y_k \mathbb{1}_{\{K\geq k\}} + ak\mathbb{1}_{\{K<k\}}, \quad k \in \mathbb{N}.$$

Increasing d if necessary we can assume that $\mathbb{E}[K] > 1$ and then pick $\theta > 0$ and a such that

$$m(\theta) = \mathbb{E}\Big[\sum_{k\geq 1} e^{-\theta X_k}\Big] = \mathbb{E}[K]\mathbb{E}\big[e^{-\theta Y}\big] + (1-e^{-\theta a})^{-1}\mathbb{E}\big[e^{-\theta a(K+1)}\big] = 1.$$

This entails

$$\kappa = m(\alpha\theta) = \mathbb{E}\Big[\sum_{k\geq 1} e^{-\alpha\theta X_k}\Big] = \mathbb{E}[K]\mathbb{E}\big[e^{-\alpha\theta Y}\big] + (1-e^{-\alpha\theta a})^{-1}\mathbb{E}\big[e^{-\alpha\theta a(K+1)}\big] < 1,$$

so that (1.4) holds. By Proposition 4.3 in [9]

$$\mathbb{P}\Big(\sum_{k\geq 1} e^{-\theta Y_k}\mathbb{1}_{\{K\geq k\}} > x\Big) \sim (\mathbb{E}[e^{-\theta Y}])^\alpha \, dx^{-\alpha} \quad \text{as } x \to \infty.$$

Since $\sum_{k\geq 1} e^{-\theta ak}\mathbb{1}_{\{K<k\}} = (1-e^{-\theta a})^{-1}e^{-\theta a(K+1)}$ is almost surely nonnegative and bounded, we infer

$$\mathbb{P}(W_1(\theta) > x) = \mathbb{P}\Big(\sum_{k\geq 1} e^{-\theta Y_k}\mathbb{1}_{\{K\geq k\}} + \sum_{k\geq 1} e^{-\theta ak}\mathbb{1}_{\{K<k\}} > x\Big) \sim (\mathbb{E}[e^{-\theta Y}])^\alpha dx^{-\alpha}$$

as $x \to \infty$, that is, (1.5) holds.

2. Tail behavior in the branching random walk

An important ingredient in the proof of Theorem 1.1 is the following result on the tail behavior of the martingale $(W_n(\theta))_{n\in\mathbb{N}_0}$, which we believe is interesting in its own right. As usual, for a real number x, we define $x^\pm := (\pm x) \vee 0$.

Theorem 2.1. *Suppose there exist $\alpha \in (1, 2)$, $\varepsilon > 0$ and a function ℓ slowly varying at ∞ such that* (1.4) *holds, that*

$$m((\alpha+\varepsilon)\theta) < \infty \tag{2.1}$$

and that

$$\mathbb{P}(W_1(\theta) > x) \;\sim\; x^{-\alpha}\ell(x) \quad \text{as } x \to \infty. \tag{2.2}$$

Then, for any bounded sequence $(a_j)_{j\in\mathbb{N}_0}$, the series $\sum_{j\geq 0} a_j(W_{j+1}(\theta) - W_j(\theta))$ converges almost surely and in L_p for $p \in [1, \alpha)$. Furthermore, as $x \to \infty$,

$$\mathbb{P}\Big(\sum_{j\geq 0} a_j(W_{j+1}(\theta) - W_j(\theta)) > x\Big) \sim \textstyle\sum_{j\geq 0} \kappa^j (a_j^+)^\alpha \, \mathbb{P}(W_1(\theta) > x) \tag{2.3}$$

and

$$\mathbb{P}\bigg(\sum_{j\geq 0} a_j(W_{j+1}(\theta)-W_j(\theta)) < -x\bigg) \sim \textstyle\sum_{j\geq 0} \kappa^j (a_j^-)^\alpha\, \mathbb{P}(W_1(\theta)>x). \tag{2.4}$$

If (2.2) *holds with* $\lim_{x\to\infty} \ell(x) = c$ *for some* $c > 0$*, that is, if* (1.5) *holds, then* (1.4) *is sufficient for* (2.3) *and* (2.4) *(i.e.,* (2.1) *is not needed).*

Remark 2.2. Since $W_0(\theta) = 1$ almost surely, (2.3) with $a_j = 1$ for $j \in \mathbb{N}_0$ yields

$$\mathbb{P}(W(\theta) > x) \sim (1-\kappa)^{-1}\mathbb{P}(W_1(\theta) > x) \quad \text{as } x \to \infty. \tag{2.5}$$

This relation can be found in earlier literature in various guises. If $\mathbb{P}(N < \infty) = 1$, then $(W_n(0))_{n\in\mathbb{N}_0}$ is a supercritical normalized Galton-Watson process. In this case, (2.5) was proved in [4] for non-integer $\alpha > 1$ and in [7] for integer $\alpha \geq 2$. If $\theta > 0$, $\mathbb{P}(N < \infty) = 1$ and $\mathcal{Z}((-\infty, -\theta^{-1}\log m(\theta))) = 0$ almost surely, $W(\theta)$ can be viewed as a limit random variable in the Crump-Mode branching process. In this case, (2.5) was obtained in [5] for non-integer $\alpha > 1$. In the setting of the branching random walks a proof of relation (2.5) was sketched in [16]. A complete proof for non-integer $\alpha > 1$ along similar lines was given in 2003 in an unpublished diploma paper of Polotskiy (Kyiv). The techniques exploited in the aforementioned works are based on Laplace-Stieltjes transforms and Abelian and Tauberian theorems. In the more general setting of weighted branching processes limit theorems for triangular arrays were exploited in [21] to prove (2.5) under the extra assumption that the positions of the first generation individuals are almost surely bounded. An alternative probabilistic proof of (2.5) based on martingale theory was given in [13]. Unfortunately, this proof is flawed, and one purpose of the present paper is to give a correct probabilistic proof of (2.5) under optimal assumptions.

The rest of the paper is organized as follows. Theorems 2.1 and 1.1 are proved in Sections 3 and 4, respectively.

3. Proof of Theorem 2.1

Henceforth, we shall abbreviate $W_n(\theta)$ and $W(\theta)$ by W_n and W, respectively. Set $Y_u := e^{-\theta S(u)}/m^{|u|}(\theta)$ for $u \in \mathcal{I}$, so that $W_n = \sum_{|u|=n} Y_u$ for $n \in \mathbb{N}_0$.

Under the assumptions of Theorem 2.1, the function $m_\theta(p) := \mathbb{E}\big[\sum_{|u|=1} Y_u^p\big]$ is log-convex on $(1, \alpha)$, $m_\theta(1) = 1$ and $m_\theta(\alpha) = \kappa < 1$. Hence, $m_\theta(p) < 1$ for all $p \in (1, \alpha)$. We can thus choose $\delta \in (0, \alpha - 1)$ such that $m_\theta(\alpha + \delta) < 1$ and further

$$\mathbb{E}\bigg[\sum_{|u|=n} Y_u^{\alpha-\delta}\bigg] = m_\theta(\alpha-\delta)^n < 1 \quad \text{and} \quad \mathbb{E}\bigg[\sum_{|u|=n} Y_u^{\alpha+\delta}\bigg] = m_\theta(\alpha+\delta)^n < 1. \tag{3.1}$$

The second inequality in (3.1) implies in particular that

$$\sum_{|u|=n} Y_u^p < \infty \quad \text{a.s.} \tag{3.2}$$

for all $p \in [1, \alpha + \delta]$.

For $k \in \mathbb{N}_0$, the random variable W_k is a function of the family $(\mathcal{Z}_v)_{v\in\mathcal{I}}$. For any $u \in \mathcal{I}$, we define $W_k^{(u)}$ to be the same function applied to the family $(\mathcal{Z}_{uv})_{v\in\mathcal{I}}$, and $W^{(u)} := \lim_{k\to\infty} W_k^{(u)}$ a.s. We shall use the decomposition

$$W_{n+1} - W_n = \sum_{|u|=n} Y_u(W_1^{(u)} - 1).$$

Observe that the Y_u, $|u|=n$ are $\mathcal{F}_n$-measurable, whereas the $W_1^{(u)}$, $|u|=n$ are i.i.d., independent of $\mathcal{F}_n$ and have the same law as W_1. In what follows, we write $\mathbb{P}_n(\cdot)$ and $\mathbb{E}_n[\cdot]$ for $\mathbb{P}(\cdot|\mathcal{F}_n)$ and $\mathbb{E}[\cdot|\mathcal{F}_n]$, respectively, and set $F(x) := \mathbb{P}(|W_1-1|\leq x)$, $x\in\mathbb{R}$.

Put $R_n := \sum_{j=0}^n a_j(W_{j+1}-W_j)$ for $n\in\mathbb{N}_0$. The sequence $(R_n,\mathcal{F}_{n+1})_{n\in\mathbb{N}_0}$ is a martingale. To ensure that the martingale converges a.s. and in L_p for $p\in(1,\alpha)$ it suffices to show that it is L_p-bounded. The L_p-boundedness follows from

$$\begin{aligned}\sup_{n\geq 0}\mathbb{E}[|R_n|^p] &\leq 4\sum_{n\geq 1}\mathbb{E}[|R_n-R_{n-1}|^p]+\mathbb{E}[|R_0|^p]\leq 4\sum_{n\geq 0}|a_n|^p\mathbb{E}[|W_{n+1}-W_n|^p]\\ &= 4\sum_{n\geq 0}|a_n|^p\mathbb{E}\Big[\mathbb{E}_n\Big|\sum_{|u|=n}Y_u(W_1^{(u)}-1)\Big|^p\Big]\\ &\leq 16\mathbb{E}[|W_1-1|^p]\sum_{n\geq 0}|a_n|^p m_\theta(p)^n<\infty\end{aligned}$$

where the first and third inequalities are obtained with the help of the Topchii-Vatutin inequality for martingales [24, Theorem 2], and $\mathbb{E}[|W_1-1|^p]<\infty$ is a consequence of (2.2).

Throughout the rest of this section we assume, without loss of generality, that $\sup_{j\geq 0}|a_j|\leq 1$. Passing to the proof of (2.3) we first show that there exists some $x_0>0$ that does not depend on n such that for all $x\geq x_0$, we have

$$\frac{\mathbb{P}_n(|\sum_{j\geq n}a_j(W_{j+1}-W_j)|>x)}{1-F(x)}\leq C\sum_{j\geq n}|a_j|^{\alpha-\delta}\mathbb{E}_n[\Xi_j]\quad\text{a.s.}\tag{3.3}$$

where C is a finite, deterministic constant that does not depend on n or x_0 and

$$\Xi_n=\sum_{|u|=n}Y_u^{\alpha-\delta}+\sum_{|u|=n}Y_u^{\alpha+\delta}\tag{3.4}$$

for some δ satisfying (3.1). Note that

$$\mathbb{E}[\Xi_n]=m_\theta(\alpha-\delta)^n+m_\theta(\alpha+\delta)^n<\infty\quad\text{and}\quad\mathbb{E}\Big[\sum_{n\geq 0}\Xi_n\Big]<\infty.\tag{3.5}$$

For typographical ease, set $Q:=W_1-1$, $Q_u:=W_1^{(u)}-1$ and $Y_{u,a}:=a_{|u|}Y_u$. For any fixed $n\in\mathbb{N}_0$ and $x>0$, we infer

$$\begin{aligned}&\mathbb{P}_n\Big(\Big|\sum_{j\geq n}a_j(W_{j+1}-W_j)\Big|>x\Big)\\ &=\mathbb{P}_n\Big(\Big|\sum_{|u|\geq n}Y_{u,a}Q_u\Big|>x,\ \sup_{|u|\geq n}|Y_{u,a}Q_u|>x\Big)\\ &\quad+\mathbb{P}_n\Big(\Big|\sum_{|u|\geq n}Y_{u,a}Q_u\Big|>x,\ \sup_{|u|\geq n}|Y_{u,a}Q_u|\leq x\Big)\\ &\leq\mathbb{P}_n\Big(\sup_{|u|\geq n}|Y_{u,a}Q_u|>x\Big)+\mathbb{P}_n\Big(\Big|\sum_{|u|\geq n}Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}\Big|>x\Big).\end{aligned}$$

We set $\mathbb{P}_n\big(\sup_{|u|\geq n}|Y_{u,a}Q_u|>x\big)=:I_1(n,x)$ and

$$\begin{aligned}\mathbb{P}_n&\bigg(\Big|\sum_{|u|\geq n}Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}\Big|>x\bigg)\\ &=\ \mathbb{P}_n\bigg(\Big|\sum_{|u|\geq n}\big(Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}-\mathbb{E}_{|u|}[Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}]\big)\Big|>\frac{x}{2}\bigg)\\ &\quad+\mathbb{P}_n\bigg(\Big|\sum_{|u|\geq n}\mathbb{E}_{|u|}[Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}]\Big|>\frac{x}{2}\bigg)=:I_2(n,x)+I_3(n,x).\end{aligned}$$

Put $T(x):=\int_{[0,\,x]}y^2\,\mathrm{d}F(y)$ and $R(x):=\int_{(x,\infty)}y\,\mathrm{d}F(y)$ for $x>0$. By Karamata's theorem (Theorems 1.6.4 and 1.6.5 in [6])

$$T(x)\sim\tfrac{\alpha}{2-\alpha}x^2(1-F(x))\sim\tfrac{\alpha}{2-\alpha}x^{2-\alpha}\ell(x)$$

and

$$R(x)\sim\tfrac{\alpha}{\alpha-1}x(1-F(x))\sim\tfrac{\alpha}{\alpha-1}x^{1-\alpha}\ell(x)$$

as $x\to\infty$. For any $A>0$ and $\delta>0$ satisfying (3.1), there exists $x_0>0$ such that, whenever $x\geq x_0$, we have

$$x^{\alpha+\delta}(1-F(x))\geq 1/A; \tag{3.6}$$

$$x^{\alpha-2+\delta}T(x)\geq 1/A; \tag{3.7}$$

$$x^{\alpha-1+\delta}R(x)\geq 1/A; \tag{3.8}$$

$$T(x)\leq\big(A+\tfrac{\alpha}{2-\alpha}\big)x^2(1-F(x)):=B_1x^2(1-F(x)); \tag{3.9}$$

$$R(x)\leq\big(A+\tfrac{\alpha}{\alpha-1}\big)x(1-F(x)):=B_2x(1-F(x)). \tag{3.10}$$

Also, x_0 can be chosen so large that (with the same δ as before) whenever $x\wedge(ux)\geq x_0$, we have

$$\tfrac{1-F(ux)}{1-F(x)}\leq A(u^{-\alpha+\delta}\vee u^{-\alpha-\delta}); \tag{3.11}$$

$$\tfrac{T(ux)}{T(x)}\leq A(u^{2-\alpha+\delta}\vee u^{2-\alpha-\delta}); \tag{3.12}$$

$$\tfrac{R(ux)}{R(x)}\leq A(u^{1-\alpha+\delta}\vee u^{1-\alpha-\delta}). \tag{3.13}$$

Inequalities (3.11) through (3.13) follow from Potter's bound (Theorem 1.5.6(iii) in [6]). While constructing bounds for I_1, I_2 and I_3 below we tacitly assume that $x\geq x_0$.

A BOUND FOR I_1. Write

$$\begin{aligned}\frac{I_1(n,x)}{1-F(x)}&=\frac{\mathbb{P}_n\big(\sup_{|u|\geq n}|Y_{u,a}Q_u|>x\big)}{1-F(x)}\leq\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{\mathbb{P}_{|u|}(|Y_{u,a}Q_u|>x)}{1-F(x)}\bigg]\\ &=\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{1-F(x/Y_{u,a})}{1-F(x)}\bigg]=\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{1-F(x/Y_{u,a})}{1-F(x)}\mathbb{1}_{\{|Y_{u,a}|>x/x_0\}}\bigg]\\ &\quad+\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{1-F(x/Y_{u,a})}{1-F(x)}\mathbb{1}_{\{|Y_{u,a}|\leq x/x_0\}}\bigg]=:I_{11}(n,x)+I_{12}(n,x).\end{aligned}$$

For $|u|\geq n$, we have

$$|Y_{u,a}|^{\alpha+\delta}\geq|Y_{u,a}|^{\alpha+\delta}\mathbb{1}_{\{|Y_{u,a}|>x/x_0\}}\geq(x/x_0)^{\alpha+\delta}\mathbb{1}_{\{|Y_{u,a}|>x/x_0\}}.$$

From this, we conclude that

$$I_{11}(n,x) \leq x_0^{\alpha+\delta}\mathbb{E}_n\bigg[\frac{\sum_{|u|\geq n}|Y_{u,a}|^{\alpha+\delta}}{x^{\alpha+\delta}(1-F(x))}\bigg] \leq Ax_0^{\alpha+\delta}\mathbb{E}_n\bigg[\sum_{|u|\geq n}|Y_{u,a}|^{\alpha+\delta}\bigg]$$

by (3.6). Further, we obtain with the help of (3.11)

$$I_{12}(n,x) \leq A\mathbb{E}_n\bigg[\sum_{|u|\geq n}\big(|Y_{u,a}|^{\alpha-\delta}\vee|Y_{u,a}|^{\alpha+\delta}\big)\bigg] \leq A\mathbb{E}_n\bigg[\sum_{j\geq n}|a_j|^{\alpha-\delta}\Xi_j\bigg].$$

A BOUND FOR I_2. By Markov's inequality

$$\begin{aligned}(x/2)^2 I_2(n,x) &\leq \mathbb{E}_n\bigg[\bigg(\sum_{|u|\geq n}\big(Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}} - \mathbb{E}_{|u|}[Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}]\big)\bigg)^2\bigg]\\ &\leq \mathbb{E}_n\bigg[\sum_{|u|\geq n}(Y_{u,a})^2Q_u^2\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}\bigg],\end{aligned}$$

as the expectations of the cross terms vanish. By virtue of (3.9) we get

$$\begin{aligned}\frac{I_2(n,x)}{4(1-F(x))} &\leq \mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{Y_{u,a}^2\int_0^{x/|Y_{u,a}|}y^2\,\mathrm{d}F(y)}{x^2(1-F(x))}\bigg] \leq B_1\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{Y_{u,a}^2T(x/|Y_{u,a}|)}{T(x)}\bigg]\\ &= B_1\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{Y_{u,a}^2T(\frac{x}{|Y_{u,a}|})}{T(x)}\mathbb{1}_{\{|Y_{u,a}|>\frac{x}{x_0}\}} + \sum_{|u|\geq n}\frac{Y_{u,a}^2T(\frac{x}{|Y_{u,a}|})}{T(x)}\mathbb{1}_{\{|Y_{u,a}|\leq\frac{x}{x_0}\}}\bigg]\\ &=: B_1(I_{21}(n,x)+I_{22}(n,x)).\end{aligned}$$

We use (3.7) and the trivial inequality $T(x)\leq x^2$ for $x\geq 0$ to obtain

$$\begin{aligned}I_{21}(n,x) &= \mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{|Y_{u,a}|^{\alpha+\delta}(x/|Y_{u,a}|)^{\alpha-2+\delta}T(x/|Y_{u,a}|)}{x^{\alpha-2+\delta}T(x)}\mathbb{1}_{\{Y_{u,a}>x/x_0\}}\bigg]\\ &\leq A\max_{y\in[0,x_0]}(y^{\alpha-2+\delta}T(y))\mathbb{E}_n\bigg[\sum_{|u|\geq n}|Y_{u,a}|^{\alpha+\delta}\bigg] \leq Ax_0^{\alpha+\delta}\mathbb{E}_n\bigg[\sum_{|u|\geq n}|Y_{u,a}|^{\alpha+\delta}\bigg].\end{aligned}$$

Further, as a consequence of (3.12),

$$I_{22}(n,x) \leq A\mathbb{E}_n\bigg[\sum_{|u|\geq n}Y_{u,a}^2(|Y_{u,a}|^{\alpha-2-\delta}\vee|Y_{u,a}|^{\alpha-2+\delta})\bigg] \leq A\mathbb{E}_n\bigg[\sum_{j\geq n}|a_j|^{\alpha-\delta}\Xi_j\bigg].$$

A BOUND FOR I_3. We first observe that for $|u|\geq n$

$$\begin{aligned}\mathbb{E}_n[Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}] &= \mathbb{E}_n\bigg[Y_{u,a}\int_{\{|y|\leq x/|Y_{u,a}|\}}y\,\mathrm{d}\mathbb{P}(Q\leq y)\bigg]\\ &= -\mathbb{E}_n\bigg[Y_{u,a}\int_{\{|y|>x/|Y_{u,a}|\}}y\,\mathrm{d}\mathbb{P}(Q\leq y)\bigg]\end{aligned}$$

whence

$$|\mathbb{E}_n[Y_{u,a}Q_u\mathbb{1}_{\{|Y_{u,a}Q_u|\leq x\}}]| \leq \mathbb{E}_n\Big[|Y_{u,a}|\int_{(x/|Y_{u,a}|,\infty)}y\,\mathrm{d}F(y)\Big] = \mathbb{E}_n\Big[|Y_{u,a}|R(x/|Y_{u,a}|)\Big].$$

Consequently, by Markov's inequality and (3.10),

$$\begin{aligned}\frac{I_3(n,x)}{2(1-F(x))} &\leq \mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{|Y_{u,a}|R(x/|Y_{u,a}|)}{x(1-F(x))}\bigg] \leq B_2\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{|Y_{u,a}|R(x/|Y_{u,a}|)}{R(x)}\bigg] \\ &= B_2\mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{|Y_{u,a}|R(x/|Y_{u,a}|)}{R(x)}\mathbb{1}_{\{|Y_{u,a}|>x/x_0\}} \\ &\qquad + \sum_{|u|\geq n}\frac{|Y_{u,a}|R(x/|Y_{u,a}|)}{R(x)}\mathbb{1}_{\{|Y_{u,a}|\leq x/x_0\}}\bigg] \\ &=: B_2(I_{31}(n,x)+I_{32}(n,x)).\end{aligned}$$

Using (3.8) and the fact that $R(x)$ is nonincreasing we conclude that

$$\begin{aligned}I_{31}(n,x) &= \mathbb{E}_n\bigg[\sum_{|u|\geq n}\frac{|Y_{u,a}|_u^{\alpha+\delta}(x/|Y_{u,a}|)^{\alpha-1+\delta}R(x/|Y_{u,a}|)}{x^{\alpha-1+\delta}R(x)}\mathbb{1}_{\{|Y_{u,a}|>x/x_0\}}\bigg] \\ &\leq A\mathbb{E}_n\bigg[\max_{y\in[0,x_0]}(y^{\alpha-1+\delta}R(y))\sum_{|u|\geq n}|Y_{u,a}|^{\alpha+\delta}\bigg] \\ &\leq A\,\mathbb{E}[|W_1-1|]x_0^{\alpha-1+\delta}\mathbb{E}_n\bigg[\sum_{|u|\geq n}|Y_{u,a}|^{\alpha+\delta}\bigg].\end{aligned}$$

Finally, by (3.13),

$$\begin{aligned}I_{32}(n,x) &\leq A\mathbb{E}_n\bigg[\sum_{|u|\geq n}|Y_{u,a}|\Big(|Y_{u,a}|^{\alpha-1-\delta}\vee|Y_{u,a}|^{\alpha-1+\delta}\Big)\bigg] \\ &\leq A\mathbb{E}_n\bigg[\sum_{|u|\geq n}|Y_{u,a}|^{\alpha-\delta}+\sum_{|u|\geq n}|Y_{u,a}|^{\alpha+\delta}\bigg].\end{aligned}$$

The preceding inequalities imply (3.3) with Ξ_k as defined in (3.4).

Now some preparatory work has to be done for the next part of the proof. Since $\mathbb{E}[W_1-1]=0$, $\mathbb{P}(|W_1-1|>x)\sim x^{-\alpha}\ell(x)$ by (2.2), $\mathbb{P}(W_1-1<-x)=0$ for $x>1$ and $\sum_{|u|=n}Y_u^{\alpha-\delta}<\infty$ a.s. for any $n\in\mathbb{N}$ as a consequence of (3.2), Lemma A.3 in [18] or Theorem 2.2 in [15] give that, as $x\to\infty$,

$$\mathbb{P}_n(W_{n+1}-W_n>x)\sim\textstyle\sum_{|u|=n}Y_u^{\alpha}(1-F(x))\quad\text{a.s.}\tag{3.14}$$

This in combination with (3.3) and Lebesgue's dominated convergence theorem enables us to conclude that, as $x\to\infty$,

$$\mathbb{P}(W_{n+1}-W_n>x)\sim m_\theta(\alpha)^n(1-F(x)),\quad n\in\mathbb{N}_0.\tag{3.15}$$

Alternatively, using an inductive argument relation (3.15) can be deduced from Theorem 2.1 in [19] and the remark following Theorem 2.2 in [19].

We are ready to finish the proof of (2.3). We claim that

$$\mathbb{P}\bigg(\sum_{j=0}^{k}a_j(W_{j+1}(\theta)-W_j(\theta))>x\bigg)\sim\sum_{j=0}^{k}\kappa^j(a_j^+)^\alpha\,\mathbb{P}(W_1(\theta)>x)\tag{3.16}$$

for $k\in\mathbb{N}_0$. This will be proved by induction on k.
For $k=0$, (3.16) is (2.2), which is an assumption.

Now suppose that (3.16) holds for fixed $k \in \mathbb{N}$. Then, for $x > 0$ and $\rho \in (0,1)$,

$$
\begin{aligned}
&\mathbb{P}\Big(\sum_{j=0}^{k+1} a_j(W_{j+1}-W_j) > x\Big) \\
&\quad \leq \mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1-\rho)x\Big) + \mathbb{P}\big(a_{k+1}(W_{k+2}-W_{k+1}) > (1-\rho)x\big) \\
&\qquad + \mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > \rho x, a_{k+1}(W_{k+2}-W_{k+1}) > \rho x\Big) \\
&\quad = \mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1-\rho)x\Big) + \mathbb{P}\big(a_{k+1}(W_{k+2}-W_{k+1}) > (1-\rho)x\big) \\
&\qquad + \mathbb{E}\Big[\mathbb{1}_{\{\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > \rho x\}} \mathbb{P}_{k+1}\big(a_{k+1}(W_{k+2}-W_{k+1}) > \rho x\big)\Big],
\end{aligned}
$$

where we used the fact that the variable $\sum_{j=0}^{k} a_j(W_{j+1}-W_j)$ is $\mathcal{F}_{k+1}$-measurable. Set $\zeta_1 := 0$, $\zeta_2 := ((a_{k+1}^+)/\rho)^\alpha \sum_{|u|=k+1} Y_u^\alpha$ and, for $x > 0$,

$$\zeta_1(x) := \mathbb{1}_{\{\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > \rho x\}} \frac{\mathbb{P}_{k+1}(a_{k+1}(W_{k+2}-W_{k+1}) > \rho x)}{1-F(x)},$$

$$\zeta_2(x) := \frac{\mathbb{P}_{k+1}(a_{k+1}(W_{k+2}-W_{k+1}) > \rho x)}{1-F(x)}.$$

In view of (3.14), we have $\lim_{x\to\infty} \zeta_1(x) = \zeta_1$ a. s. and $\lim_{x\to\infty} \zeta_2(x) = \zeta_2$ a. s. Further, $\lim_{x\to\infty} \mathbb{E}[\zeta_2(x)] = \mathbb{E}[\zeta_2]$ by (3.15). Since, for $x > 0$, we have $0 \leq \zeta_1(x) \leq \zeta_2(x)$ a. s., we can invoke Pratt's lemma [20] to get $\lim_{x\to\infty} \mathbb{E}[\zeta_1(x)] = \mathbb{E}[\zeta_1]$. Hence,

$$\lim_{x\to\infty} \frac{\mathbb{P}\big(\sum_{i=0}^{k} a_i(W_{i+1}-W_i) > \rho x, a_{k+1}(W_{k+2}-W_{k+1}) > \rho x\big)}{1-F(x)} = 0. \tag{3.17}$$

By the induction hypothesis, (3.15) and (3.17)

$$\limsup_{x\to\infty} \frac{\mathbb{P}\big(\sum_{j=0}^{k+1} a_j(W_{j+1}-W_j) > x\big)}{1-F(x)} \leq (1-\rho)^{-\alpha} \sum_{j=0}^{k+1} m_\theta(\alpha)^j (a_j^+)^\alpha.$$

Letting $\rho \downarrow 0$ yields

$$\limsup_{x\to\infty} \frac{\mathbb{P}\big(\sum_{j=0}^{k+1} a_j(W_{j+1}-W_j) > x\big)}{1-F(x)} \leq \sum_{j=0}^{k+1} m_\theta(\alpha)^j (a_j^+)^\alpha. \tag{3.18}$$

We now derive the corresponding inequality for the limit inferior. To this end, for $x > 0$ and $\rho > 0$, we write

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{j=0}^{k+1} a_j(W_{j+1}-W_j) > x\Big) \\
&\quad \geq \mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1+\rho)x, |a_{k+1}(W_{k+2}-W_{k+1})| \leq \rho x\Big) \\
&\qquad + \mathbb{P}\Big(a_{k+1}(W_{k+2}-W_{k+1}) > (1+\rho)x, \Big|\sum_{j=0}^{k} a_j(W_{j+1}-W_j)\Big| \leq \rho x\Big) \\
&\quad = \mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1+\rho)x\Big) \\
&\qquad - \mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1+\rho)x, |a_{k+1}(W_{k+2}-W_{k+1})| > \rho x\Big) \\
&\qquad + \mathbb{P}\big(a_{k+1}(W_{k+2}-W_{k+1}) > (1+\rho)x\big) \\
&\qquad - \mathbb{P}\Big(a_{k+1}(W_{k+2}-W_{k+1}) > (1+\rho)x, |\sum_{j=0}^{k} a_j(W_{j+1}-W_j)| > \rho x\Big).
\end{aligned}$$

The argument that led to (3.17) applies here as well. It gives

$$\lim_{x\to\infty} \frac{\mathbb{P}\big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1+\rho)x, |a_{k+1}(W_{k+2}-W_{k+1})| > \rho x\big)}{1-F(x)} = 0 \quad (3.19)$$

and

$$\lim_{x\to\infty} \frac{\mathbb{P}\big(a_{k+1}(W_{k+2}-W_{k+1}) > (1+\rho)x, |\sum_{j=0}^{k} a_j(W_{j+1}-W_j)| > \rho x\big)}{1-F(x)} = 0. \quad (3.20)$$

By the induction hypothesis, (3.15), (3.19) and (3.20)

$$\liminf_{x\to\infty} \frac{\mathbb{P}\big(\sum_{j=0}^{k+1} a_j(W_{j+1}-W_j) > x\big)}{1-F(x)} \geq (1+\rho)^{-\alpha} \sum_{j=0}^{k+1} m_\theta(\alpha)^j (a_j^+)^\alpha.$$

Upon letting $\rho \downarrow 0$, we obtain

$$\liminf_{x\to\infty} \frac{\mathbb{P}\big(\sum_{j=0}^{k+1} a_j(W_{j+1}-W_j) > x\big)}{1-F(x)} \geq \sum_{j=0}^{k+1} m_\theta(\alpha)^j (a_j^+)^\alpha. \quad (3.21)$$

Combining (3.18) and (3.21) gives (3.16) for $k+1$, thereby proving (3.16) in general.

To check (2.3) we fix $k \in \mathbb{N}_0$, $x > 0$ and $\rho \in (0,1)$, and write

$$\mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1+\rho)x\Big) - \mathbb{P}\Big(\Big|\sum_{j\geq k+1} a_j(W_{j+1}-W_j)\Big| > \rho x\Big) \quad (3.22)$$
$$\leq \mathbb{P}\Big(\sum_{j\geq 0} a_j(W_{j+1}-W_j) > x\Big)$$
$$\leq \mathbb{P}\Big(\sum_{j=0}^{k} a_j(W_{j+1}-W_j) > (1-\rho)x\Big) + \mathbb{P}\Big(\Big|\sum_{j\geq k+1} a_j(W_{j+1}-W_j)\Big| > \rho x\Big).$$

From (3.22), (3.16) and (3.3), we infer

$$(1+\rho)^{-\alpha}\sum_{j=0}^{k} m_\theta(\alpha)^j (a_j^+)^\alpha - C\rho^{-\alpha}\sum_{j\geq k+1} |a_j|^{\alpha-\delta}\mathbb{E}[\Xi_j]$$
$$\leq \liminf_{x\to\infty} \frac{\mathbb{P}(\sum_{j\geq 0} a_j(W_{j+1}-W_j) > x)}{1-F(x)}$$
$$\leq \limsup_{x\to\infty} \frac{\mathbb{P}(\sum_{j\geq 0} a_j(W_{j+1}-W_j) > x)}{1-F(x)}$$
$$\leq (1-\rho)^{-\alpha}\sum_{j=0}^{k} m_\theta(\alpha)^j (a_j^+)^\alpha + C\rho^{-\alpha}\sum_{j\geq k+1} |a_j|^{\alpha-\delta}\mathbb{E}[\Xi_j]$$

Letting $k\to\infty$ and then $\rho \downarrow 0$, we arrive at (2.3). The proof of (2.4) is analogous, hence omitted.

A perusal of the proof above reveals that the need for condition (2.1) is only motivated by the use of Potter's bound, see (3.11), (3.12) and (3.13). If $\lim_{x\to\infty} \ell(x) = c$, that is, condition (1.5) holds, inequality (3.11) can be replaced by the following: for any $A > 1$ there exists $x_0 > 0$ such that whenever $x \geq x_0$ and $ux \geq x_0$,

$$\tfrac{1-F(ux)}{1-F(x)} \leq Au^{-\alpha},$$

likewise for (3.12) and (3.13) (with the same x_0 as x_0 can be increased if necessary). This shows that condition (2.1) is no longer needed, (1.4) being sufficient. The proof of Theorem 2.1 is complete.

4. Proof of Theorem 1.1

Our proof of Theorem 1.1 is essentially based on the following result in combination with Theorem 2.1.

Lemma 4.1. *Let $V = f((\mathcal{Z}(u))_{u\in\mathcal{I}})$ for a measurable function f such that $\mathbb{E}[V] = 0$ and*

$$\mathbb{P}(V > x) \sim c_1 x^{-\alpha} \quad \text{and} \quad \mathbb{P}(-V > x) \sim c_2 x^{-\alpha}, \quad x\to\infty \quad (4.1)$$

for some $\alpha \in (1,2)$ and finite $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$. Further, suppose that $m(\alpha\theta) < \infty$ ((1.4) is not required). For $n \in \mathbb{N}$, set

$$\Theta_n := m(\alpha\theta)^{-n/\alpha}\sum_{|u|=n} e^{-\theta S(u)} V^{(u)}, \quad (4.2)$$

where $V^{(u)} = f((\mathcal{Z}(uv))_{v\in\mathcal{I}})$ *for* $u \in \mathcal{I}$. *Then, for* $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}\big[\exp(\mathrm{i}t\Theta_n)\big] = \mathbb{E}\Big[\exp\Big(\frac{\Gamma(2-\alpha)}{\alpha-1}W(\alpha\theta)|t|^\alpha\Big((c_1+c_2)\cos\Big(\frac{\pi\alpha}{2}\Big) - \mathrm{i}(c_1-c_2)\sin\Big(\frac{\pi\alpha}{2}\Big)\,\mathrm{sign}(t)\Big)\Big)\Big]. \tag{4.3}$$

Proof. Since, conditionally given $\mathcal{F}_n$, Θ_n is a weighted sum of i.i.d. random variables, (4.3) follows from the classical limit theory for triangular arrays.

Suppose we can check that, for every $x > 0$,

$$L(x) := -\lim_{n\to\infty} \sum_{|u|=n} \mathbb{P}_n\Big(\frac{e^{-\theta S(u)}V^{(u)}}{m(\alpha\theta)^{n/\alpha}} > x\Big) = -c_1 x^{-\alpha}W(\alpha\theta) \quad \text{a. s.}; \tag{4.4}$$

$$L(-x) := \lim_{n\to\infty} \sum_{|u|=n} \mathbb{P}_n\Big(\frac{e^{-\theta S(u)}V^{(u)}}{m(\alpha\theta)^{n/\alpha}} \leq -x\Big) = c_2 x^{-\alpha}W(\alpha\theta) \quad \text{a. s.}; \tag{4.5}$$

$$\sigma^2 := \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} \sum_{|u|=n} \mathbb{V}\mathrm{ar}_n\left[\frac{e^{-\theta S(u)}V^{(u)}}{m(\alpha\theta)^{n/\alpha}}\mathbb{1}_{\left\{\frac{e^{-\theta S(u)}|V^{(u)}|}{m(\alpha\theta)^{n/\alpha}}\leq\varepsilon\right\}}\right] = 0 \quad \text{a. s.} \tag{4.6}$$

and

$$a_0(\tau) := \lim_{n\to\infty} \sum_{|u|=n} \mathbb{E}_n\left[\frac{e^{-\theta S(u)}V^{(u)}}{m(\alpha\theta)^{n/\alpha}}\mathbb{1}_{\{|e^{-\theta S(u)}V^{(u)}|\leq\tau m(\alpha\theta)^{n/\alpha}\}}\right] = -\tau^{1-\alpha}\frac{\alpha(c_1-c_2)}{\alpha-1}W(\alpha\theta) \quad \text{a. s.} \tag{4.7}$$

for each $\tau > 0$. Then, according to Theorem 1 on p. 116 in [10],

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_n\big[\mathrm{i}t\Theta_n\big] \\
&= \exp\Big(\mathrm{i}at - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}}\Big(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\Big)\,\mathrm{d}L(x)\Big) \\
&= \exp\Big(-\alpha c_2 W(\alpha\theta)\Big(\frac{\mathrm{i}\pi t}{2\cos(\frac{\pi\alpha}{2})} - \int_{-\infty}^0\Big(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\Big)|x|^{-\alpha-1}\,\mathrm{d}x\Big)\Big) \\
&\quad\cdot\exp\Big(\alpha c_1 W(\alpha\theta)\Big(\frac{\mathrm{i}\pi t}{2\cos(\frac{\pi\alpha}{2})} + \int_0^\infty\Big(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\Big)x^{-\alpha-1}\,\mathrm{d}x\Big)\Big) \text{ a. s.}
\end{aligned} \tag{4.8}$$

for $t \in \mathbb{R}$. Here,

$$a := a_0(\tau) - \int_{[-\tau,\,\tau]}\frac{x^3}{1+x^2}\,\mathrm{d}L(x) + \int_{\mathbb{R}\setminus[-\tau,\,\tau]}\frac{x}{1+x^2}\,\mathrm{d}L(x) = \frac{\alpha(c_1-c_2)\pi W(\alpha\theta)}{2\cos(\frac{\pi\alpha}{2})}$$

as a consequence of

$$\int_0^\tau \frac{x^{2-\alpha}}{1+x^2}\,\mathrm{d}x - \int_\tau^\infty \frac{x^{-\alpha}}{1+x^2}\,\mathrm{d}x = \int_0^\infty \frac{x^{2-\alpha}}{1+x^2}\,\mathrm{d}x - \int_\tau^\infty x^{-\alpha}\,\mathrm{d}x = -\frac{\pi}{2\cos(\frac{\pi\alpha}{2})} - \frac{\tau^{1-\alpha}}{\alpha-1}.$$

The last equality follows from

$$\int_0^\infty \frac{x^{2-\alpha}}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^1 x^{(1-\alpha)/2}(1-x)^{-(3-\alpha)/2}\,\mathrm{d}x = \frac{1}{2}\Gamma\Big(\frac{3-\alpha}{2}\Big)\Gamma\Big(1-\frac{3-\alpha}{2}\Big)$$
$$= \frac{\pi}{2\sin\big(\frac{\pi(3-\alpha)}{2}\big)} = -\frac{\pi}{2\cos(\frac{\pi\alpha}{2})}. \tag{4.9}$$

In view of (4.9) the right-hand side of (4.8) equals

$$\exp\left(\alpha c_2 W(\alpha\theta)\int_{-\infty}^0 \big(e^{\mathrm{i}tx}-1-\mathrm{i}tx\big)|x|^{-\alpha-1}\,\mathrm{d}x\right)\Big)$$
$$\cdot\exp\left(\alpha c_1 W(\alpha\theta)\int_0^\infty \big(e^{\mathrm{i}tx}-1-\mathrm{i}tx\big)x^{-\alpha-1}\,\mathrm{d}x\right)\Big)$$
$$= \exp\left(\frac{\Gamma(2-\alpha)}{\alpha-1}c_2 W(\alpha\theta)|t|^\alpha\big(\cos(\tfrac{\pi\alpha}{2})+\mathrm{i}\sin(\tfrac{\pi\alpha}{2})\ \mathrm{sign}(t)\big)\right)$$
$$\cdot\exp\left(\frac{\Gamma(2-\alpha)}{\alpha-1}c_1 W(\alpha\theta)|t|^\alpha\big(\cos(\tfrac{\pi\alpha}{2})-\mathrm{i}\sin(\tfrac{\pi\alpha}{2})\ \mathrm{sign}(t)\big)\right)$$
$$= \exp\left(\frac{\Gamma(2-\alpha)}{\alpha-1}W(\alpha\theta)|t|^\alpha\big((c_1+c_2)\cos(\tfrac{\pi\alpha}{2})-\mathrm{i}(c_1-c_2)\sin(\tfrac{\pi\alpha}{2})\ \mathrm{sign}(t)\big)\right)$$

having utilized the first formula given on p. 170 in [10] for the penultimate equality. Now (4.3) is secured by (4.8), the last displayed formula and Lebesgue's dominated convergence theorem.

Next, we are passing to the proofs of (4.4) through (4.7).

Proofs of (4.4) and (4.5). We start by recalling that, by Theorem 3 in [3],

$$\lim_{n\to\infty}\sup_{|u|=n}\frac{e^{-\theta S(u)}}{m(\alpha\theta)^{n/\alpha}} = 0 \quad \text{a. s.} \tag{4.10}$$

Using this in combination with (4.1) gives, for any $x>0$,

$$\sum_{|u|=n}\mathbb{P}_n\Big(\frac{e^{-\theta S(u)}V^{(u)}}{m(\alpha\theta)^{n/\alpha}}>x\Big) \sim \sum_{|u|=n} c_1\big(xe^{\theta S(u)}m(\alpha\theta)^{n/\alpha}\big)^{-\alpha}$$
$$= c_1 x^{-\alpha}W_n(\alpha\theta) \ \to\ c_1 x^{-\alpha}W(\alpha\theta) \quad \text{a. s.}$$

as $n\to\infty$. This proves (4.4). The proof of (4.5) is analogous.

Proof of (4.6). For $\varepsilon>0$,

$$\sum_{|u|=n}\mathbb{V}\mathrm{ar}_n\Big[\frac{e^{-\theta S(u)}V^{(u)}}{m(\alpha\theta)^{n/\alpha}}\mathbb{1}_{\big\{\frac{e^{-\theta S(u)}|V^{(u)}|}{m(\alpha\theta)^{n/\alpha}}\le\varepsilon\big\}}\Big]$$
$$= \sum_{|u|=n}\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\mathbb{E}_n\big[(V^{(u)})^2\mathbb{1}_{\{|V^{(u)}|\le e^{\theta S(u)}m(\alpha\theta)^{n/\alpha}\varepsilon\}}\big]$$
$$= \sum_{|u|=n}\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\int_{[0,e^{\theta S(u)}m(\alpha\theta)^{n/\alpha}\varepsilon]} y^2\,\mathrm{d}\mathbb{P}(|V|\le y\}.$$

Observe that (4.1) entails

$$\mathbb{P}(|V|>x)\sim(c_1+c_2)x^{-\alpha} \quad \text{as } x\to\infty.$$

Integration by parts thus leads to

$$\int_{[0,\,x]} y^2 \,\mathrm{d}\mathbb{P}(|V| \leq y) \sim \tfrac{\alpha(c_1+c_2)}{2-\alpha} x^{2-\alpha} \quad \text{as } x \to \infty.$$

Using this and (4.10), we conclude that, as $n \to \infty$,

$$\begin{aligned}
&\sum_{|u|=n} \mathbb{V}\mathrm{ar}_n\Big[\frac{e^{-\theta S(u)} V^{(u)}}{m(\alpha\theta)^{n/\alpha}} \mathbb{1}_{\{|V^{(u)}| \leq e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \varepsilon\}}\Big] \\
&\quad = \sum_{|u|=n} \frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}} \int_{[0, e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \varepsilon]} y^2 \,\mathrm{d}\mathbb{P}(|V| \leq y) \\
&\quad \sim \sum_{|u|=n} \frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}} \frac{\alpha(c_1+c_2)}{2-\alpha} \big(e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \varepsilon\big)^{2-\alpha} \\
&\quad = \varepsilon^{2-\alpha} \frac{\alpha(c_1+c_2)}{2-\alpha} \sum_{|u|=n} \frac{e^{-\alpha\theta S(u)}}{m(\alpha\theta)^n} \\
&\quad = \varepsilon^{2-\alpha} \frac{\alpha(c_1+c_2)}{2-\alpha} W_n(\alpha\theta) \ \to \ \varepsilon^{2-\alpha} \frac{\alpha(c_1+c_2)}{2-\alpha} W(\alpha\theta) \quad \text{a.s.}
\end{aligned}$$

This last expression vanishes as $\varepsilon \downarrow 0$ which proves (4.6).

Proof of (4.7). For every $\tau > 0$, since $\mathbb{E}[V] = 0$, we have

$$\begin{aligned}
&\sum_{|u|=n} \mathbb{E}_n\Big[\frac{e^{-\theta S(u)} V^{(u)}}{m(\alpha\theta)^{n/\alpha}} \mathbb{1}_{\{|\frac{e^{-\theta S(u)} V^{(u)}}{m(\alpha\theta)^{n/\alpha}}| \leq \tau\}}\Big] \\
&\quad = -\sum_{|u|=n} \mathbb{E}_n\Big[\frac{e^{-\theta S(u)} V^{(u)}}{m(\alpha\theta)^{n/\alpha}} \mathbb{1}_{\{|\frac{e^{-\theta S(u)} V^{(u)}}{m(\alpha\theta)^{n/\alpha}}| > \tau\}}\Big] \\
&\quad = -\sum_{|u|=n} \frac{e^{-\theta S(u)}}{m(\alpha\theta)^{n/\alpha}} \mathbb{E}_n\Big[V^{(u)} \mathbb{1}_{\{|V^{(u)}| > \tau \frac{m(\alpha\theta)^{n/\alpha}}{e^{-\theta S(u)}}\}}\Big] \\
&\quad = -\sum_{|u|=n} \frac{e^{-\theta S(u)}}{m(\alpha\theta)^{n/\alpha}} \int_{\mathbb{R} \setminus [-e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \tau, e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \tau]} y \,\mathrm{d}\mathbb{P}(V \leq y).
\end{aligned}$$

Using (4.1) and integration by parts, we infer

$$\int_{\mathbb{R} \setminus [-x,x]} y \,\mathrm{d}\mathbb{P}(V \leq y) \sim \tfrac{\alpha(c_1-c_2)}{\alpha-1} x^{1-\alpha} \quad \text{as } x \to \infty.$$

This asymptotic relation together with (4.10) implies that, as $n \to \infty$,

$$\begin{aligned}
&\sum_{|u|=n} \mathbb{E}_n\Big[\frac{e^{-\theta S(u)} V^{(u)}}{m(\alpha\theta)^{n/\alpha}} \mathbb{1}_{\{|\frac{e^{-\theta S(u)} V^{(u)}}{m(\alpha\theta)^{n/\alpha}}| \leq \tau\}}\Big] \\
&\quad = -\sum_{|u|=n} \frac{e^{-\theta S(u)}}{m(\alpha\theta)^{n/\alpha}} \int_{\mathbb{R} \setminus [-e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \tau, e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \tau]} y \,\mathrm{d}\mathbb{P}(V \leq y) \\
&\quad \sim -\sum_{|u|=n} \frac{e^{-\theta S(u)}}{m(\alpha\theta)^{n/\alpha}} \frac{\alpha(c_1-c_2)}{\alpha-1} \big(e^{\theta S(u)} m(\alpha\theta)^{n/\alpha} \tau\big)^{1-\alpha} \\
&\quad = -\tau^{1-\alpha} \frac{\alpha(c_1-c_2)}{\alpha-1} \sum_{|u|=n} \frac{e^{-\alpha\theta S(u)}}{m(\alpha\theta)^n} \ \to \ -\tau^{1-\alpha} \frac{\alpha(c_1-c_2)}{\alpha-1} W(\alpha\theta) \quad \text{a.s.}
\end{aligned}$$

This proves (4.7). □

Proof of Theorem 1.1. We show that, for any $r \in \mathbb{N}_0$,

$$\kappa^{-n/\alpha}\big((W-W_n),\ldots,\kappa^{r/\alpha}(W-W_{n-r})\big) \stackrel{\mathrm{d}}{\to} W(\alpha\theta)^{1/\alpha}\,(U_0,\ldots,U_r)\,.$$

By the Cramér-Wold device, this is equivalent to proving the following: for any $\beta_0,\ldots,\beta_r$ and $t \in \mathbb{R}$,

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\bigg[\exp\bigg(\mathrm{i}t\sum_{j=0}^{r}\beta_j\kappa^{-(n-j)/\alpha}(W-W_{n-j})\bigg)\bigg]\\&\quad=\mathbb{E}\bigg[\exp\bigg(\mathrm{i}tW(\alpha\theta)^{1/\alpha}\sum_{j=0}^{r}\beta_jU_j\bigg)\bigg]=\mathbb{E}\bigg[\Phi(\gamma_0W(\alpha\theta)^{1/\alpha}t)\prod_{i=1}^{r}\Psi\big(\gamma_iW(\alpha\theta)^{\frac{1}{\alpha}}t\big)\bigg],\end{aligned}\tag{4.11}$$

where

$$\gamma_i := \sum_{j=i}^{r}\beta_j\kappa^{(j-i)/\alpha}\tag{4.12}$$

for $i = 0,\ldots,r$ and, for $t \in \mathbb{R}$,

$$\begin{aligned}\Phi(t) &:= \mathbb{E}\big[\exp(\mathrm{i}tU_0)\big] = \exp\Big(\frac{\Gamma(2-\alpha)}{\alpha-1}\,\frac{c|t|^\alpha}{1-\kappa}\Big(\cos\Big(\frac{\pi\alpha}{2}\Big)-\mathrm{i}\sin\Big(\frac{\pi\alpha}{2}\Big)\operatorname{sign}(t)\Big)\Big),\\ \Psi(t) &:= \mathbb{E}\big[\exp(\mathrm{i}tQ_1)\big] = \exp\Big(\frac{\Gamma(2-\alpha)}{\alpha-1}c|t|^\alpha\Big(\cos\Big(\frac{\pi\alpha}{2}\Big)-\mathrm{i}\sin\Big(\frac{\pi\alpha}{2}\Big)\operatorname{sign}(t)\Big)\Big)\end{aligned}$$

(see (1.2) and (1.3)). Using the representation

$$U_j = \kappa^{j/\alpha}U_0 + \kappa^{(j-1)/\alpha}Q_1 + \ldots + \kappa^{1/\alpha}Q_{j-1} + Q_j$$

for $j \in \mathbb{N}$, we obtain

$$\sum_{j=0}^{r}\beta_jU_j = \sum_{j=0}^{r}\beta_j\kappa^{j/\alpha}U_0 + \sum_{i=1}^{r}\sum_{j=i}^{r}\beta_j\kappa^{(j-i)/\alpha}Q_i$$

which justifies the last equality in (4.11).

Let $n \geq r$. For notational convenience, we set $\beta_j = 0$ for $j < 0$. Then we have

$$\begin{aligned}\sum_{j=0}^{r}\beta_j\kappa^{-(n-j)/\alpha}(W-W_{n-j}) &= \sum_{j=0}^{r}\beta_j\kappa^{-(n-j)/\alpha}\sum_{i\geq n-j}(W_{i+1}-W_i)\\&= \sum_{i\geq n-r}(W_{i+1}-W_i)\sum_{j=n-i}^{r}\beta_j\kappa^{-(n-j)/\alpha}\\&= \sum_{i\geq 0}(W_{i+1+n-r}-W_{i+n-r})\sum_{j=r-i}^{r}\beta_j\kappa^{-(n-j)/\alpha}\\&= m(\alpha\theta)^{-(n-r)/\alpha}\sum_{|u|=n-m}e^{-\theta S(u)}\sum_{i\geq 0}(W_{i+1}^{(u)}-W_i^{(u)})\sum_{j=r-i}^{r}\beta_j\kappa^{-(r-j)/\alpha}.\end{aligned}$$

The last expression equals Θ_{n-r} defined in (4.2) with

$$V = \sum_{i\geq 0}\kappa^{-i/\alpha}\gamma_{r-i}(W_{i+1}-W_i),$$

where for the negative integers we set $\gamma_{-i} := \kappa^{i/\alpha}\gamma_0$. Observe that the so defined V is centered. Further, since the sequence $a_i := \kappa^{-i/\alpha}\gamma_{r-i}$ is eventually constant, by Theorem 2.1, the distribution of V satisfies (4.1) with

$$c_1 = c\sum_{i=-\infty}^{r}(\gamma_i^+)^\alpha \quad \text{and} \quad c_2 = c\sum_{i=-\infty}^{r}(\gamma_i^-)^\alpha.$$

According to relation (4.3) from Lemma 4.1 with these c_1 and c_2, we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\Big[\exp\Big(\mathrm{i}t\sum_{j=0}^{r}\beta_j\kappa^{-(n-j)/\alpha}(W-W_{n-j})\Big)\Big]\\
&=\mathbb{E}\Big[\exp\Big(B|t|^\alpha\sum_{j=-\infty}^{r}\Big(|\gamma_j|^\alpha\big(\cos(\tfrac{\pi\alpha}{2})-\mathrm{i}\sin(\tfrac{\pi\alpha}{2})\,\mathrm{sign}(\gamma_j t)\big)\Big)\Big)\Big]\\
&=\mathbb{E}\Big[\exp\Big((A|\gamma_0 t|^\alpha\big(\cos(\tfrac{\pi\alpha}{2})-\mathrm{i}\sin(\tfrac{\pi\alpha}{2})\,\mathrm{sign}(\gamma_0 t)\big)\Big)\\
&\qquad\cdot\prod_{j=1}^{r}\exp\Big(B|\gamma_j t|^\alpha\big(\cos(\tfrac{\pi\alpha}{2})-\mathrm{i}\sin(\tfrac{\pi\alpha}{2})\,\mathrm{sign}(\gamma_j t)\big)\Big)\Big]\\
&=\mathbb{E}\Big[\Phi(\gamma_0 W(\alpha\theta)^{1/\alpha}t)\prod_{j=1}^{r}\Psi(\gamma_j W(\alpha\theta)^{1/\alpha}t)\Big],
\end{aligned}$$

where

$$A := \frac{\Gamma(2-\alpha)}{\alpha-1}\frac{cW(\alpha\theta)}{1-\kappa} \quad \text{and} \quad B := \frac{\Gamma(2-\alpha)}{\alpha-1}cW(\alpha\theta).$$

The proof of Theorem 1.1 is complete. □

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Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, 01601 Kyiv, Ukraine
E-mail address: iksan@univ.kiev.ua

Universität Innsbruck, Austria and Uniwersytet Wrocławski, Wrocław, Poland
E-mail address: kolesko@math.uni.wroc.pl

Universität Innsbruck, Austria
E-mail address: matthias.meiners@uibk.ac.at