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FLUCTUATIONS OF BIGGINS' MARTINGALES AT COMPLEX PARAMETERS

Abstract. The long-term behavior of a supercritical branching random walk can be described and analyzed with the help of Biggins' martingales, parametrized by real or complex numbers. The study of these martingales with complex parameters is a rather recent topic. Assuming that certain sufficient conditions for the convergence of the martingales to non-degenerate limits hold, we investigate the fluctuations of the martingales around their limits. We discover three different regimes. First, we show that for parameters with small absolute values, the fluctuations are Gaussian and the limit laws are scale mixtures of the real or complex standard normal laws. We also cover the boundary of this phase. Second, we find a region in the parameter space in which the martingale fluctuations are determined by the extremal positions in the branching random walk. Finally, there is a critical region (typically on the boundary of the set of parameters for which the martingales converge to a non-degenerate limit) where the fluctuations are stable-like and the limit laws are the laws of randomly stopped Lévy processes satisfying invariance properties similar to stability.

Keywords: branching random walk; central limit theorem; complex martingales; minimal position; point processes; rate of convergence; stable processes
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1. Introduction

We consider a discrete-time supercritical branching random walk on the real line. The distribution of the branching random walk is governed by a point process $\mathcal{Z}$ on $\mathbb{R}$. Although there are numerous papers in which $\mathcal{Z}(\mathbb{R})$ is allowed to be infinite with positive probability, the standing assumption of the present paper is $\mathcal{Z}(\mathbb{R}) < \infty$ almost surely (a. s.). At time 0, the process starts with one individual (also called particle), the ancestor, which resides at the origin. At time 1, the ancestor dies and simultaneously places offspring on the real line with positions given by the points of the point process $\mathcal{Z}$. The offspring of the ancestor form the first generation of the branching random walk. At time 2, each particle of the first generation dies and has offspring with positions relative to their parent's position given by an independent copy of $\mathcal{Z}$. The individuals produced by the first generation particles form the second generation of the process, and so on.

The sequence of (random) Laplace transforms of the point process of the nth generation positions, evaluated at an appropriate $\lambda \in \mathbb{C}$ and suitably normalized, forms a martingale. This martingale that we denote by $(Z_n(\lambda))_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N} := \{1, 2, \ldots\}$, is called additive martingale or Biggins' martingale. These martingales play a key role in the study of the branching random walk, see e.g. [7, Theorem 4] where the spread of the nth generation particles is described in terms of additive martingales. In the same paper, Biggins showed that subject

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to some mild conditions, $Z_n(\lambda)$ converges almost surely to some limit $Z(\lambda)$ locally uniformly in λ from a certain open domain $\Lambda \subseteq \mathbb{C}$. Biggins' result was extended by two of the three present authors [29] to (parts of) the boundary of the set Λ. It is natural to ask for the rate of this convergence, i.e., for the fluctuations of $Z(\lambda) - Z_n(\lambda)$.

A partial answer to this question was given by Iksanov and Kabluchko [21], who proved a functional central limit theorem with a random centering for Biggins' martingale for real and sufficiently small λ. The counterpart of this result in the context of the complex branching Brownian motion energy model has been derived by Hartung and Klimovsky [16]. Another related statement for branching Brownian motion can be found in the recent paper by Maillard and Pain [35], where the fluctuations of the derivative martingale for branching Brownian motion are studied. The authors of the present paper also investigated in [22] fluctuations of $Z(\lambda) - Z_n(\lambda)$ for real λ in the regime where the distribution of $Z_1(\lambda)$ belongs to the normal domain of attraction of an α–stable law with $\alpha \in (1, 2)$. We refer to the end of Section 2 for a detailed account of the existing literature.

The aim of the paper at hand is to give a complete description of the fluctuations of Biggins' martingales whenever they converge while making only minimal moment assumptions. It turns out that, apart from the *Gaussian regime* studied in [21], there are two further cases. There is an *extremal regime*, where the fluctuations are determined by the particles close to the minimal position in the branching random walk. In this regime, the fluctuations are exponentially small with a polynomial correction. And finally, there is a *critical stable regime* with fluctuations of polynomial order.

2. Model description and main results

We continue with the formal definition of the branching random walk and a review of the results on which our work is based.

2.1. Model description and known results.

The model. Set $\mathcal{I} := \bigcup_{n \geq 0} \mathbb{N}^n$. We use the standard Ulam-Harris notation, that is, for $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, we also write $u_1 \ldots u_n$. Further, if $v = (v_1, \ldots, v_m) \in \mathbb{N}^m$, we write uv for $(u_1, \ldots, u_n, v_1, \ldots, v_m)$. For $k \leq n$, denote $u_1 \ldots u_k$, the ancestor of u in generation k, by $u|_k$. The ancestor of the whole population is identified with the empty tuple $\varnothing$ and its position is $S(\varnothing) = 0$. Let $(\mathcal{Z}(u))_{u \in \mathcal{I}}$ be a family of i.i.d. copies of the basic reproduction point process $\mathcal{Z}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We write $\mathcal{Z}(u) = \sum_{j=1}^{N(u)} X_j(u)$, where $N(u) = \mathcal{Z}(u)(\mathbb{R})$, $u \in \mathcal{I}$. We assume that $\mathcal{Z}(\varnothing) = \mathcal{Z}$. In general, we drop the argument $\varnothing$ for quantities derived from $\mathcal{Z}(\varnothing)$, for instance, $N = N(\varnothing)$. Generation 0 of the population is given by $\mathcal{G}_0 := \{\varnothing\}$ and, recursively,

$$\mathcal{G}_{n+1} := \{uj \in \mathbb{N}^{n+1} : u \in \mathcal{G}_n \text{ and } 1 \leq j \leq N(u)\}$$

is generation $n+1$ of the process. Define the set of all individuals by $\mathcal{G} := \bigcup_{n \in \mathbb{N}_0} \mathcal{G}_n$. The position of an individual $u = u_1 \ldots u_n \in \mathcal{G}_n$ is

$$S(u) := X_{u_1}(\varnothing) + \ldots + X_{u_n}(u_1 \ldots u_{n-1}).$$

The point process of the positions of the nth generation individuals will be denoted by $\mathcal{Z}_n$, that is,

$$\mathcal{Z}_n = \sum_{|u|=n} \delta_{S(u)}$$

where here and in what follows, we write $|u| = n$ for $u \in \mathcal{G}_n$. The sequence of point processes $(\mathcal{Z}_n)_{n\in\mathbb{N}_0}$ is called a *branching random walk.*

We assume that $(\mathcal{Z}_n)_{n\in\mathbb{N}_0}$ is supercritical, i. e., $\mathbb{E}[N] = \mathbb{E}[\mathcal{Z}(\mathbb{R})] > 1$. Then the generation sizes $\mathcal{Z}_n(\mathbb{R})$, $n \in \mathbb{N}_0$ form a supercritical Galton-Watson process and thus $\mathbb{P}(\mathcal{S}) > 0$ for the survival set

$$\mathcal{S} := \{\#\mathcal{G}_n > 0 \text{ for all } n \in \mathbb{N}\} = \{\mathcal{Z}_n(\mathbb{R}) > 0 \text{ for all } n \in \mathbb{N}\}.$$

The Laplace transform of the intensity measure μ of $\mathcal{Z}$ is the function

$$\lambda \;\mapsto\; m(\lambda) := \int_{\mathbb{R}} e^{-\lambda x}\,\mu(\mathrm{d}x) = \mathbb{E}\bigg[\sum_{|u|=1} e^{-\lambda S(u)}\bigg], \qquad \lambda \in \mathbb{C} \tag{2.1}$$

where $\lambda = \theta + \mathrm{i}\eta$ with $\theta, \eta \in \mathbb{R}$. (We adopt the convention from [7] and always write θ for $\mathrm{Re}(\lambda)$ and η for $\mathrm{Im}(\lambda)$.) Throughout the paper, we assume that

$$\mathcal{D} = \{\lambda \in \mathbb{C} : m(\lambda) \text{ converges absolutely}\} = \{\theta \in \mathbb{R} : m(\theta) < \infty\} + \mathrm{i}\mathbb{R} \;\text{ is non-empty.}$$

For $\lambda \in \mathcal{D}$ and $n \in \mathbb{N}_0$, let

$$Z_n(\lambda) := \frac{1}{m(\lambda)^n} \int_{\mathbb{R}} e^{-\lambda x}\,\mathcal{Z}_n(\mathrm{d}x) = \frac{1}{m(\lambda)^n} \sum_{|u|=n} e^{-\lambda S(u)}.$$

Denote by $\mathcal{F}_n := \sigma(\mathcal{Z}(u) : u \in \bigcup_{k=0}^{n-1} \mathbb{N}^k)$, and let $\mathcal{F}_\infty := \sigma(\mathcal{F}_n : n \in \mathbb{N}_0)$. It is well known and easy to check that $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ forms a complex-valued martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$. It is called *additive martingale in the branching random walk* and also *Biggins' martingale* in honor of Biggins' seminal contribution [6].

Convergence of complex martingales. Convergence of these martingales has been investigated by various authors in the case $\lambda = \theta \in \mathbb{R}$, see e. g. [5, 6, 32]. For the complex case, the most important sources for us are [7] and [29]. Theorem 1 of [7] states that if

$$\mathbb{E}[Z_1(\theta)^\gamma] < \infty \quad \text{for some } \gamma \in (1, 2] \tag{B1}$$

and

$$\tfrac{m(p\theta)}{|m(\lambda)|^p} < 1 \quad \text{for some } p \in (1, \gamma], \tag{B2}$$

then $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges a. s. and in L^p to a limit variable $Z(\lambda)$. Theorem 2 in the same source gives that this convergence is locally uniform (a. s. and in mean) on the set $\Lambda = \bigcup_{\gamma\in(1,2]} \Lambda_\gamma$ where $\Lambda_\gamma = \Lambda_\gamma^1 \cap \Lambda_\gamma^3$ and, for $\gamma \in (1, 2]$,

$$\Lambda_\gamma^1 = \mathrm{int}\{\lambda \in \mathcal{D} : \mathbb{E}[Z_1(\theta)^\gamma] < \infty\} \text{ and } \Lambda_\gamma^3 = \mathrm{int}\big\{\lambda \in \mathcal{D} : \inf_{1\le p\le\gamma} \tfrac{m(p\theta)}{|m(\lambda)|^p} < 1\big\}.$$

In [29], convergence of the martingales $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ for parameters λ from the boundary $\partial\Lambda$ is investigated. Theorem 2.1 in the cited article states that subject to the conditions

$$\tfrac{m(\alpha\theta)}{|m(\lambda)|^\alpha} = 1 \;\text{ and }\; \mathbb{E}\big[\textstyle\sum_{|u|=1} \theta S(u) \tfrac{e^{-\alpha\theta S(u)}}{|m(\lambda)|^\alpha}\big] \ge -\log(|m(\lambda)|) \text{ for some } \alpha \in (1, 2) \tag{C1}$$

and

$$\mathbb{E}[|Z_1(\lambda)|^\alpha \log_+^{2+\epsilon}(|Z_1(\lambda)|)] < \infty \quad \text{for some } \epsilon > 0 \tag{C2}$$

with the same α as in (C1), there is convergence of $Z_n(\lambda)$ to some limit variable $Z(\lambda)$. The convergence holds a. s. and in L^p for any $p < \alpha$.

As has already been mentioned the fluctuations of $Z_n(\lambda)$ around $Z(\lambda)$ as $n \to \infty$ are the subject of the present paper. More precisely, we find (complex) scaling constants $a_n = a_n(\lambda) \neq 0$ such that $a_n(Z(\lambda) - Z_n(\lambda))$ converges in distribution to a non-degenerate limit as $n \to \infty$.

2.2. **Main results.** We give an example before we state our main results.

Example. There are three fundamentally different regimes for the fluctuations of $Z_n(\lambda)$ around its limit $Z(\lambda)$. These regimes are best understood via an example, which is in close analogy to branching Brownian motion.

Example 2.1 (Binary splitting and Gaussian increments)**.** Consider a branching random walk with binary splitting and independent standard Gaussian increments, that is, $\mathcal{Z} = \delta_{X_1} + \delta_{X_2}$ where X_1, X_2 are independent standard normals. Then $m(\lambda) = 2\exp(\lambda^2/2)$ for $\lambda \in \mathbb{C}$. For every $\theta \in \mathbb{R}$ and $\gamma > 1$, we have $\mathbb{E}[Z_1(\theta)^\gamma] < \infty$. Hence $\Lambda = \{\lambda \in \mathbb{C} : m(p\theta)/|m(\lambda)|^p < 1 \text{ for some } p \in (1,2]\}$. Thus, $\lambda \in \Lambda$ if and only if there exists some $p \in (1,2]$ with $m(p\theta)/|m(\lambda)|^p < 1$. The latter inequality is equivalent to

$$(1-p)2\log 2 + p^2\theta^2 - p(\theta^2 - \eta^2) < 0. \tag{2.2}$$

It follows from the discussion in [29, Example 3.1] that $\lambda \in \Lambda$ iff $|\theta| \leq \sqrt{2\log 2}/2$ and $\theta^2 + \eta^2 < \log 2$, or $\sqrt{2\log 2}/2 \leq |\theta| < \sqrt{2\log 2}$ and $|\eta| < \sqrt{2\log 2} - |\theta|$. Corollary 1.1 of [21] applies to parameters $\theta \in \mathbb{R}$ satisfying $m(2\theta)/m(\theta)^2 < 1$

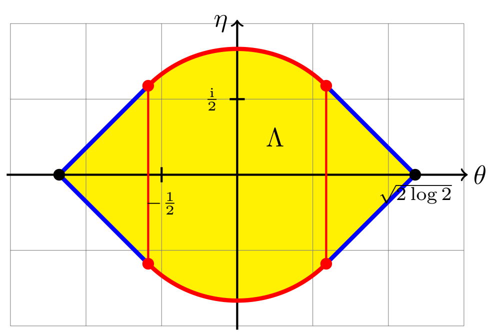

FIGURE 1. The figure shows the different regimes of fluctuations of Biggins' martingales in the branching random walk with binary splitting and independent standard Gaussian increments.

or, equivalently, $|\theta| < \sqrt{\log 2}$. In this case, the corollary gives convergence in distribution of $(\sqrt{2}\exp(-\theta^2/2))^n(Z(\theta) - Z_n(\theta))$ to a constant multiple of $\sqrt{Z(2\theta)} \cdot X$ where X is real standard normal and independent of $Z(2\theta)$. According to [32], $Z(2\theta)$ is non-degenerate iff $|\theta| < \sqrt{2\log 2}/2$, i.e., the limit in Corollary 1.1 of [21] is non-degenerate only for θ which are situated on the real axis strictly between the two red vertical lines in Figure 1. Our first result, Theorem 2.2 below, extends Corollary 1.1 from [21] and, in this particular example, gives the convergence of $(\sqrt{2}\exp((\lambda^2/2 - \theta^2))^n(Z(\lambda) - Z_n(\lambda))$ to a constant multiple of $\sqrt{Z(2\theta)} \cdot X$ in the whole bounded yellow domain surrounded by red arcs and lines. Here, X is independent of $Z(2\theta)$ and complex standard normal if $\operatorname{Im}(\lambda) \neq 0$. For parameters λ

from the red vertical lines, the same limit relation holds, but the limit is degenerate as $Z(2\theta) = 0$ a. s. Indeed, 2θ is then one of the two black dots in the figure. This problem can be resolved with the help of Seneta-Heyde norming. From [2] we know that $\sqrt{n}Z_n(2\theta)$ converges in probability to a constant multiple of the (non-degenerate) limit D_∞ of the derivative martingale. Modifying the scaling in Theorem 2.2 by the additional prefactor $n^{1/4}$ gives a nontrivial limit theorem where the limit is a constant multiple of $\sqrt{D_\infty} \cdot X$ with the same X as before which is further independent of D_∞. This is the content of Theorem 2.3 which applies to the λ from the vertical red lines.

A similar trick does not work for parameters from the open yellow domains surrounded by the two triangles consisting of red vertical lines and diagonal blue lines. There, the contribution of the minimal positions in the branching random walk to $Z(\lambda) - Z(\lambda)$ is too large for a limit theorem with a (randomly scaled) normal or stable limit. Instead, it can be checked that our Theorem 2.5 applies. The most tedious part here is to show that $\mathbb{E}[|Z(\lambda)|^p] < \infty$ for some suitable p, but this can be achieved by checking that the sufficient conditions (B1) and (B2) are fulfilled. Theorem 2.5 is based on the convergence of the point process of the branching random walk seen from its tip [34]. The correct scaling factors provided by the theorem are $n^{3\lambda/(2\vartheta)} \cdot (2\exp(\lambda^2/2)/(4^{\lambda/\vartheta}))^n$ with $\vartheta = \sqrt{2\log 2}$ and the limit distribution has a random series representation involving the limit process of branching random walk seen from its tip.

Finally, on the blue lines, it holds that the distribution of the martingale limit $Z(\lambda)$ is in the domain of attraction of a stable law and hence $Z(\lambda) - Z_n(\lambda)$ exhibits stable-like fluctuations. This regime is covered by Theorem 2.9, which shows that $n^{\frac{\lambda}{2\alpha\theta}}(Z(\lambda) - Z_n(\lambda))$ converges in distribution to a Lévy process independent of $\mathcal{F}_\infty$ satisfying an invariance property similar to α-stability (the details are explained in Example 2.7) evaluated at the limit D_∞ of the derivative martingale, where $\alpha = \sqrt{2\log 2}/\theta \in (1, 2)$ for $\theta \in (\frac{1}{2}\sqrt{2\log 2}, \sqrt{2\log 2})$.

Weak convergence almost surely and in probability. If $\zeta, \zeta_1, \zeta_2, \ldots$ are random variables taking values in $\mathbb{C}$, we write

$$\mathcal{L}(\zeta_n|\mathcal{F}_n) \overset{\mathrm{w}}{\to} \mathcal{L}(\zeta|\mathcal{F}_\infty) \quad \text{in } \mathbb{P}\text{-probability} \tag{2.3}$$

(in words, the distribution of ζ_n given $\mathcal{F}_n$ converges weakly to the distribution of ζ given $\mathcal{F}_\infty$ in $\mathbb{P}$-probability) if for every bounded continuous function $\phi : \mathbb{C} \to \mathbb{R}$ it holds that $\mathbb{E}[\phi(\zeta_n)|\mathcal{F}_n]$ converges to $\mathbb{E}[\phi(\zeta)|\mathcal{F}_\infty]$ in $\mathbb{P}$-probability as $n \to \infty$. Notice that (2.3) implies that ζ_n converges to ζ in distribution as $n \to \infty$ as for any bounded and continuous function $\phi : \mathbb{C} \to \mathbb{R}$ and every strictly increasing sequence of positive integers, we can extract a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\mathbb{E}[\phi(\zeta_{n_k})|\mathcal{F}_{n_k}]$ converges to $\mathbb{E}[\phi(\zeta)|\mathcal{F}_\infty]$ a. s. Hence, by the dominated convergence theorem,

$$\mathbb{E}[\phi(\zeta_{n_k})] = \mathbb{E}[\mathbb{E}[\phi(\zeta_{n_k})|\mathcal{F}_{n_k}]] \to \mathbb{E}[\mathbb{E}[\phi(\zeta)|\mathcal{F}_\infty]] = \mathbb{E}[\phi(\zeta)] \quad \text{as } k \to \infty.$$

This implies $\mathbb{E}[\phi(\zeta_n)] \to \mathbb{E}[\phi(\zeta)]$ as $n \to \infty$ and, therefore, $\zeta_n \overset{\mathrm{d}}{\to} \zeta$.

Analogously, we write

$$\mathcal{L}(\zeta_n|\mathcal{F}_n) \overset{\mathrm{w}}{\to} \mathcal{L}(\zeta|\mathcal{F}_\infty) \quad \mathbb{P}\text{-a. s.} \tag{2.4}$$

(in words, the distribution of ζ_n given $\mathcal{F}_n$ converges a. s. to the distribution of ζ given $\mathcal{F}_\infty$) if for every bounded continuous function $\phi : \mathbb{C} \to \mathbb{R}$ it holds that $\mathbb{E}[\phi(\zeta_n)|\mathcal{F}_n]$ converges to $\mathbb{E}[\phi(\zeta)|\mathcal{F}_\infty]$ a. s. as $n \to \infty$. Clearly, also $\mathcal{L}(\zeta_n|\mathcal{F}_n) \overset{\mathrm{w}}{\to} \mathcal{L}(\zeta|\mathcal{F}_\infty)$ $\mathbb{P}$-a. s. implies $\zeta_n \overset{\mathrm{d}}{\to} \zeta$.

Henceforth, we shall assume that $\lambda \in \mathcal{D}$ satisfies $\theta \geq 0$. This simplifies the presentation of our results but is not a restriction of generality. Indeed, if $\theta < 0$, we may replace the point process $\mathcal{Z} = \sum_{j=1}^{N} \delta_{X_j}$ by $\sum_{j=1}^{N} \delta_{-X_j}$ and θ by $-\theta > 0$.

Small $|\lambda|$: Gaussian fluctuations. Our first result is an extension of Corollary 1.1 in [21] to the complex case. For $\lambda \in \mathcal{D}$ with $m(\lambda) \neq 0$, we set

$$\sigma_\lambda^2 := \mathbb{E}[|Z_1(\lambda) - 1|^2] = \mathbb{E}[|Z_1(\lambda)|^2] - 1 \in [0, \infty]. \tag{2.5}$$

Notice that $\sigma_\theta^2 < \infty$ implies $\sigma_\lambda^2 < \infty$ since $|Z_1(\lambda)| \leq \frac{m(\theta)}{|m(\lambda)|} Z_1(\theta)$.

Throughout the paper, we call a complex random variable $\zeta = \xi + \mathrm{i}\tau$ with $\xi = \mathrm{Re}(\zeta)$ and $\tau = \mathrm{Im}(\zeta)$ standard normal if ξ and τ are independent, identically distributed centered normal random variables with $\mathbb{E}[|\zeta|^2] = \mathbb{E}[\xi^2] + \mathbb{E}[\tau^2] = 1$.

Theorem 2.2 (Gaussian case)**.** *Assume that $\lambda \in \mathcal{D}$ with $m(\lambda) \neq 0$ is such that $\sigma_\theta^2 < \infty$, $\sigma_\lambda^2 > 0$ and $m(2\theta) < |m(\lambda)|^2$. Define*

$$m = \begin{cases} m(2\theta) & \text{if } |m(2\lambda)| < m(2\theta), \\ m(2\lambda) & \text{if } |m(2\lambda)| = m(2\theta). \end{cases}$$

Then

$$\mathcal{L}\Big(\tfrac{m(\lambda)^n}{m^{n/2}}(Z(\lambda) - Z_n(\lambda))\Big|\mathcal{F}_n\Big) \stackrel{\mathrm{w}}{\to} \mathcal{L}\Big(\tfrac{\sigma_\lambda}{\sqrt{1-m(2\theta)/|m(\lambda)|^2}}\sqrt{Z(2\theta)}X\Big|\mathcal{F}_\infty\Big) \text{ in } \mathbb{P}\text{-probability} \tag{2.6}$$

where X is independent of $\mathcal{F}_\infty$. Here, X is complex standard normal if $|m(2\lambda)| < m(2\theta)$ whereas X is real standard normal if $|m(2\lambda)| = m(2\theta)$.

If, additionally, either $2\theta \in \Lambda$ or $Z(2\theta) = 0$ a. s., then the weak convergence in $\mathbb{P}$-probability in (2.6) *can be strengthened to weak convergence $\mathbb{P}$-a. s.*

A perusal of the proof of Theorem 2.2 reveals that the theorem still holds when $\mathcal{Z}(\mathbb{R}) = \infty$ with positive probability, that is, our standing assumption $\mathcal{Z}(\mathbb{R}) < \infty$ a. s. is not needed for this result.

Further, notice that the limit in Theorem 2.2 may vanish a. s., namely, if $Z(2\theta) = 0$ a. s. Equivalent conditions for $(Z_n(2\theta))_{n \in \mathbb{N}_0}$ to be uniformly integrable or equivalently

$$\mathbb{P}(Z(2\theta) > 0) > 0 \tag{2.7}$$

are given in [32] and [5, Theorem 1.3]. For instance,

$$\mathbb{E}[Z_1(2\theta) \log_+(Z_1(2\theta))] < \infty \quad \text{and} \quad 2\theta m'(2\theta)/m(2\theta) < \log(m(2\theta)) \tag{2.8}$$

imply (2.7). In particular, the condition $2\theta \in \Lambda$ comfortably ensures (2.7).

However, there may be a region of $\lambda \in \Lambda$ for which $m(2\theta) < |m(\lambda)|^2$ and $\sigma_\theta^2 < \infty$ but $Z(2\theta)$ is degenerate at 0 as it is the case in Example 2.1[1]. In this situation, the assertion of Theorem 2.2 holds but the limit is degenerate at 0. This means that $2\theta \notin \Lambda$. Typically, there is a real parameter $\vartheta > 0$ on the boundary of Λ such that

$$\vartheta m'(\vartheta)/m(\vartheta) = \log(m(\vartheta)) \tag{2.9}$$

and either $2\theta = \vartheta$ or $2\theta > \vartheta$. The second case leads to a non-Gaussian regime in which the extremal positions dominate the fluctuations on $Z_n(\lambda)$ around $Z(\lambda)$. This case will be dealt with further below. In the first case, under mild moment assumptions, a polynomial correction factor is required and a different martingale limit figures, namely, the limit of the *derivative martingale*. More precisely, we

[1] In the example, the corresponding region is $\theta^2 + \eta^2 < \log 2$ and $\theta \geq \frac{1}{2}\sqrt{2\log 2}$.

suppose that (2.8) is violated because $2\theta = \vartheta$ where $\vartheta > 0$ is as in (2.9). Then, for $n \in \mathbb{N}_0$ and $u \in \mathcal{G}_n$, we define

$$V(u) := \vartheta S(u) + n\log(m(\vartheta)). \tag{2.10}$$

By definition and by (2.9),

$$\mathbb{E}\bigg[\sum_{|u|=1} e^{-V(u)}\bigg] = 1 \quad \text{and} \quad \mathbb{E}\bigg[\sum_{|u|=1} V(u)e^{-V(u)}\bigg] = 0. \tag{2.11}$$

The branching random walk $((V(u))_{u\in\mathcal{G}_n})_{n\geq 0}$ is said to be in the *boundary case*. Then $W_n := \sum_{|u|=n} e^{-V(u)} = Z_n(\vartheta) \to 0$ a.s., but the derivative martingale

$$\partial W_n := \sum_{|u|=n} e^{-V(u)}V(u) \tag{2.12}$$

converges $\mathbb{P}$-a.s. under appropriate assumptions to some random variable D_∞ satisfying $D_\infty > 0$ a.s. on the survival set $\mathcal{S}$, see [10] for details. Due to a result by Aïdékon and Shi [2, Theorem 1.1], the limit D_∞ also appears as the limit in probability of the rescaled martingale W_n, namely,

$$\sqrt{n}W_n \overset{\mathbb{P}}{\to} \sqrt{\frac{2}{\pi\sigma^2}}D_\infty \tag{2.13}$$

where

$$\sigma^2 = \mathbb{E}\bigg[\sum_{|u|=1} V(u)^2 e^{-V(u)}\bigg] \in (0,\infty). \tag{2.14}$$

Relation (2.13) holds subject to the conditions (2.11), (2.14) and

$$\mathbb{E}[W_1\log_+^2(W_1)] < \infty \quad \text{and} \quad \mathbb{E}[\tilde{W}_1\log_+(\tilde{W}_1)] < \infty \tag{2.15}$$

where $\tilde{W}_1 := \sum_{|u|=1} e^{-V(u)}V(u)_+$ and $x_\pm := \max(\pm x, 0)$. For the case where (2.13) holds, we have the following result.

Theorem 2.3 (Gaussian boundary case)**.** *Suppose that $\vartheta > 0$ satisfies (2.9) and that (2.11), (2.14) and (2.15) hold for $V(u) = \vartheta S(u) + |u|\log(m(\vartheta))$, $u \in \mathcal{G}$. Further, assume that $\lambda \in \mathcal{D}$ with $m(\lambda) \neq 0$ is such that $\sigma_\theta^2 < \infty$, $\sigma_\lambda^2 > 0$, $m(2\theta) < |m(\lambda)|^2$ and $2\theta = \vartheta$. Define*

$$m = \begin{cases} m(2\theta) & \text{if } |m(2\lambda)| < m(2\theta), \\ m(2\lambda) & \text{if } |m(2\lambda)| = m(2\theta) \end{cases}$$

and $a_n := n^{1/4}\frac{m(\lambda)^n}{m^{n/2}}$ for $n \in \mathbb{N}$. Then

$$\mathcal{L}\big(a_n(Z(\lambda) - Z_n(\lambda))\big|\mathcal{F}_n\big) \overset{\mathrm{w}}{\to} \mathcal{L}\Big(\tfrac{\sqrt{\frac{2}{\pi}}\frac{\sigma_\lambda}{\sigma}}{\sqrt{1-m(2\theta)/|m(\lambda)|^2}}\sqrt{D_\infty}X\Big|\mathcal{F}_\infty\Big) \quad \text{in } \mathbb{P}\text{-probability} \tag{2.16}$$

where X is independent of $\mathcal{F}_\infty$. Here, X is complex standard normal if $|m(2\lambda)| < m(2\theta)$ whereas X is real standard normal if $|m(2\lambda)| = m(2\theta)$.

The regime in which the extremal positions dominate. Again suppose that $\vartheta > 0$ satisfies (2.9) and that (2.11), (2.14) and (2.15) hold for $V(u) = \vartheta S(u) + |u|\log(m(\vartheta))$, $u \in \mathcal{G}$. Further, assume that $\lambda \in \Lambda$, but $2\theta > \vartheta$. Then, typically, $Z_n(2\theta) \to 0$ because (2.8) is violated because

$$2\theta m'(2\theta)/m(2\theta) > \log(m(2\theta)).$$

It is known, see e.g. [37], that $\min_{|u|=n} V(u)$ is of the order $\frac{3}{2}\log n$ as $n \to \infty$. It will turn out that this is too slow for a result in the spirit of Theorem 2.2 in the sense that the contributions of the particles with small positions in the nth generation to $Z(\lambda) - Z_n(\lambda)$ are substantial, and hence no (conditionally) infinitely divisible limit distribution can be expected. Instead, the description of the fluctuations $Z(\lambda) - Z_n(\lambda)$ will follow from Madaule's work [34], where the behavior of the point processes

$$\mu_n := \sum_{|u|=n} \delta_{V_n(u)}$$

with $V_n(u) := V(u) - \frac{3}{2}\log n$ was studied. For the reader's convenience, we state in detail a consequence of the main result in [34].

Proposition 2.4. *Suppose the branching random walk $(V(u))_{u\in\mathcal{G}}$ satisfies* (2.11)*,* (2.14) *and* (2.15)*. Further, suppose that*

$$\text{The branching random walk } (V(u))_{u\in\mathcal{G}} \text{ is non-lattice.} \tag{A1}$$

Then there is a point process $\mu_\infty = \sum_{k\in\mathbb{N}} \delta_{P_k}$ such that $\mu_\infty((-\infty, 0])$ is a. s. finite and μ_n converges in distribution to μ_∞ (in the space of locally finite point measures equipped with the topology of vague convergence).

Source. This can be derived from [34, Theorem 1.1]. □

Let $Z^{(1)}(\lambda), Z^{(2)}(\lambda), \ldots$ denote independent random variables with the same distribution as $Z(\lambda) - 1$ which are independent of μ_∞. We consider the following series

$$X_{\text{ext}} := \sum_k e^{-\frac{\lambda P_k^*}{\vartheta}} Z^{(k)}(\lambda) = \lim_{n\to\infty} \sum_{k=1}^{n} e^{-\frac{\lambda P_k^*}{\vartheta}} Z^{(k)}(\lambda), \tag{2.17}$$

where $-\infty < P_1^* \le P_2^* \le \ldots$ are the atoms of μ_∞ arranged in increasing order.

Theorem 2.5 (Domination by extremal positions)**.** *Suppose that $\vartheta > 0$ satisfies* (2.9) *and that* (2.11)*,* (2.14)*,* (2.15) *and* (A1) *hold for $V(u) = \vartheta S(u) + |u|\log(m(\vartheta))$, $u \in \mathcal{G}$. Let $\lambda \in \Lambda$ and assume that $\theta \in (\frac{\vartheta}{2}, \vartheta)$. If there is $p \in (\frac{\vartheta}{\theta}, 2]$ satisfying $\mathbb{E}[|Z(\lambda)|^p] < \infty$, then the series X_{ext} defined by* (2.17) *converges a. s. to a non-degenerate limit. Moreover,*

$$n^{\frac{3\lambda}{2\vartheta}} \Big(\frac{m(\lambda)}{m(\vartheta)^{\lambda/\vartheta}}\Big)^n (Z(\lambda) - Z_n(\lambda)) \stackrel{\mathrm{d}}{\to} X_{\text{ext}}.$$

Sufficient conditions for $\mathbb{E}[|Z(\lambda)|^p] < \infty$, which are easy to check, are (B1) and (B2). Further sufficient conditions for $\mathbb{E}[|Z(\lambda)|^p] < \infty$ are given in Proposition 2.6 below. Finally, we should mention the upcoming paper [23] in which conditions for the convergence in L^p for $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ are provided.

The boundary of Λ: Stable fluctuations. It has been shown in [29, Theorem 2.1] that the martingale $Z_n(\lambda)$ converges on a part of the boundary of Λ. More precisely, consider the condition

$$\frac{m(\alpha\theta)}{|m(\lambda)|^\alpha} = 1 \quad \text{and} \quad \frac{\theta m'(\theta\alpha)}{|m(\lambda)|^\alpha} = \log(|m(\lambda)|) \tag{C1}$$

and define

$$\partial\Lambda^{(1,2)} := \{\lambda \in \partial\Lambda \cap \mathcal{D} : \text{(C1) holds with } \alpha \in (1,2)\}.$$

Theorem 2.1 in [29] says that if $\lambda \in \mathcal{D}$ satisfies (C1) (actually, Theorem 2.1 in [29] requires a weaker assumption) and if $\mathbb{E}[|Z_1(\lambda)|^\alpha \log_+^{2+\epsilon}(|Z_1(\lambda)|)] < \infty$ for some $\epsilon > 0$, then $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges a.s. and in L^p for every $p < \alpha$ to some limit $Z(\lambda)$ satisfying $\mathbb{E}[Z(\lambda)] = 1$. If an additional moment assumption holds, then a simplified version of the proof of Theorem 2.1 in [29] gives the following result.

Proposition 2.6. *Suppose that $\lambda \in \mathcal{D}$ satisfies*

$$\frac{m(\alpha\theta)}{|m(\lambda)|^\alpha} = 1 \quad \text{and} \quad \frac{\theta m'(\theta\alpha)}{|m(\lambda)|^\alpha} \leq \log(|m(\lambda)|)$$

for some $\alpha \in (1,2)$. If, additionally, $\mathbb{E}[|Z_1(\lambda)|^\gamma] < \infty$ for some $\alpha < \gamma \leq 2$, then $Z_n(\lambda) \to Z(\lambda)$ in L^p for all $p < \alpha$ and there exists a constant $C > 0$ such that

$$\mathbb{P}(|Z(\lambda)| \geq t) \leq Ct^{-\alpha} \tag{2.18}$$

for all $t > 0$.

For the rest of this section, we assume that $\lambda \in \partial\Lambda^{(1,2)}$ and that $\alpha \in (1,2)$ satisfies (C1). Notice that if $\vartheta > 0$ is defined via (2.9), then $\alpha\theta = \vartheta$ in the given situation. To determine the fluctuations of $Z_n(\lambda)$ around $Z(\lambda)$ in this setting, we require stronger assumptions than those of Theorem 2.1 in [29]. First of all, as before, we define $V(u)$ via (2.10), i.e., $V(u) := \vartheta S(u) + n\log(m(\vartheta))$ for $n \in \mathbb{N}_0$ and $u \in \mathcal{G}_n$. Then (C1) becomes (2.11). Further, we shall require that the following conditions hold:

$$\mathbb{E}[Z_1(\theta)^\gamma] < \infty \text{ for some } \gamma \in (\alpha, 2], \tag{2.19}$$

$$\mathbb{E}\big[Z_1(\kappa\theta)^2\big] < \infty \text{ for some } \kappa \in (\tfrac{\alpha}{2}, 1). \tag{2.20}$$

We denote by $\mathbb{U} = \mathbb{U}(\lambda)$ the smallest closed subgroup of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that

$$\mathbb{P}\Big(\frac{e^{-\lambda S(u)}}{m(\lambda)} \in \mathbb{U} \text{ for all } u \in \mathcal{I}\Big) = 1.$$

Furthermore, for simplicity of presentation, we assume that

$$\{|z| : z \in \mathbb{U}\} = \mathbb{R}_> := (0,\infty). \tag{2.21}$$

Let us now briefly describe the structure of $\mathbb{U}$. If the subgroup $\mathbb{U}_1 = \mathbb{U} \cap \{|z| = 1\}$ coincides with the unit sphere $\{|z| = 1\}$, then $\mathbb{U}$ is the whole multiplicative group $\mathbb{C}^*$. Otherwise, $\mathbb{U}_1$ is a finite group and $\mathbb{U}$ consists of finitely many connected components. By $\mathbb{U}_\mathbb{R}$ we denote the one-parameter subgroup of $\mathbb{U}$ which is either $\mathbb{R}_>$ if $\mathbb{U} = \mathbb{C}^*$ or it is the connected component of $\mathbb{U}$ that contains 1 if $\mathbb{U} \neq \mathbb{C}^*$. Clearly, $\mathbb{U}_\mathbb{R}$ is a subgroup isomorphic to the multiplicative group $\mathbb{R}_>$. By γ_t we denote the canonical parametrization of $\mathbb{U}_\mathbb{R}$ satisfying $|\gamma_t| = t$. We infer that there exists some $w \in \mathbb{C}$ with $\mathrm{Re}(w) = 1$ such that

$$\gamma_t = t^w = \exp(w \log t) \text{ for all } t > 0. \tag{2.22}$$

Clearly, $w = 1$ if $\mathbb{U}_\mathbb{R} = \mathbb{R}_>$. It is also worth mentioning that $\mathbb{U}_1 \times \mathbb{R}_> \simeq \mathbb{U}$ via the isomorphism $T : \mathbb{U}_1 \times \mathbb{R}_> \to \mathbb{U}$, $(z,t) \mapsto z\gamma_t = zt^w$. For illustration purposes, we interrupt the setup and discuss an example.

Example 2.7 (Binary splitting and Gaussian increments revisited)**.** Again we consider a branching random walk with binary splitting and independent standard Gaussian increments, that is, $\mathcal{Z} = \delta_{X_1} + \delta_{X_2}$ with independent standard normal random variables X_1, X_2. Recall that, in this situation, we have $m(\lambda) = 2\exp(\lambda^2/2)$ for $\lambda \in \mathbb{C}$. The parameter region we are interested in is $\sqrt{2\log 2}/2 < \theta < \sqrt{2\log 2}$ and $\eta = \sqrt{2\log 2} - \theta$, see Figure 1. For $\lambda = \theta + \mathrm{i}\eta$ from this region, we have

$$m(\lambda) = 2\exp((\theta+\mathrm{i}\eta)^2/2) = \exp(\sqrt{2\log 2}\theta + \mathrm{i}\theta\eta).$$

Therefore, $\mathbb{U}$ is generated by the set

$$\{e^{\theta(x-\sqrt{2\log 2})+\mathrm{i}\eta(x-\theta)} : x \in \mathbb{R}\}$$

which we may rewrite as

$$\{e^{\mathrm{i}\eta^2} \cdot e^{(\theta+\mathrm{i}\eta)x} : x \in \mathbb{R}\}.$$

In particular, (2.21) holds. Moreover, $\mathbb{U}_1$ is the closed (multiplicative) subgroup of the unit circle generated by $e^{\mathrm{i}\eta^2}$. This group is finite if and only if $\frac{1}{2\pi}\eta^2 \in \mathbb{Q}$, and $\mathbb{U}_1 = \{z \in \mathbb{C} : |z| = 1\}$, otherwise. As θ varies over $(\sqrt{2\log 2}/2, \sqrt{2\log 2})$, the square of the imaginary part, η^2, ranges over the whole interval $(0, \frac{1}{2}\log 2)$. Thus, for all but countably many θ, the group $\mathbb{U}$ equals $\mathbb{C}^*$, but for countably many θ, $\mathbb{U}$ will consist of a finite family of 'snails' as depicted in the figure below. Finally,

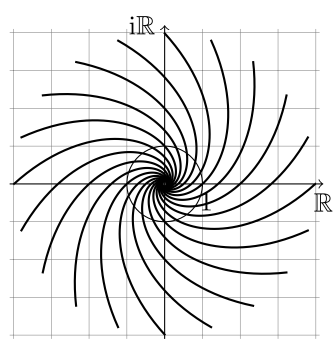

FIGURE 2. The group $\mathbb{U}$ in the case $\theta = \sqrt{2\log 2} - \sqrt{\frac{\pi}{10}}$ and $\eta = \sqrt{\frac{\pi}{10}}$.

whenever $\mathbb{U} = \mathbb{C}^*$, the scaling exponent can be chosen as $w = 1$. When $\mathbb{U} \neq \mathbb{C}^*$, then $\mathbb{U}_1$ is finite and the connected component of $\mathbb{U}_1$ containing 1 is

$$\{e^{(\theta+\mathrm{i}\eta)x} : x \in \mathbb{R}\} = \{e^{\lambda x} : x \in \mathbb{R}\} = \{t^{\lambda/\theta} : t > 0\}$$

so that $w = \lambda/\theta$ in this case.

By ϱ we denote the Haar measure on $\mathbb{U}$ satisfying the normalization condition

$$\varrho(\{z \in \mathbb{C} : 1 \leq |z| < e\}) = 1, \tag{2.23}$$

i.e., ϱ is the image of the measure $\varrho_1 \times \frac{\mathrm{d}t}{t}$, where ϱ_1 is the uniform distribution on $\mathbb{U}_1$, via the isomorphism T.

To understand the fluctuations of $Z(\lambda) - Z_n(\lambda)$, one needs to know the tail behavior of $Z(\lambda)$. The following theorem, which is interesting in its own right, provides the information required. For its formulation, we introduce some additional notation. We write $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the one-point (Alexandroff) compactification of $\mathbb{C}$. Further, we denote by $C_c^2(\hat{\mathbb{C}} \setminus \{0\})$ the set of real-valued, twice continuously partially differentiable functions on $\hat{\mathbb{C}} \setminus \{0\}$ with compact support. Finally, we remind the reader that a measure ν on $\mathbb{C}$ is called a *Lévy measure* if $\nu(\{0\}) = 0$ and $\int_{\mathbb{C}} (|z|^2 \wedge 1) \, \nu(\mathrm{d}z) < \infty$. A Lévy measure ν is called $(\mathbb{U}, \alpha)$-invariant if $\nu(uB) = |u|^{-\alpha} \nu(B)$ for all $u \in \mathbb{U}$ and all Borel sets $B \subseteq \mathbb{C} \setminus \{0\}$. The Lévy measure ν is called non-zero if $\nu(B) > 0$ for some Borel set B as above.

Theorem 2.8. *Let $\lambda \in \mathcal{D}$ satisfy* (C1) *with $\alpha \in (1, 2)$. Further, suppose that* (2.19), (2.20) *and* (2.21) *hold. Then there is a non-zero $(\mathbb{U}, \alpha)$-invariant Lévy measure ν on $\mathbb{C}$ such that*

$$\lim_{\substack{|z| \to 0, \\ z \in \mathbb{U}}} |z|^{-\alpha} \mathbb{E}[\phi(zZ(\lambda))] = \int \phi \, \mathrm{d}\nu$$

for all $\phi \in C_c^2(\hat{\mathbb{C}} \setminus \{0\})$.

We denote by $(X_t)_{t \geq 0}$ a complex-valued Lévy process which is independent of $\mathcal{F}_\infty$ and has characteristic exponent

$$\Psi(x) = \int \left(e^{\mathrm{i}\langle x, z\rangle} - 1 - \mathrm{i}\langle x, z\rangle\right) \nu(\mathrm{d}z), \quad x \in \mathbb{C}.$$

Notice that Ψ is well-defined as integration by parts gives

$$\int_{\{|z| \geq 1\}} (|z| - 1) \, \nu(\mathrm{d}z) = \int_1^\infty \nu(\{|z| \geq t\}) \, \mathrm{d}t = \nu(\{|z| \geq 1\}) \int_1^\infty t^{-\alpha} \, \mathrm{d}t < \infty.$$

Therefore, $(X_t)_{t \geq 0}$ is the Lévy process associated with the Lévy-Khintchine characteristics $(0, -\int_{\{|z|>1\}} z \, \nu(\mathrm{d}z), \nu)$ (cf. [27, p. 291, Corollary 15.8]).

Now we are ready to describe the fluctuations of $Z(\lambda) - Z_n(\lambda)$ for $\lambda \in \partial\Lambda^{(1,2)}$.

Theorem 2.9. *Suppose that the assumptions of Theorem 2.8 hold. Then there exists $w \in \mathbb{C}$ such that* $\mathrm{Re}(w) = 1$ *(see* (2.22) *for the definition of w) and*

$$\mathcal{L}\left(n^{\frac{w}{2\alpha}}(Z(\lambda) - Z_n(\lambda)) \mid \mathcal{F}_n\right) \stackrel{\mathrm{w}}{\to} \mathcal{L}(X_{cD_\infty} | \mathcal{F}_\infty) \quad \textit{in } \mathbb{P}\textit{-probability} \tag{2.24}$$

for $c := \sqrt{\frac{2}{\pi\sigma^2}}$, σ^2 defined by (2.14) *with $V(u)$ as in* (2.10) *and D_∞ being the a. s. limit of the derivative martingale defined in* (2.12).

Related literature. The martingale convergence theorem guarantees the almost sure convergence of $Z_n(\theta)$, but its limit $Z(\theta)$ may vanish a. s. Equivalent conditions for $\mathbb{P}(Z(\theta) = 0) < 1$ can be found in [6, Lemma 5], [32] and [5, Theorem 1.3]. Convergence in distribution of $a_n(Z(\lambda) - Z_n(\lambda))$ as $n \to \infty$ for constants $a_n > 0$ can be viewed as a result on the rate of convergence. In [3, 21, 22, 24, 25] the rate of convergence of $Z_n(\theta)$ to $Z(\theta)$ has been investigated in the regime $\mathbb{P}(Z(\theta)=0)<1$. The papers [3, 24, 25] deal with the issue of convergence of the infinite series

$$\textstyle\sum_{n=0}^{\infty} a_n(Z(\theta)-Z_n(\theta)). \tag{2.25}$$

More precisely, in [3], necessary conditions and sufficient conditions for the convergence in L^p of the infinite series in (2.25) are given in the situation where $a_n = e^{an}$ for some $a>0$. Sufficient conditions for the almost sure convergence of the series in (2.25) have been provided in the case where $a_n = e^{an}$ for some $a>0$ in [24] and in the case where $(a_n)_{n\in\mathbb{N}_0}$ is regularly varying at $+\infty$ in [25].

The papers [21, 22] are in the spirit of the article at hand. In these works, for $\alpha\in(1,2]$, it is shown that if $\kappa := m(\alpha\theta)/m(\theta)^\alpha < 1$, then $\kappa^{-n/\alpha}(Z(\theta)-Z_n(\theta))$ converges in distribution to a random variable $Z(\alpha\theta)^{1/\alpha}U$ where U is a (non-degenerate) centered α–stable random variable (normal, if $\alpha=2$) independent of $Z(\alpha\theta)$. Specifically, the case where $\alpha\in(1,2)$ and $\mathbb{P}(Z_1(\theta)>x)\sim cx^{-\alpha}$ for some $c>0$ is covered in [22, Corollary 1.3] whereas the case $\alpha=2$ and $\mathbb{E}[Z_1(\theta)^2]<\infty$ is investigated in [21, Corollary 1.1]. Both papers actually contain functional versions of these convergences. The aforementioned assertions are extensions of the corresponding results for Galton-Watson processes [13, 18, 19].

The counterpart of our Theorem 2.2, which gives the fluctuations of Biggins' martingales for small parameters has a natural analogue in the complex branching Brownian motion energy model. The corresponding statement in the latter model is [16, Theorem 1.4].

It is well known that if $\theta m'(\theta)/m(\theta) = \log(m(\theta))$, then $Z_n(\theta)$ converges to 0 a. s. In this case a natural object to study is the derivative martingale $(D_n)_{n\in\mathbb{N}_0}$. In order to study the fluctuations of D_n around its limit D_∞ one needs an additional correction term of order $(\log n)/\sqrt{n}$. The corresponding result, again in the context of branching Brownian motion, is given in [35], where it is shown that $\sqrt{n}(D_\infty - D_n + \frac{\log n}{\sqrt{2\pi n}}D_\infty) \xrightarrow{\mathrm{d}} S_{D_\infty}$ for an independent 1–stable Lévy process $(S_t)_{t\geq 0}$.

The martingale limits $Z(\lambda)$ solve smoothing equations, namely,

$$Z(\lambda) = \sum_{|u|=1} \frac{e^{-\lambda S(u)}}{m(\lambda)}[Z(\lambda)]_u \quad \text{a. s.} \tag{2.26}$$

where the $[Z(\lambda)]_u$, $u\in\mathbb{N}$ are independent copies of $Z(\lambda)$ which are independent of the positions $S(u)$, $|u|=1$. If U is centered α–stable and independent of $Z(\alpha\theta)$, then the limit variable $Z(\alpha\theta)^{1/\alpha}U$ in [21, Corollary 1.1] and [22, Corollary 1.3] satisfies

$$Z(\alpha\theta)^{1/\alpha}\,U = \Bigg(\sum_{|v|=1}\frac{e^{-\alpha\theta S(v)}}{m(\alpha\theta)}[Z(\lambda)]_v\Bigg)^{\frac{1}{\alpha}} U \overset{\text{law}}{=} \sum_{|v|=1}\frac{e^{-\theta S(v)}}{m(\alpha\theta)^{1/\alpha}}[Z(\lambda)]_v^{1/\alpha}\,U_v$$

where $(U_v)_{v\in\mathbb{N}}$ is a family of independent copies of U which is independent of all other random variables appearing on the right-hand side of the latter distributional equality. Hence, the distribution of $Z(\alpha\theta)^{1/\alpha}\,U$ is a solution to the following fixed-point equation of the smoothing transformation:

$$X \overset{\text{law}}{=} \sum_{j\geq 1} T_j X_j \tag{2.27}$$

where $T_j := \mathbb{1}_{\{j \in \mathcal{G}_1\}} \frac{e^{-\theta S(j)}}{m(\alpha\theta)^{1/\alpha}}$ and the X_j, $j \in \mathbb{N}$ are independent copies of the random variable X. In (2.27), which should be seen as an equation for the distribution of X rather than the random variable X itself, $T_1, T_2, \ldots$ are considered given whereas the distribution of X is considered unknown. Equation (2.27) has been studied in depth in the case where the T_j and X_j are nonnegative, see [4] for the most recent contribution and an overview of earlier results. If, however, we consider complex $Z(\lambda)$ at complex parameters, (2.26) becomes an equation between complex random variables and it is reasonable to conjecture that the limiting distributions of $a_n(Z(\lambda) - Z_n(\lambda))$ are solutions to (2.27) with complex-valued T_j and X_j. A systematic study of (2.27) in the case where T_j and X_j are complex-valued has been addressed only recently in [36].

3. Preliminaries

In this section, we fix some notation and set the stage for the proofs of our main results.

3.1. **Notation.**

Complex numbers. Throughout the paper, we identify $\mathbb{C}$ and $\mathbb{R}^2$. For instance, for $z \in \mathbb{C}$, we sometimes write z_1 for $\mathrm{Re}(z)$ and z_2 for $\mathrm{Im}(z)$. Further, we sometimes identify $z \in \mathbb{C}$ with the column vector $(z_1, z_2)^\mathsf{T}$ and write z^T for the row vector (z_1, z_2). As usual, we write $\overline{z}$ for the complex conjugate of $z \in \mathbb{C}$, i.e., $\overline{z} = z_1 - \mathrm{i} z_2$. In some proofs, we identify a complex number $z = re^{\mathrm{i}\varphi}$ with the matrix $rR(\varphi)$ where $R(\varphi)$ is the 2×2 rotation matrix

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

By $\hat{\mathbb{C}}$ we denote the one-point compactification of $\mathbb{C}$, i.e., $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and a set $K \subseteq \hat{\mathbb{C}}$ is relatively compact if it is relatively compact in $\mathbb{C}$ or the complement of a bounded subset of $\mathbb{C}$. A function $\phi : \hat{\mathbb{C}} \to \mathbb{R}$ is differentiable at ∞ if $\psi : \mathbb{C} \to \mathbb{R}$ with $\psi(z) = \phi(1/z)$ for $z \neq 0$ and $\psi(0) = \phi(\infty)$ is differentiable at 0.

Conditional expectations. Throughout the paper, we write $\mathbb{P}_n(\cdot)$ for $\mathbb{P}(\cdot|\mathcal{F}_n)$ for every $n \in \mathbb{N}_0$. The corresponding (conditional) expectation and variance are denoted $\mathbb{E}_n[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_n]$ and $\mathbb{V}\mathrm{ar}_n[\cdot] := \mathbb{V}\mathrm{ar}[\cdot|\mathcal{F}_n]$. We further write $\mathbb{E}[X; A]$ for $\mathbb{E}[X\mathbb{1}_A]$, $\mathbb{V}\mathrm{ar}[X; A]$ for $\mathbb{V}\mathrm{ar}[X\mathbb{1}_A]$, and $\mathrm{Cov}[X; A]$ for the covariance matrix of the vector $X\mathbb{1}_A$. If X is a complex random variable, we write $\mathrm{Cov}[X]$ for the covariance matrix of the vector $(\mathrm{Re}(X), \mathrm{Im}(X))^\mathsf{T}$. We also use the analogous notation with $\mathbb{E}$, $\mathbb{V}\mathrm{ar}$ and Cov replaced by $\mathbb{E}_n$, $\mathbb{V}\mathrm{ar}_n$ and Cov_n.

The martingale. Further, when $\lambda \in \Lambda$ is fixed, we sometimes write Z_n for $Z_n(\lambda)$ and Z for $Z(\lambda)$ in order to unburden the notation.

3.2. **Background and relevant results from the literature.**

Recursive decomposition of tail martingales. Throughout the paper, we denote by $[\cdot]_u$, $u \in \mathcal{I}$ the canonical shift operators, that is, for any function Ψ of $(\mathcal{Z}(v))_{v\in\mathcal{I}}$, we write $[\Psi]_u$ for the same function applied to the family $(\mathcal{Z}(uv))_{v\in\mathcal{I}}$. Using this notation, we obtain the following decomposition of $Z(\lambda) - Z_n(\lambda)$:

$$Z(\lambda) - Z_n(\lambda) = m(\lambda)^{-n} \sum_{|u|=n} e^{-\lambda S(u)}([Z(\lambda)]_u - 1) \quad \text{a.s.}, \tag{3.1}$$

which is valid for every $n \in \mathbb{N}_0$. Therefore, with respect to $\mathbb{P}_n$, $Z(\lambda) - Z_n(\lambda)$ is a sum of i.i.d. centered random variables. This explains the appearance of (randomly scaled) normal or stable distributions in our main theorems.

Minimal position: First order. If $\theta > 0$ with $m(\theta) < \infty$, then [9, Theorem 3] gives

$$\sup_{|u|=n} \tfrac{e^{-\theta S(u)}}{m(\theta)^n} \to 0 \quad \text{a.s. as } n \to \infty. \tag{3.2}$$

4. The Gaussian regime

Before we prove Theorems 2.2 and 2.3, we recall some basic facts about complex random variables.

Covariance calculations. The proofs of Theorems 2.2 and 2.3 are based on covariance calculations for complex random variables. We remind the reader of some simple but useful facts in this context. If $\zeta = \xi + \mathrm{i}\tau$ is a complex random variable with $\xi = \mathrm{Re}(\zeta)$ and $\tau = \mathrm{Im}(\zeta)$, then a simple calculation shows that the covariance matrix of ζ can be represented as

$$\mathrm{Cov}[\zeta] = \begin{pmatrix} \mathbb{E}[\xi^2] & \mathbb{E}[\xi\tau] \\ \mathbb{E}[\xi\tau] & \mathbb{E}[\tau^2] \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{Re}(\mathbb{E}[|\zeta|^2] + \mathbb{E}[\zeta^2]) & \mathrm{Im}(\mathbb{E}[|\zeta|^2] + \mathbb{E}[\zeta^2]) \\ \mathrm{Im}(\mathbb{E}[|\zeta|^2] + \mathbb{E}[\zeta^2]) & \mathrm{Re}(\mathbb{E}[|\zeta|^2] - \mathbb{E}[\zeta^2]) \end{pmatrix}. \tag{4.1}$$

Thus covariance calculations can be reduced to second moment calculations.

Proof of Theorems 2.2 and 2.3. Throughout the paragraph, for $n \in \mathbb{N}_0$ and $u \in \mathcal{G}_n$, we set $Y_u := e^{-\lambda S(u)}/m^{n/2}$. We start with a lemma.

Lemma 4.1. *Suppose that $\lambda \in \mathcal{D}$ with $m(\lambda) \neq 0$ is such that $\sigma_\theta^2 < \infty$, $\sigma_\lambda^2 > 0$ and $m(2\theta) < |m(\lambda)|^2$. then*

$$\mathbb{E}[|Z(\lambda)-1|^2] = \frac{\mathbb{E}[|Z_1-1|^2]}{1-\frac{m(2\theta)}{|m(\lambda)|^2}} < \infty \quad \textit{and} \quad \mathbb{E}[(Z(\lambda)-1)^2] = \frac{\mathbb{E}[(Z_1-1)^2]}{1-\frac{m(2\lambda)}{m(\lambda)^2}}. \tag{4.2}$$

Proof. Observe that (B1) and (B2) are satisfied with $\gamma = p = 2$. Consequently, $\mathbb{E}[|Z - 1|^2] < \infty$. In the next step, we calculate $\mathbb{E}[|Z - 1|^2]$ and $\mathbb{E}[(Z - 1)^2]$. (Actually, the calculations below again give $\mathbb{E}[|Z - 1|^2] < \infty$.) As the increments of square-integrable martingales are uncorrelated,

$$\begin{aligned}\mathbb{E}[|Z - 1|^2] &= \lim_{n\to\infty} \mathbb{E}[(Z_n - 1)(\overline{Z}_n - 1)] = \sum_{n=0}^{\infty} \mathbb{E}[|Z_{n+1} - Z_n|^2] \\ &= \mathbb{E}[|Z_1 - 1|^2] \sum_{n=0}^{\infty} \mathbb{E}\Big[\sum_{|u|=n} \frac{e^{-2\theta S(u)}}{|m(\lambda)|^{2n}}\Big] = \frac{\mathbb{E}[|Z_1 - 1|^2]}{1 - m(2\theta)/|m(\lambda)|^2}.\end{aligned}$$

Analogously, we infer

$$\mathbb{E}[(Z-1)^2] = \mathbb{E}[(Z_1-1)^2]\sum_{n=0}^{\infty}\mathbb{E}\bigg[\sum_{|u|=n}\frac{e^{-2\lambda S(u)}}{m(\lambda)^{2n}}\bigg] = \frac{\mathbb{E}[(Z_1-1)^2]}{1-m(2\lambda)/m(\lambda)^2}.$$

□

Our combined proof of Theorems 2.2 and 2.3 is based on an application of the Lindeberg-Feller central limit theorem.

Proof of Theorems 2.2 and 2.3. Recall that $m = m(2\theta)$ if $|m(2\lambda)| < m(2\theta)$ and $m = m(2\lambda)$ if $|m(2\lambda)| = m(2\theta)$. For $n \in \mathbb{N}$, define $c_n = 1$ in the situation of Theorem 2.2 and $c_n := n^{1/4}$ in the situation of Theorem 2.3. Further, let $a_n := c_n \frac{m(\lambda)^n}{m^{n/2}}$ for $n \in \mathbb{N}$. Then (3.1) can be rewritten in the form

$$a_n(Z-Z_n) = c_n \sum_{|u|=n} Y_u([Z]_u - 1). \tag{4.3}$$

The right-hand side of (4.3) given $\mathcal{F}_n$ is the sum of independent centered random variables. We show that the distribution of this sum given $\mathcal{F}_n$ converges in probability to the distribution of a complex or real normal random variable. To this end, we check the Lindeberg-Feller condition. For any $\varepsilon > 0$, using that $|m| = m(2\theta)$, we obtain

$$\begin{aligned}\sum_{|u|=n}\mathbb{E}_n[|c_nY_u([Z]_u-1)|^2\mathbb{1}_{\{|c_nY_u([Z]_u-1)|^2>\varepsilon\}}] &= c_n^2\sum_{|u|=n}|Y_u|^2\sigma_\lambda^2(\varepsilon c_n^{-2}|Y_u|^{-2}) \\ &= c_n^2\sum_{|u|=n}\frac{e^{-2\theta S(u)}}{m(2\theta)^n}\sigma_\lambda^2(\varepsilon c_n^{-2}|Y_u|^{-2})\end{aligned}$$

where, for $x \geq 0$,

$$\sigma_\lambda^2(x) := \mathbb{E}[|Z-1|^2\mathbb{1}_{\{|Z-1|^2>x\}}].$$

By Lemma 4.1, we have $\mathbb{E}[|Z-1|^2] < \infty$. The dominated convergence theorem thus yields $\sigma_\lambda^2(x) \downarrow 0$ as $x \uparrow \infty$. Moreover, in the situation of Theorem 2.2,

$$c_n^2 \sup_{|u|=n}|Y_u|^2 = \sup_{|u|=n}\frac{e^{-2\theta S(u)}}{m(2\theta)^n} \to 0 \quad \text{a.s. as } n\to\infty$$

by (3.2) (applied with θ replaced by 2θ). In the situation of Theorem 2.3,

$$c_n^2 \sup_{|u|=n}|Y_u|^2 = n^{1/2}\sup_{|u|=n} e^{-V(u)} \to 0 \quad \text{in } \mathbb{P}\text{-probability as } n\to\infty$$

by Proposition A.3. In any case, we conclude that

$$\begin{aligned}&\sum_{|u|=n}\mathbb{E}_n[|c_nY_u([Z]_u-1)|^2\mathbb{1}_{\{|c_nY_u([Z]_u-1)|^2>\varepsilon\}}] \\ &\qquad \leq c_n^2 Z_n(2\theta)\sigma_\lambda^2(\varepsilon(c_n\sup\nolimits_{|u|=n}|Y_u|)^{-2}) \to 0\end{aligned}$$

as $n\to\infty$ a.s. or in $\mathbb{P}$-probability, respectively, having utilized (2.13) for the convergence in $\mathbb{P}$-probability. By (4.1), covariance calculations can be reduced to calculations for the second absolute (conditional) moment and the second (conditional) moment of $\frac{m(\lambda)^n}{m^{n/2}}(Z-Z_n)$:

$$\begin{aligned}\mathbb{E}_n\big[|a_n(Z-Z_n)|^2\big] &= c_n^2\mathbb{E}_n\bigg[\bigg(\sum_{|u|=n} Y_u([Z]_u-1)\bigg)\bigg(\sum_{|v|=n}\overline{Y}_v([\overline{Z}]_v-1)\bigg)\bigg]\\ &= c_n^2\mathbb{E}_n\bigg[\sum_{|u|=n}|Y_u|^2|[Z]_u-1|^2\bigg] = \mathbb{E}[|Z-1|^2]c_n^2\sum_{|u|=n}|Y_u|^2\\ &= \mathbb{E}[|Z-1|^2]c_n^2 Z_n(2\theta). \end{aligned} \tag{4.4}$$

where the second equation follows from the fact that, for $|u|=|v|=n$ with $u\neq v$, $[Z]_u-1$ and $[Z]_v-1$ are independent and centered, and hence the cross terms vanish. The right-hand side of (4.4) converges to $\mathbb{E}[|Z-1|^2]Z(2\theta)$ a. s. in the situation of Theorem 2.2 and to $\mathbb{E}[|Z-1|^2](\frac{2}{\pi\sigma^2})^{1/2}D_\infty$ in $\mathbb{P}$-probability in the situation of Theorem 2.3. An analogous calculation gives

$$\mathbb{E}_n\big[(a_n(Z-Z_n))^2\big] = \mathbb{E}[(Z-1)^2]c_n^2\sum_{|u|=n} Y_u^2. \tag{4.5}$$

We shall find the limit of the right-hand side of (4.5), thereby verifying that the conditions [17, Eqs. (2.5)–(2.7)] are fulfilled. The claimed convergence then follows from the cited source and the Cramér-Wold device [27, p. 87, Corollary 5.5]. In the situation of Theorem 2.2, if $Z_n(2\theta)\to 0$ a. s., then $\sum_{|u|=n} Y_u^2\to 0$ a. s., so that nothing remains to be shown. Thus, for the remainder of the proof, we suppose that $Z_n(2\theta)$ converges a. s. and in L^1 to $Z(2\theta)$ or that (2.13) holds. We distinguish two cases.

Case 1: Let $|m(2\lambda)|<m(2\theta)$. We apply Lemma A.4 with $(L(u))_{u\in\mathcal{G}} = (Y_u^2)_{u\in\mathcal{G}}$. In this case

$$\mathbb{E}\bigg[\sum_{|u|=1}|L(u)|\bigg] = \mathbb{E}\bigg[\sum_{|u|=1}|Y_u|^2\bigg] = \mathbb{E}[Z_1(2\theta)] = 1.$$

Further,

$$a := \mathbb{E}\bigg[\sum_{|u|=1} L(u)\bigg] = \mathbb{E}\bigg[\sum_{|u|=1} Y_u^2\bigg] = \frac{m(2\lambda)}{m(2\theta)}$$

satisfies $|a|<1$. When the assumptions of Theorem 2.2 hold, Lemma A.4(b) applies (with condition (i) satisfied) and yields $\sum_{|u|=n} Y_u^2\to 0$ in $\mathbb{P}$-probability. If, additionally, $2\theta\in\Lambda$, then

$$\mathbb{E}\bigg[\sum_{|u|=1}|L(u)|^p\bigg] = \mathbb{E}\bigg[\sum_{|u|=1}|Y_u|^{2p}\bigg] = \frac{m(p2\theta)}{m(2\theta)^p} < 1$$

for some $p\in(1,2]$. Hence, $\sum_{|u|=n} Y_u^2\to 0$ a. s. by Lemma A.4(a). When the assumptions of Theorem 2.3 hold, we obtain $n^{1/2}\sum_{|u|=n} Y_u^2\to 0$ in $\mathbb{P}$-probability by another appeal to Lemma A.4(b) (this time with condition (ii) satisfied). Thus, under the assumptions of both theorems, the limit of the right-hand side of (4.5) vanishes.

Case 2: Let $|m(2\lambda)| = m(2\theta)$. Then there exists some $\varphi\in[0,2\pi)$ such that $m(2\lambda) = m(2\theta)e^{\mathrm{i}\varphi}$. This implies $e^{-2\mathrm{i}\eta S(u)} = e^{\mathrm{i}\varphi}$ for all $|u|=1$ a. s., equivalently, $S(u)\in\frac{-\varphi}{2\eta}+\frac{\pi}{\eta}\mathbb{Z}$ for all $|u|=1$ a. s. Therefore, a. s. for every $u\in\mathcal{G}$,

$$e^{-\lambda S(u)} = e^{-\theta S(u)}e^{-\mathrm{i}\eta S(u)} = \pm e^{\mathrm{i}\varphi/2}e^{-\theta S(u)}$$

and thereupon $m(\lambda) = e^{\mathrm{i}\varphi/2}q$ where $q \in \mathbb{R}$ with $0 < |q| \leq m(\theta)$. Consequently, $Z_n(\lambda) \in \mathbb{R}$ a. s. for every $n \in \mathbb{N}_0$. Thus, also $Z(\lambda) \in \mathbb{R}$ a. s. Further $m(\lambda)^n/m^{n/2} = m(\lambda)^n/m(2\lambda)^{n/2} = q^n/m(2\theta)^{n/2} \in \mathbb{R}$. Hence, all terms in (4.4) and (4.5) coincide and so do their limits. □

It is worth noting that in Case 2, in order to arrive at the stronger statement (weak convergence a. s.), we do not need $2\theta \in \Lambda$, but only require the uniform integrability of $(Z_n(2\theta))_{n\in\mathbb{N}_0}$ or equivalently (2.7).

5. The regime in which the extremal positions dominate

First recall that $V(u)$ is defined by (2.10) and that $V_n(u) = V(u) - \frac{3}{2}\log n$ for $u \in \mathcal{G}_n$. Further, for each $K \in \mathbb{R}$ define $f_K : \mathbb{R} \to [0, 1]$ by

$$f_K(x) := \begin{cases} 1 & \text{for } x \leq K, \\ K + 1 - x & \text{for } K \leq x \leq K + 1, \\ 0 & \text{for } x \geq K + 1. \end{cases} \tag{5.1}$$

Our proof of Theorem 2.5 is based on two lemmas about the processes μ_n, $n \in \mathbb{N}$ and related point processes. Proposition 2.4 tells us that

$$\textstyle\int f \, \mathrm{d}\mu_n \stackrel{\mathrm{d}}{\to} \int f \, \mathrm{d}\mu_\infty \quad \text{as } n \to \infty \tag{5.2}$$

for all continuous and compactly supported $f : \mathbb{R} \to [0, \infty)$. This taken together with information about the left tail of μ_n for large n provided by [1, Theorem 1.1] enables us to show that relation (5.2) holds for a wider class of functions f. This is the content of Lemma 5.1.

Lemma 5.1. *Suppose that the assumptions of Theorem 2.5 are satisfied. Then relation* (5.2) *holds for all continuous functions* $f : \mathbb{R} \to [0, \infty)$ *with* $f(x) = 0$ *for all sufficiently large* x*.*

Proof. Pick an arbitrary continuous function $f : \mathbb{R} \to [0, \infty)$ satisfying $f(x) = 0$ for all sufficiently large x. For any fixed $K \in \mathbb{R}$, the function $g_K(x) := f(x)(1 - f_K(x))$ is continuous and has a compact support. Therefore, $\int g_K \mathrm{d}\mu_n \stackrel{\mathrm{d}}{\to} \int g_K \mathrm{d}\mu_\infty$ as $n \to \infty$ by Proposition 2.4. Since $\mu_\infty((-\infty, a]) < \infty$ a. s. for any $a \in \mathbb{R}$ by another appeal to Proposition 2.4, we infer

$$\lim_{K\to-\infty} \int g_K \, \mathrm{d}\mu_\infty = \int f \, \mathrm{d}\mu_\infty \quad \text{a. s.} \tag{5.3}$$

On the other hand, for any $\varepsilon > 0$,

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{P}\Big(\Big| \int f(x) f_K(x) \, \mu_n(\mathrm{d}x) \Big| > \varepsilon\Big) &\leq \limsup_{n\to\infty} \mathbb{P}\big(\mu_n((-\infty, K + 1]) \geq 1\big) \\ &= \limsup_{n\to\infty} \mathbb{P}\Big(\min_{|u|=n} V(u) - \tfrac{3}{2}\log n \leq K + 1\Big) \end{aligned}$$

where $\min_\emptyset := \infty$. By [1, Theorem 1.1],

$$\lim_{K\to-\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\min_{|u|=n} V(u) - \tfrac{3}{2}\log n \leq K + 1\Big) = 0.$$

The latter limit relation, (5.3) and [12, Theorem 4.2] imply $\int f \mathrm{d}\mu_n \stackrel{\mathrm{d}}{\to} \int f \mathrm{d}\mu_\infty$. □

With Lemma 5.1 at hand we can now show that for any $\gamma > 1$

$$\textstyle\int e^{-\gamma x}\,\mu_\infty(\mathrm{d}x) = \sum_k e^{-\gamma P_k} < \infty \quad \text{a.s.} \tag{5.4}$$

To see this, pick $M > 0$ and consider the following chain of inequalities:

$$\begin{aligned}\mathbb{P}\Big(\sum_j e^{-\gamma P_j} > M\Big) &= \sup_{K\in\mathbb{N}} \mathbb{P}\Big(\sum_j e^{-\gamma P_j} f_K(P_j) > M\Big)\\ &\leq \sup_{K\in\mathbb{N}} \liminf_{n\to\infty} \mathbb{P}\Big(\sum_{|u|=n} e^{-\gamma V_n(u)} f_K(V_n(u)) > M\Big)\\ &\leq \limsup_{n\to\infty} \mathbb{P}\Big(\sum_{|u|=n} e^{-\gamma V_n(u)} > M\Big)\end{aligned}$$

where Lemma 5.1 and the Portmanteau theorem have been used for the first inequality. The latter lim sup tends to 0 as $M \to \infty$ by [34, Proposition 2.1].

Recall that $(Z^{(k)})_{k\in\mathbb{N}}$ denotes a sequence of independent copies of $Z(\lambda)-1$ which are also independent of $\mu_\infty = \sum_k \delta_{P_k}$. We define point processes on $\mathbb{R}\times\mathbb{C}$ by

$$\mu_\infty^* := \sum_k \delta_{(P_k, Z^{(k)})} \quad \text{and} \quad \mu_n^* := \sum_{|u|=n} \delta_{(V_n(u), [Z(\lambda)]_u - 1)}, \quad n \in \mathbb{N}.$$

Lemma 5.2. *Suppose that the assumptions of Theorem 2.5 are satisfied. Then* $\int f\,\mathrm{d}\mu_n^* \stackrel{\mathrm{d}}{\to} \int f\,\mathrm{d}\mu_\infty^*$ *for all bounded continuous function* $f : \mathbb{R}\times\mathbb{C} \to \mathbb{C}$ *such that* $f(x,z) = 0$ *whenever* x *is sufficiently large.*

Proof. We derive the assertion from Lemma 5.1. More precisely, first let $f : \mathbb{R}\times\mathbb{C} \to [0,\infty)$ be an arbitrary continuous function such that $f(x,z) = 0$ for all $z \in \mathbb{C}$ whenever x is sufficiently large. Since the convergence $\int f\,\mathrm{d}\mu_n^* \stackrel{\mathrm{d}}{\to} \int f\,\mathrm{d}\mu_\infty^*$ is equivalent to the convergence of the corresponding Laplace transforms it suffices to show that the Laplace functional of μ_n^* at f converges to the Laplace functional of μ_∞^* at f. To this end, define $\varphi(x) := \mathbb{E}[\exp(-f(x, Z^{(1)}))]$ for $x \in \mathbb{R}$. Clearly, $0 < \varphi \leq 1$. Further, the continuity of f together with the dominated convergence theorem imply that φ is continuous. Therefore, $-\log\varphi : \mathbb{R} \to [0,\infty)$ is continuous. Since $f(x,z) = 0$ for all sufficiently large x, the same is true for $-\log\varphi$. Lemma 5.1 implies that $\int(-\log\varphi(x))\,\mu_n(\mathrm{d}x) \stackrel{\mathrm{d}}{\to} \int(-\log\varphi(x))\,\mu_\infty(\mathrm{d}x)$. Using this, we find that the Laplace functional of μ_n^* evaluated at f satisfies

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\int f(x,y)\,\mu_n^*(\mathrm{d}x,\mathrm{d}y)\Big)\Big] &= \mathbb{E}\Big[\mathbb{E}_n\Big[\exp\Big(-\sum_{|u|=n} f(V_n(u),[Z(\lambda)]_u-1)\Big)\Big]\Big]\\ &= \mathbb{E}\Big[\prod_{|u|=n}\varphi(V_n(u))\Big]\\ &= \mathbb{E}\Big[\exp\Big(-\sum_{|u|=n}(-\log\varphi(V_n(u)))\Big)\Big]\\ &= \mathbb{E}\Big[\exp\Big(-\int(-\log\varphi(x))\,\mu_n(\mathrm{d}x)\Big)\Big]\\ &\to \mathbb{E}\Big[\exp\Big(-\int(-\log\varphi(x))\,\mu_\infty(\mathrm{d}x)\Big)\Big]\\ &= \mathbb{E}\Big[\exp\Big(-\int f(x,y)\,\mu_\infty^*(\mathrm{d}x,\mathrm{d}y)\Big)\Big].\end{aligned}$$

This completes the proof for nonnegative f. For the general case, we decompose $f = f_1 - f_2 + \mathrm{i}(f_3 - f_4)$ with $f_j : \mathbb{R}\times\mathbb{C} \to [0,\infty)$ vanishing for large x. Then for any nonnegative λ_j, from the first part, we conclude

$$\int(\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4)\,\mathrm{d}\mu_n^* \stackrel{\mathrm{d}}{\to} \int(\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4)\,\mathrm{d}\mu_\infty^*$$

and, in particular, we infer $(\int f_j\,\mathrm{d}\mu_n^*)_{j=1,\ldots,4} \stackrel{\mathrm{d}}{\to} (\int f_j\,\mathrm{d}\mu_\infty^*)_{j=1,\ldots,4}$ from which we deduce the convergence $\int f\,\mathrm{d}\mu_n^* \stackrel{\mathrm{d}}{\to} \int f\,\mathrm{d}\mu_\infty^*$. ☐

We now make the final preparations for the proof of Theorem 2.5. We have to show that $z_n(Z(\lambda) - Z_n(\lambda))$ converges in distribution where

$$z_n := n^{\frac{3\lambda}{2\vartheta}}\Big(\frac{m(\lambda)}{m(\vartheta)^{\lambda/\vartheta}}\Big)^n, \quad n\in\mathbb{N}.$$

We shall use the decomposition

$$\begin{aligned}z_n(Z(\lambda)-Z_n(\lambda)) &= \frac{z_n}{m(\lambda)^n}\sum_{|u|=n} e^{-\lambda S(u)}([Z(\lambda)]_u-1)\\ &= \frac{z_n}{m(\lambda)^n} e^{n\frac{\lambda}{\vartheta}\log m(\vartheta)}\sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V(u)}([Z(\lambda)]_u-1)\\ &= \sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)}([Z(\lambda)]_u-1)\\ &= \sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)} f_K(V_n(u))([Z(\lambda)]_u-1)\\ &\quad+\sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)}\big((1-f_K(V_n(u)))\cdot([Z(\lambda)]_u-1)\big)\\ &=: Y_{n,K} + R_{n,K}.\end{aligned}$$

We first check that the contribution of $R_{n,K}$ is negligible as K tends to infinity.

Lemma 5.3. *If the assumptions of Theorem 2.5 hold, then, for any* $\delta > 0$ *and every measurable* $h_K : \mathbb{R} \to [0,1]$ *satisfying* $0 \leq h_K \leq \mathbb{1}_{[K,\infty)}$

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\bigg(\bigg|\sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)} h_K(V_n(u))([Z(\lambda)]_u - 1)\bigg| > \delta\bigg) = 0.$$

Proof. Let $\varepsilon, \delta > 0$ and $1 < \beta < \beta_0 := p \cdot \frac{\theta}{\vartheta}$. From Proposition 2.1 in [34], we know that the sequence of distributions of the random variables $\sum_{|u|=n} e^{-\beta V_n(u)}$, $n \in \mathbb{N}$ is tight. Therefore, there is an $M > 0$ such that $\sup_{n\in\mathbb{N}} \mathbb{P}(\mathcal{Q}_n) \leq \varepsilon$ where

$$\mathcal{Q}_n := \Big\{ \sum_{|u|=n} e^{-\beta V_n(u)} > M \Big\}.$$

Then

$$\mathbb{P}\bigg(\bigg|\sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)} h_K(V_n(u))([Z(\lambda)]_u - 1)\bigg| > \delta\bigg)$$
$$\leq \mathbb{P}\bigg(\bigg|\sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)} h_K(V_n(u))([Z(\lambda)]_u - 1)\bigg| > \delta, \mathcal{Q}_n^{\mathsf{c}}\bigg) + \varepsilon.$$

We estimate the above probability using the following strategy. First, we use Markov's inequality for the function $x \mapsto |x|^p$. Then, given $\mathcal{F}_n$, we apply Lemma A.1. This gives

$$\begin{aligned}
&\mathbb{P}\bigg(\bigg|\sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)} h_K(V_n(u))([Z(\lambda)]_u - 1)\bigg| > \delta, \mathcal{Q}_n^{\mathsf{c}}\bigg)\\
&\leq \frac{4}{\delta^p} \cdot \mathbb{E}[|Z(\lambda) - 1|^p] \cdot \mathbb{E}\bigg[\sum_{|u|=n} e^{-\beta_0 V_n(u)} h_K(V_n(u))^p \mathbb{1}_{\mathcal{Q}_n^{\mathsf{c}}}\bigg]\\
&\leq \frac{4}{\delta^p} \cdot \mathbb{E}[|Z(\lambda) - 1|^p] \cdot e^{(\beta-\beta_0)K} \mathbb{E}\bigg[\sum_{|u|=n} e^{-\beta V_n(u)} \mathbb{1}_{\mathcal{Q}_n^{\mathsf{c}}}\bigg]\\
&\leq \frac{4}{\delta^p} \cdot \mathbb{E}[|Z(\lambda) - 1|^p] \cdot e^{(\beta-\beta_0)K} M. \qquad (5.5)
\end{aligned}$$

The above bound does not depend on n and, moreover, tends to 0 as $K \to \infty$. The latter is obvious since $\beta < \beta_0$ and thus $\lim_{K\to\infty} e^{(\beta-\beta_0)K} = 0$.
We conclude that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\bigg(\bigg|\sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)} h_K(V_n(u))([Z(\lambda)]_u - 1)\bigg| > \delta\bigg) \leq \varepsilon.$$

The assertion follows as we may choose ε arbitrarily small. □

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. Define

$$\hat{Y}_0 := 0 \quad \text{and} \quad \hat{Y}_n := \sum_{k=1}^{n} e^{-\frac{\lambda}{\vartheta}P_k^*} Z^{(k)}(\lambda), \quad n \in \mathbb{N}$$

and recall the notation $\beta_0 = p \cdot \frac{\theta}{\vartheta} > 1$. Given μ_∞, for each $n \in \mathbb{N}$, the random variable $\hat{Y}_n$ is the sum of complex-valued independent centered random variables.

An application of Lemma A.1 yields

$$\mathbb{E}\big[|\hat{Y}_n|^p\big|\mu_\infty\big] \leq 4\ \mathbb{E}[|Z(\lambda)-1|^p] \cdot \sum_{k=1}^{n} e^{-\beta_0 P_k^*} \leq 4\ \mathbb{E}[|Z(\lambda)-1|^p] \cdot \sum_k e^{-\beta_0 P_k},$$

and the latter sum is almost surely finite by (5.4). This shows that $(\hat{Y}_n)_{n\in\mathbb{N}_0}$, conditionally on μ_∞, is an L^p-bounded martingale. We conclude that $\hat{Y}_n$ converges a. s. conditionally on μ_∞, hence, also unconditionally thereby proving the first part of Theorem 2.5.

The proof of the second part is based on an application of Theorem 4.2 in [12] and the decomposition

$$z_n(Z(\lambda) - Z_n(\lambda)) = Y_{n,K} + R_{n,K}.$$

In view of Lemma 5.3, the cited theorem gives the assertion once we have shown the following two assertions:

1. $Y_{n,K} \stackrel{\mathrm{d}}{\to} Y_K$ as $n \to \infty$ for every fixed $K > 0$ where Y_K is some finite random variable;
2. $Y_K \stackrel{\mathrm{d}}{\to} X_{\mathrm{ext}}$ as $K \to \infty$.

The first assertion is a consequence of Lemma 5.2. Indeed, the function $(x, z) \mapsto e^{-\frac{\lambda}{\vartheta}x} f_K(x)z$ is continuous and vanishes for all sufficiently large x. Therefore, Lemma 5.2 yields

$$\begin{aligned} Y_{n,K} &= \sum_{|u|=n} e^{-\frac{\lambda}{\vartheta}V_n(u)} f_K(V_n(u))([Z(\lambda)]_u - 1) \\ &= \int e^{-\frac{\lambda}{\vartheta}x} f_K(x)z\, \mu_n^*(\mathrm{d}x, \mathrm{d}z) \stackrel{\mathrm{d}}{\to} \int e^{-\frac{\lambda}{\vartheta}x} f_K(x)z\, \mu_\infty^*(\mathrm{d}x, \mathrm{d}z) =: Y_K. \end{aligned}$$

Set $\hat{Y} := \sum_k e^{-\frac{\lambda}{\vartheta}P_k} Z^{(k)}(\lambda)$ and note that $\hat{Y} \stackrel{\text{law}}{=} X_{\mathrm{ext}}$. To see that the second assertion holds, we prove that $\mathbb{E}[|\hat{Y} - Y_K|^p|\mu_\infty] \to 0$ a. s. as $K \to \infty$ which entails $Y_K \stackrel{\mathbb{P}}{\to} \hat{Y}$ as $K \to \infty$. To this end, we use (an infinite version of) Lemma A.1 to obtain

$$\begin{aligned} \mathbb{E}\big[|\hat{Y} - Y_K|^p|\mu_\infty\big] &\leq 4\mathbb{E}[|Z(\lambda) - 1|^p\big] \cdot \sum_k e^{-\beta_0 P_k}\big(1 - f_K(P_k)\big)^p \\ &\leq 4\mathbb{E}[|Z(\lambda) - 1|^p\big] \cdot \sum_k e^{-\beta_0 P_k} \mathbb{1}_{\{P_k>K\}}. \end{aligned}$$

In view of (5.4) the right-hand side converges to zero a. s. as $K \to \infty$. The proof of Theorem 2.5 is complete. □

6. The boundary $\partial\Lambda^{(1,2)}$

Throughout this section, we fix $\lambda \in \mathcal{D}$ and suppose that

$$\tfrac{m(\alpha\theta)}{|m(\lambda)|^\alpha} = 1 \quad \text{and} \quad \tfrac{\theta m'(\theta\alpha)}{|m(\lambda)|^\alpha} = \log(|m(\lambda)|) \tag{C1}$$

holds with $\alpha \in (1, 2)$, i.e., $\lambda \in \partial\Lambda^{(1,2)}$ and that there are $\gamma \in (\alpha, 2]$ and $\kappa \in (\frac{\alpha}{2}, 1)$ such that

$$(2.19) \quad \mathbb{E}[Z_1(\theta)^\gamma] < \infty \qquad \text{and} \qquad (2.20) \quad \mathbb{E}\big[Z_1(\kappa\theta)^2\big] < \infty.$$

As before, for $n \in \mathbb{N}_0$ and $u \in \mathcal{G}_n$, we set $L(u) := e^{-\lambda S(u)}/m(\lambda)^n$, and abbreviate $Z_n(\lambda)$ and $Z(\lambda)$ by Z_n and Z, respectively. Notice that $\sum_{|u|=n} |L(u)|^\alpha =$

$\sum_{|u|=n} e^{-V(u)}$ for each $n \in \mathbb{N}_0$ where $V(u)$ is defined in (2.10), i.e., $V(u) := \alpha\theta S(u) + |u|\log(m(\alpha\theta))$. The assumptions of Theorem 2.8 guarantee that (2.13) holds, that is,

$$\sqrt{n}\sum_{|u|=n}|L(u)|^\alpha \xrightarrow{\mathbb{P}} \sqrt{\frac{2}{\pi\sigma^2}}D_\infty. \tag{6.1}$$

Indeed, (C1) and (2.19) entail conditions (2.11), (2.14) and (2.15), which are sufficient for (2.13) to hold. To be more precise, (C1) implies (2.11). Further, the function

$$\mathbb{R} \ni t \mapsto m_V(t) = \mathbb{E}\bigg[\sum_{|u|=1} e^{-tV(u)}\bigg] = \frac{m(\alpha\theta t)}{m(\alpha\theta)^t}$$

is finite at $t = 1/\alpha < 1$ since $\lambda \in \mathcal{D}$ satisfies (C1) and at $t = \gamma/\alpha > 1$ since, by superadditivity,

$$\bigg(\sum_{|u|=1} e^{-\theta S(u)}\bigg)^\gamma \geq \sum_{|u|=1} e^{-\gamma\theta S(u)},$$

and (2.19) holds. Therefore, m_V is finite on $[1/\alpha, \gamma/\alpha]$ and analytic on $(1/\alpha, \gamma/\alpha)$. In particular, the second derivative is finite at $t = 1$, which yields (2.14). Again by superadditivity, we conclude that

$$\bigg(\sum_{|u|=1} e^{-\theta S(u)}\bigg)^\gamma \geq \bigg(\sum_{|u|=1} e^{-\alpha\theta S(u)}\bigg)^{\gamma/\alpha}.$$

Thus, (2.19) implies the first condition in (2.15). To see that the second condition in (2.15) also holds, pick $\delta > 0$ such that $\alpha - \delta > 1$ and use

$$\bigg(\sum_{|u|=1} e^{-\theta S(u)}\bigg)^\gamma \geq \bigg(\sum_{|u|=1} e^{-(\alpha-\delta)\theta S(u)}\bigg)^{\frac{\gamma}{\alpha-\delta}} \geq \delta^{\frac{\gamma}{\alpha-\delta}} \cdot \bigg(\sum_{|u|=1} e^{-\alpha\theta S(u)}(\theta S(u))_+\bigg)^{\frac{\gamma}{\alpha-\delta}}.$$

6.1. **Martingale fluctuations on $\partial\Lambda^{(1,2)}$.** First, we show how from the knowledge of the tail behaviour of $Z(\lambda)$ we can deduce Theorem 2.9. To this end, suppose that the assumptions of Theorem 2.8 are satisfied. Set $W = Z - 1$ and observe that W has the same tail behavior as Z, i.e.,

$$\lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} \mathbb{E}[|z|^{-\alpha}\phi(zW)] = \int \phi \,\mathrm{d}\nu \tag{6.2}$$

for any $\phi \in C_c^2(\hat{\mathbb{C}}\setminus\{0\})$. To see that this is true, first notice that, for any $w \in \hat{\mathbb{C}}\setminus\{0\}$ and $z \in \mathbb{U}$ such that $|z| \leq 1$, we have

$$|\phi(w) - \phi(w-z)| = |z|\bigg|\frac{\phi(w) - \phi(w-z)}{z}\bigg| \leq |z|\sup_u|\nabla\phi(u)|\mathbb{1}_{P_1}(w),$$

where P_j is the j-neighborhood of $\operatorname{supp}\phi$, i.e., $P_j = \{u : |u-t| \leq j \text{ for some } t \in \operatorname{supp}\phi\}$. Setting $\chi(w) := \sup_u|\nabla\phi(u)|\mathbb{1}_{P_2} * \chi_0(w)$ where $\chi_0 : \mathbb{C} \to [0,\infty)$ is a probability density function smooth on $\mathbb{C}$ and supported by the unit disc, we infer that $\chi \in C_c^2(\hat{\mathbb{C}}\setminus\{0\})$ and

$$|\phi(w) - \phi(w-z)| \leq |z|\chi(w).$$

Hence,

$$\Big|\lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} \mathbb{E}[|z|^{-\alpha}\phi(zW)] - \lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} \mathbb{E}[|z|^{-\alpha}\phi(zZ)]\Big| \leq \lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} |z|\cdot\mathbb{E}[|z|^{-\alpha}\chi(zZ)] = 0$$

where Theorem 2.8 has been used.

Our first result in this section is a corollary of Theorem 2.8. Recall that, for a complex number $z\in\mathbb{C}$, we sometimes write $z_1 = \mathrm{Re}(z)$ and $z_2 = \mathrm{Im}(z)$.

Corollary 6.1. *In the situation of Theorem 2.8, for every $h>0$ with $\nu(\{y : |y| = h\}) = 0$ and every $j,k = 1,2$, we have*

$$\lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} |z|^{-\alpha}\mathbb{E}[(zW)_j(zW)_k; |zW|\leq h] = \int_{\{|y|<h\}} y_j y_k\,\nu(\mathrm{d}y) \tag{6.3}$$

$$\textit{and}\quad \lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} |z|^{-\alpha}\mathbb{E}[zW; |zW|\leq h] = -\int_{\{|y|>h\}} y\,\nu(\mathrm{d}y). \tag{6.4}$$

Proof. We start with some preparations. Throughout the proof, when letting $|z|\to 0$ it is tacitly assumed that $z\in\mathbb{U}$. First, observe that

$$\limsup_{|z|\to 0} |z|^{-\alpha}\mathbb{E}[|zW|^2; |zW|<\delta] \to 0 \quad \text{as } \delta\to 0. \tag{6.5}$$

To see this, first choose a nonnegative function $\phi\in C_c^2(\hat{\mathbb{C}}\setminus\{0\})$ satisfying $\phi \geq \mathbb{1}_{\{|z|\geq 1\}}$. Then, by (6.2),

$$|z|^{-\alpha}\mathbb{P}(|zW|>1) \leq |z|^{-\alpha}\mathbb{E}[\phi(zW)] \to \textstyle\int \phi\,\mathrm{d}\nu$$

as $|z|\to 0$. In particular, there is a finite constant $C>0$ such that

$$\sup_{0<|z|\leq 1} |z|^{-\alpha}\mathbb{P}(|zW|>1) \leq C. \tag{6.6}$$

In order to prove (6.5), pick $\delta\in(0,1)$. We may suppose that $0<|z|<\delta$. Then

$$\begin{aligned}|z|^{-\alpha}\mathbb{E}[|zW|^2; |zW|<\delta] &\leq |z|^{-\alpha}\mathbb{E}[(|zW|\wedge\delta)^2]\\ &= |z|^{-\alpha}\int_0^{|z|} 2t\mathbb{P}(|zW|>t)\,\mathrm{d}t + |z|^{-\alpha}\int_{|z|}^{\delta} 2t\mathbb{P}(|zW|>t)\,\mathrm{d}t.\end{aligned}$$

The first integral can be bounded above by

$$|z|^{-\alpha}\int_0^{|z|} 2t\,\mathrm{d}t = |z|^{2-\alpha} \leq \delta^{2-\alpha}.$$

Regarding the second integral, use (6.6) to arrive at

$$|z|^{-\alpha}\int_{|z|}^{\delta} 2t\mathbb{P}(|zW|>t)\,\mathrm{d}t = 2\int_{|z|}^{\delta} t^{1-\alpha}\Big(\frac{|z|}{t}\Big)^{-\alpha}\mathbb{P}\Big(\Big|\frac{z}{t}W\Big|>1\Big)\,\mathrm{d}t \leq \frac{2C\delta^{2-\alpha}}{2-\alpha}.$$

In conclusion, (6.5) holds. Further, we have to show that

$$\limsup_{|z|\to 0} |z|^{-\alpha}\mathbb{E}[|zW|; |zW|>K] \to 0 \quad \text{as } K\to\infty. \tag{6.7}$$

Indeed, in view of (6.6), we find

$$\begin{aligned}
&\limsup_{|z|\to 0} |z|^{-\alpha}\mathbb{E}[|zW|; |zW|>K] \\
&= \limsup_{|z|\to 0} \Big[\int_K^\infty t^{-\alpha}\Big|\frac{z}{t}\Big|^{-\alpha}\mathbb{P}\Big(\Big|\frac{z}{t}W\Big|>1\Big)\,\mathrm{d}t + K^{1-\alpha}\Big|\frac{z}{K}\Big|^{-\alpha}\mathbb{P}\Big(\Big|\frac{z}{K}W\Big|>1\Big)\Big] \\
&\leq CK^{1-\alpha}\Big(\frac{1}{\alpha-1}+1\Big) = \frac{CK^{1-\alpha}\alpha}{\alpha-1},
\end{aligned}$$

which tends to zero as $K\to\infty$. Hence, (6.7) holds.

We are ready to prove (6.4). To this end, observe that $\mathbb{E}[zW; |zW|\leq h] = -\mathbb{E}[zW; |zW|>h]$ since $\mathbb{E}[W]=0$. Now pick $0<\delta<h<K$ such that $h+\delta<K$ and that

$$\nu(\{y : |y| = h\}) = 0. \tag{6.8}$$

Let $\phi\in C_{\mathrm{c}}^2(\hat{\mathbb{C}}\setminus\{0\})$ be of the form $\phi(z) = zf(|z|)$ with twice continuously differentiable $f:[0,\infty)\to[0,1]$ satisfying $f(z)=0$ for $z\leq h$ and $f(z)=1$ for $z\in[h+\delta,K]$. Then

$$\begin{aligned}
&\limsup_{|z|\to 0}\Big|\int_{\{|y|>h\}} y\,\nu(\mathrm{d}y) - |z|^{-\alpha}\mathbb{E}[zW; |zW|>h]\Big| \\
&\quad\leq \limsup_{|z|\to 0}\Big|\int \phi(y)\,\nu(\mathrm{d}y) - |z|^{-\alpha}\mathbb{E}[\phi(zW)]\Big| \\
&+ \int_{\{h<|y|<h+\delta\}} |y-\phi(y)|\,\nu(\mathrm{d}y) + \limsup_{|z|\to 0}|z|^{-\alpha}\mathbb{E}[|zW-\phi(zW)|; h<|zW|<h+\delta] \\
&\quad+ \int_{\{|y|>K\}} |y-\phi(y)|\,\nu(\mathrm{d}y) + \limsup_{|z|\to 0}|z|^{-\alpha}\mathbb{E}[|zW-\phi(zW)|; |zW|>K] \\
&\quad\leq \int_{\{h<|y|<h+\delta\}} |y|\,\nu(\mathrm{d}y) + \limsup_{|z|\to 0}|z|^{-\alpha}(h+\delta)\mathbb{P}(h<|zW|<h+\delta) \\
&\quad+ \int_{\{|y|>K\}} |y|\,\nu(\mathrm{d}y) + \limsup_{|z|\to 0}|z|^{-\alpha}\mathbb{E}[|zW|; |zW|>K]
\end{aligned}$$

having utilized (6.2) and $|y-\phi(y)|\leq y$ for $|y|\in\mathbb{C}$. The first (second) term on the right-hand side converges to zero as $\delta\to 0$ in view of (6.8) (and suitable approximation of $\mathbb{1}_{\{h<|z|<h+\delta\}}$ by twice continuously differentiable functions with subsequent application of (6.2)). The third and fourth term tend to 0 as $K\to\infty$ by (6.7) and since $\int_{\{|y|\geq 1\}}|y|\,\nu(\mathrm{d}y)<\infty$. The latter follows from the fact that $\nu(\{|y|\geq t\}) = t^{-\alpha}\nu(\{|y|\geq 1\})$ which is due to the $(\mathbb{U},\alpha)$-invariance of ν.

Turning to the proof of (6.3), we fix $h>0$ satisfying (6.8) and pick $j,k\in\{1,2\}$. For $0<\delta<h/2$, choose $f\in C_{\mathrm{c}}^2((0,\infty))$ taking values in $[0,1]$ with $f=0$ on $(0,\delta/2]$, $f=1$ on $[\delta,h-\delta]$ and $f=0$ on $[h+\delta,\infty)$. Define $\phi\in C_{\mathrm{c}}^2(\hat{\mathbb{C}}\setminus\{0\})$ via $\phi(z) = z_jz_kf(|z|)$, $z\in\hat{\mathbb{C}}$. In particular, $\phi(z)=z_jz_k$ for $\delta\leq|z|\leq h-\delta$ and $\phi(z)=0$ for $|z|>h+\delta$. Using (6.2) with this ϕ and (6.5) and arguing along the lines of the proof of (6.4), we conclude that

$$\begin{aligned}\limsup_{|z|\to 0}&\left||z|^{-\alpha}\mathbb{E}[(zW)_j(zW)_k;|zW|\leq h]-\int_{\{|y|\leq h\}}y_jy_k\,\nu(\mathrm{d}y)\right|\\&\leq\limsup_{|z|\to 0}|z|^{-\alpha}\mathbb{E}[|zW|^2;|zW|<\delta]\\&\quad+(h+\delta)^2\limsup_{|z|\to 0}|z|^{-\alpha}\mathbb{P}(h-\delta<|zW|\leq h+\delta)\\&\quad+\int_{\{|y|<\delta\}}|y|^2\,\nu(\mathrm{d}y)+(h+\delta)^2\nu(\{h-\delta<|y|\leq h+\delta\}).\end{aligned}$$

This bound tends to 0 as $\delta\to 0$. We conclude that (6.3) holds. □

We are now ready to prove Theorem 2.9.

Proof of Theorem 2.9. For any strictly increasing sequence of natural numbers, we can pass to a subsequence $(n_k)_{k\in\mathbb{N}}$ such that the convergence in (6.1) and (A.6) hold a. s. along this subsequence. Once more, we use decomposition (3.1). First, we show that the triangular array $\{n_k^{w/(2\alpha)}L(u)([Z]_u-1)\}_{|u|=n_k,k\in\mathbb{N}}$ is a null array. Indeed,

$$\sup_{|u|=n_k}\mathbb{E}_{n_k}[|n_k^{\frac{w}{2\alpha}}L(u)([Z]_u-1)|\wedge 1]\leq\mathbb{E}[|Z-1|]\cdot n_k^{\frac{1}{2\alpha}}\sup_{|u|=n_k}e^{-\frac{1}{\alpha}V(u)}\to 0\text{ a. s.}$$

as $k\to\infty$ by (A.6). According to [27, Theorem 15.28 and p. 295], it suffices to prove that, for every $h>0$ with $\nu(\{z:|z|=h\})=0$,

$$\sum_{|u|=n_k}\mathcal{L}(n_k^{\frac{w}{2\alpha}}L(u)[W]_u\mid\mathcal{F}_{n_k})\to cD_\infty\nu\text{ vaguely in }\hat{\mathbb{C}}\setminus\{0\},\tag{6.9}$$

$$\sum_{|u|=n_k}\mathrm{Cov}_{n_k}[n_k^{\frac{w}{2\alpha}}L(u)[W]_u;|n_k^{\frac{1}{2\alpha}}L(u)[W]_u|\leq h]\to cD_\infty\int_{\{|z|\leq h\}}zz^{\mathsf{T}}\,\nu(\mathrm{d}z)\text{ a. s.},\tag{6.10}$$

$$\sum_{|u|=n_k}\mathbb{E}_{n_k}[n_k^{\frac{w}{2\alpha}}L(u)[W]_u;|n_k^{\frac{1}{2\alpha}}L(u)[W]_u|\leq h]\to -cD_\infty\int_{\{|z|\leq h\}}z\,\nu(\mathrm{d}z)\text{ a. s.}\tag{6.11}$$

where $c=\sqrt{\frac{2}{\pi\sigma^2}}$. Take any $\phi\in C_c^2(\hat{\mathbb{C}}\setminus\{0\})$. Then, by (6.2),

$$\lim_{k\to\infty}\sum_{|u|=n_k}\mathbb{E}_{n_k}[\phi(n_k^{\frac{w}{2\alpha}}L(u)[W]_u)]=\lim_{k\to\infty}n_k^{1/2}\sum_{|u|=n_k}|L(u)|^\alpha\int\phi\,\mathrm{d}\nu=cD_\infty\int\phi\,\mathrm{d}\nu$$

a. s. proving (6.9). Similarly, for (6.10) and (6.11), we can apply (6.3) and (6.4), respectively. As a result we conclude that for any bounded continuous function $\psi:\mathbb{C}\to\mathbb{R}$ it holds that

$$\mathbb{E}_{n_k}[\psi(n_k^{1/2\alpha}(Z-Z_{n_k}))]\to\mathbb{E}[\psi(X_{cD_\infty})|\mathcal{F}_\infty]\quad\text{a. s.}\tag{6.12}$$

with c as before. To summarize, we have shown that from any deterministic strictly increasing sequence of positive integers, we can extract a deterministic subsequence $(n_k)_{k\in\mathbb{N}}$ such that (6.12) holds. In other words, for every bounded and continuous $\psi:\mathbb{C}\to\mathbb{R}$,

$$\mathbb{E}_n[\psi(n^{\frac{w}{2\alpha}}(Z-Z_n))]\xrightarrow{\mathbb{P}}\mathbb{E}[\psi(X_{cD_\infty})|\mathcal{F}_\infty]\quad\text{as }n\to\infty,$$

i.e., (2.24) holds. □

6.2. **The tail behavior of $Z(\lambda)$ for $\lambda \in \partial\Lambda^{(1,2)}$.**

An upper bound on the tails of the distribution of $Z(\lambda)$ for $\lambda \in \partial\Lambda^{(1,2)}$.

Proof of Proposition 2.6. Proposition 2.6 can be proved along the lines of the proof of Theorem 2.1 in [29]. Equation (4.3) in the cited source carries over to the present situation, so it suffices to show that the truncated martingale $(Z_n^{(t)}(\lambda))_{n\in\mathbb{N}_0}$ with increments

$$Z_n^{(t)} - Z_{n-1}^{(t)} = \sum_{|u|=n-1} L(u)\mathbb{1}_{\{|L(u|_j)|\leq t \text{ for } j=0,\ldots,n-1\}}([Z_1]_u - 1)$$

satisfies

$$\sup_{n\in\mathbb{N}_0} \mathbb{E}[|Z_n^{(t)} - 1|^p] \leq \text{const} \cdot t^{\gamma-\alpha}$$

where $1 \leq p < \alpha$ and the constant is independent of t. (This bound is analogous to (4.7) in [29].) To prove the above uniform bound, one may argue as in the proof of [29, Theorem 2.1] with $\phi(x) := |x|^\gamma$. What is more, the fact that this function is multiplicative and satisfies the assumptions of Lemma A.1 (Topchiĭ-Vatutin inequality for complex martingales), allows for a substantial simplification of the proof given in [29, Theorem 2.1]. A combination of the uniform moment bound above with formula (4.3) in [29] yields the desired tail bound $\mathbb{P}(|Z(\lambda)| > t) \leq \text{const} \cdot t^{-\alpha}$ for all $t > 0$. □

Existence of the Lévy measure ν. We now prove the following, more detailed version of Theorem 2.8. The claim that the Lévy measure ν is non-zero which is not covered by Theorem 2.8 will be justified in the next subsection. Recall that ϱ denotes the Haar measure on $\mathbb{U}$ normalized according to (2.23).

Theorem 6.2. *Suppose that $\lambda \in \mathcal{D}$ and that the assumptions of Theorem 2.8 are satisfied. Then there is a $(\mathbb{U},\alpha)$-invariant Lévy measure ν on $\mathbb{C}\setminus\{0\}$ such that for any $\phi \in C_c^2(\hat{\mathbb{C}}\setminus\{0\})$, we have*

$$\begin{aligned}\int \phi \,\mathrm{d}\nu &= \lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} |z|^{-\alpha}\mathbb{E}[\phi(zZ)] \\ &= -\frac{2}{\sigma^2}\int |z|^{-\alpha}\log|z|\bigg(\mathbb{E}[\phi(zZ)] - \sum_{|u|=1}\mathbb{E}[\phi(zL(u)[Z]_u)]\bigg)\,\varrho(\mathrm{d}z) \qquad (6.13)\end{aligned}$$

where $\sigma^2 = \mathbb{E}\big[\sum_{|u|=1}|L(u)|^\alpha(\log|L(u)|)^2\big]$. Moreover,

$$\int |z|^{-\alpha}\bigg(\mathbb{E}[\phi(zZ)] - \sum_{|u|=1}\mathbb{E}[\phi(zL(u)[Z]_u)]\bigg)\,\varrho(\mathrm{d}z) = 0. \qquad (6.14)$$

For the proof of Theorem 6.2, we need the following proposition.

Proposition 6.3. *Suppose that $(R_n)_{n\in\mathbb{N}_0}$ is a neighborhood recurrent multiplicative random walk on $\mathbb{U}$ such that $\mathbb{E}[\log|R_1|] = 0$ and $\sigma^2 := \mathbb{E}[(\log|R_1|)^2] \in (0,\infty)$. Further, suppose that $f, h : \mathbb{U} \to \mathbb{R}$ are continuous functions satisfying $|f(z)| \leq c_f(1 \wedge |z|^{-\delta})$ and $|h(z)| \leq c_h(|z|^\delta \wedge |z|^{-\delta})$ for some constants $c_f, c_h, \delta > 0$ and*

$$f(z) = \mathbb{E}[f(zR_1)] + h(z) \quad \text{for all } z \in \mathbb{U}.$$

If there exist a sequence $(z_n)_{n\in\mathbb{N}}$ *in* $\mathbb{U}$ *with* $z_n \to 0$ *and a continuous function* g *such that* $f(z_n z) \to g(z)$ *for all* $z \in \mathbb{U}$, *then*

$$\int h(z)\,\varrho(\mathrm{d}z) = 0 \quad \text{and} \quad \lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} f(z) = -\frac{2}{\sigma^2}\int h(z)\log|z|\,\varrho(\mathrm{d}z). \tag{6.15}$$

Proof. From our assumptions and the dominated convergence theorem, we deduce

$$g(z) = \lim_{n\to\infty} f(zz_n) = \lim_{n\to\infty}(\mathbb{E}[f(zz_nR_1)]+h(zz_n)) = \lim_{n\to\infty}\mathbb{E}[f(zz_nR_1)] = \mathbb{E}[g(zR_1)]. \tag{6.16}$$

Consequently, $(g(zR_n))_{n\in\mathbb{N}_0}$ is a bounded martingale and, therefore, converges a. s. as $n\to\infty$. On the other hand, $(R_n)_{n\in\mathbb{N}_0}$ is neighborhood recurrent on $\mathbb{U}$. Using the continuity of g, we conclude that g is constant.

Now we define stopping times $\tau := \inf\{n\in\mathbb{N} : |R_n|<1\}$, $T_0 := 0$ and, recursively, $T_n = \inf\{k \geq T_{n-1} : |R_{T_k}| \geq |R_{T_{n-1}}|\}$ for $n\in\mathbb{N}$. A variant of the duality lemma [28, Lemma 4] then yields

$$\mathbb{E}\bigg[\int_{\{|R_\tau|\leq|z|<1\}} f(z_nz)\,\varrho(\mathrm{d}z)\bigg] = -\sum_{k=0}^{\infty}\mathbb{E}\bigg[\int_{\{|z|>|R_{T_k}z_n|\}} h(z)\,\varrho(\mathrm{d}z)\bigg]. \tag{6.17}$$

The left-hand side converges to $g(1)\mathbb{E}[-\log|R_\tau|]$ as $n\to\infty$, and so does the right-hand side. On the other hand, observe that the bound on h implies that it is directly Riemann integrable (dRi) on $\mathbb{U}$, cf. [14, p. 396] for the precise definition. Moreover, for any $\rho>0$, the function $h_\rho(s) := \mathbb{1}_{(\rho,\infty)}(|s|)\int_{\{|z|>|s|\}} h(z)\,\varrho(\mathrm{d}z)$ is also dRi on $\mathbb{U}$. Hence, we infer

$$\begin{aligned}\sum_{k=0}^{\infty}&\mathbb{E}\bigg[\int_{\{|z|>|R_{T_k}z_n|\}} h(z)\,\varrho(\mathrm{d}z)\bigg]\\ &= \sum_{k=0}^{\infty}\mathbb{E}\bigg[\mathbb{1}_{\{|R_{T_k}z_n|\leq\rho\}}\int_{\{|z|>|R_{T_k}z_n|\}} h(z)\,\varrho(\mathrm{d}z)\bigg] + \sum_{k=0}^{\infty}\mathbb{E}[h_\rho(R_{T_k}z_n)].\end{aligned} \tag{6.18}$$

Now suppose that $c := \int h(z)\,\varrho(\mathrm{d}z) \neq 0$. Then choose $\rho>0$ so small that

$$\bigg|\int_{\{|z|>|s|\}} h(z)\,\varrho(\mathrm{d}z) - c\bigg| < \frac{|c|}{2}$$

for all $|s|\leq\rho$. From the renewal theorem for the group $\mathbb{U}$ [14, Theorem A.1], we conclude that

$$\sum_{k=0}^{\infty}\mathbb{E}[h_\rho(R_{T_k}z_n)] \to \frac{1}{\mathbb{E}[\log|R_{T_1}|]}\int h_\rho(s)\varrho(\mathrm{d}s) \quad \text{as } n\to\infty.$$

On the other hand, the first infinite series in (6.18) is unbounded as $n\to\infty$. This is a contradiction and, hence, $\int h(z)\,\varrho(\mathrm{d}z) = 0$, which is the first equality in (6.15). From [28, Proposition 1] we infer that the function $s\mapsto\int_{\{|z|>|s|\}} h(z)\,\varrho(\mathrm{d}z)$ is also dRi with

$$\iint_{\{|z|>|s|\}} h(z)\,\varrho(\mathrm{d}z)\,\varrho(\mathrm{d}s) = \int h(z)\log|z|\,\varrho(\mathrm{d}z).$$

An application of the renewal theorem for the group $\mathbb{U}$ [14, Theorem A.1] yields that the right-hand side of (6.17) converges to

$$\frac{-1}{\mathbb{E}[\log|R_{T_1}|]}\int h(z)\log|z|\,\varrho(\mathrm{d}z).$$

Finally, since $\mathbb{E}[\log|R_\tau|]\cdot\mathbb{E}[\log|R_{T_1}|] = -\frac{\sigma^2}{2}$ by the proof of Theorem 18.1 on p. 196 in [38], we find

$$g(1) = -\frac{2}{\sigma^2}\int h(z)\log|z|\,\varrho(\mathrm{d}z),$$

which does not depend on the sequence $(z_n)_{n\in\mathbb{N}}$. □

Further, we recall an elementary but useful fact.

Proposition 6.4. *Let $\phi \in C_c^2(\hat{\mathbb{C}}\setminus\{0\})$. Then, for any $0 < \varepsilon \le 1$, there is a finite constant $C > 0$ such that for any $n \in \mathbb{N}$ and any $x_1, \dots, x_n \in \mathbb{C}$ it holds that*

$$\Big|\phi(\textstyle\sum_{k=1}^n x_k) - \sum_{k=1}^n \phi(x_k)\Big| \le C\sum_{1\le j\ne k\le n}|x_j|^\varepsilon|x_k|^\varepsilon.$$

Source. The proposition which is almost identical with [15, Lemma 6.2] follows from the proof of the cited lemma. □

It is routine to check using induction on n that the formula

$$\mathbb{E}[f(R_n)] = \mathbb{E}\bigg[\sum_{|u|=n}|L(u)|^\alpha f(L(u))\bigg], \tag{6.19}$$

which is assumed to hold for any bounded and measurable function $f : \mathbb{C} \to \mathbb{R}$, defines (the distribution of) a multiplicative random walk $(R_n)_{n\in\mathbb{N}_0}$ on $\mathbb{U}$ with i.i.d. steps R_n/R_{n-1}, $n \in \mathbb{N}$. From (6.19) for $n = 1$ and (C1), we infer $\mathbb{E}[\log|R_1|] = 0$, i.e., the random walk $(\log|R_n|)_{n\in\mathbb{N}_0}$ on $\mathbb{R}$ has centered steps and thus is recurrent. Consequently, $(R_n)_{n\in\mathbb{N}_0}$ is neighborhood recurrent. Moreover, by (6.19) and (2.14), we have $\mathbb{E}[(\log|R_1|)^2] = \sigma^2 \in (0,\infty)$.

Theorem 6.2 will now be proved by an application of Proposition 6.3.

Proof of Theorem 6.2. For any $z \in \mathbb{U}$, we define a finite measure ν_z on the Borel sets of $\mathbb{C}$ via

$$\nu_z(A) = |z|^{-\alpha}\mathbb{P}(zZ \in A).$$

First observe that, since $\mathbb{P}[|Z| > t] \le Ct^{-\alpha}$ by Proposition 2.6, the family of measures $\{\nu_z\}_{z\in\mathbb{C}\setminus\{0\}}$ as a subset of the set of locally finite measures on $\hat{\mathbb{C}}\setminus\{0\}$ is relatively vaguely sequentially compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{U}$ satisfying $z_n \to 0$ such that ν_{z_n} converges vaguely to some measure ν.

Let $\phi \in C_c^2(\hat{\mathbb{C}}\setminus\{0\})$. Define $f(z) = |z|^{-\alpha}\mathbb{E}[\phi(zZ)]$, $h(z) = f(z) - \mathbb{E}[f(zR_1)]$ for $z \in \mathbb{C}$. We shall show that the assumptions of Proposition 6.3 are satisfied. From the proposition we then infer that the limit $\lim_{n\to\infty} f(z_n)$ (hence, ν) does not depend on the particular choice of $(z_n)_{n\in\mathbb{N}}$, which implies that $\lim_{|z|\to 0, z\in\mathbb{U}} f(z)$ exists.

First, notice that f is continuous by the dominated convergence theorem and, thus, also h is continuous again by the dominated convergence theorem. Since ϕ is bounded, we have $|f(z)| \le \|\phi\|_\infty|z|^{-\alpha}$ where $\|\phi\|_\infty := \sup_{x\in\hat{\mathbb{C}}\setminus\{0\}}|\phi(x)| < \infty$. Since, moreover, there is some $r > 0$ such that $\phi(z) = 0$ for all $|z| \le r$, we infer $|f(z)| \le |z|^{-\alpha}\|\phi\|_\infty\mathbb{P}(|zZ| > r) \le C\|\phi\|_\infty r^{-\alpha}$ for all $|z| > 0$. Hence, $|f(z)| \le c_f\cdot(1\wedge|z|^{-\alpha})$ for all $z \ne 0$ and some $c_f > 0$. Next, we show that

$$|h(z)| \le c_h(|z|^\delta \wedge |z|^{-\delta}) \tag{6.20}$$

for some $c_h, \delta > 0$ and all $z \neq 0$. In order to prove that $|h(z)| \leq c|z|^{-\delta}$ for some $c > 0$, it suffices to give a corresponding bound on $|\mathbb{E}[f(zR_1)]|$. To this end, we first notice that for any $s \in \mathbb{R}$, by (6.19), we have

$$\mathbb{E}[|R_1|^s] = \mathbb{E}\Big[\sum_{|u|=1} |L(u)|^\alpha |L(u)|^s\Big] = \mathbb{E}\Big[\sum_{|u|=1} \Big|\frac{e^{-\lambda S(u)}}{m(\lambda)}\Big|^{\alpha+s}\Big] = \frac{m((\alpha+s)\theta)}{|m(\lambda)|^{\alpha+s}}.$$

Thus, $\mathbb{E}[|R_1|^s] < \infty$ iff $m((\alpha+s)\theta) < \infty$. By assumption $m(\theta) < \infty$ and $m(\alpha\theta) < \infty$. Since m is convex, it is finite on the whole interval $[\theta, \alpha\theta]$. Therefore, for $\delta_1 \in (0, \alpha - 1)$, we have $m((\alpha - \delta_1)\theta) < \infty$ and, equivalently, $\mathbb{E}[|R_1|^{-\delta_1}] < \infty$. Consequently,

$$|\mathbb{E}[f(zR_1)]| \leq c_f \mathbb{E}[1 \wedge |zR_1|^{-\alpha}] \leq c_f \mathbb{E}[1 \wedge |zR_1|^{-\delta_1}] \leq c_f \mathbb{E}[|R_1|^{-\delta_1}] \cdot |z|^{-\delta_1}$$

for all $|z| > 0$. It remains to show that we may choose $\delta > 0$ such that also $|h(z)| \leq C|z|^\delta$. To this end, recall that $\kappa \in (\alpha/2, 1)$ is such that $\mathbb{E}[Z_1(\kappa\theta)^2] < \infty$, see (2.20). For this κ, we obtain

$$\begin{aligned}
|h(z)| = |f(z) - \mathbb{E}[f(zR_1)]| &= |z|^{-\alpha}\Big|\mathbb{E}[\phi(zZ)] - \mathbb{E}\Big[\sum_{|u|=1} \phi(zL(u)[Z]_u)\Big]\Big| \\
&= |z|^{-\alpha}\Big|\mathbb{E}\Big[\phi\Big(\sum_{|u|=1} zL(u)[Z]_u\Big) - \sum_{|u|=1} \phi(zL(u)[Z]_u)\Big]\Big| \\
&\leq C|z|^{2\kappa-\alpha}\mathbb{E}\Big[\sum_{u\neq v} |L(u)[Z]_u|^\kappa |L(v)[Z]_v|^\kappa\Big]
\end{aligned}$$

by Proposition 6.4. The expectation in the above expression can be estimated by

$$\mathbb{E}\Big[\Big(\sum_{|u|=1} |L(u)|^\kappa\Big)^2\Big] [\mathbb{E}[|Z|^\kappa]]^2 < \infty \tag{6.21}$$

where the finiteness is due to (2.20). We have shown that (6.20) holds with $\delta = \delta_1$ for any $\delta_1 \in (0, (2\kappa - \alpha) \wedge (\alpha - 1))$.

Since ν_{z_n} converges vaguely to ν, we have, for any $\phi \in C_{\mathrm{c}}^2(\hat{\mathbb{C}} \setminus \{0\})$,

$$\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} \int \phi \, \mathrm{d}\nu_{z_n} = \int \phi \, \mathrm{d}\nu. \tag{6.22}$$

Fix any $z \in \mathbb{U}$. Then

$$\lim_{n\to\infty} f(z_n z) = |z|^{-\alpha} \lim_{n\to\infty} \int \phi(zx)\, \nu_{z_n}(\mathrm{d}x) = |z|^{-\alpha} \int \phi(zx)\, \nu(\mathrm{d}x) =: g(z) \tag{6.23}$$

because the function $t \mapsto \phi(tz)$ still belongs to $C_{\mathrm{c}}^2(\hat{\mathbb{C}} \setminus \{0\})$. Finally, we observe that the function $t \mapsto g(t)$ is continuous on $\hat{\mathbb{C}} \setminus \{0\}$. Consequently, Proposition 6.3 applies and shows that (6.14) holds and that

$$\begin{aligned}
\lim_{\substack{|z|\to 0,\\ z\in\mathbb{U}}} |z|^{-\alpha}\mathbb{E}[\phi(zZ)] &= -\frac{2}{\sigma^2}\int h(z)\log|z|\,\varrho(\mathrm{d}z) \\
&= -\frac{2}{\sigma^2}\int \big(f(z) - \mathbb{E}[f(zR_1)]\big)\log|z|\,\varrho(\mathrm{d}z) \\
&= -\frac{2}{\sigma^2}\int |z|^{-\alpha}\log|z|\Big(\mathbb{E}[\phi(zZ)] - \mathbb{E}\Big[\sum_{|u|=1}\phi(zL(u)[Z]_u)\Big]\Big)\,\varrho(\mathrm{d}z).
\end{aligned}$$

This proves (6.13). The $(\mathbb{U},\alpha)$-invariance of ν follows from the fact that the right-hand sides of (6.22) and (6.23) are equal for each $z \in \mathbb{U}$. □

The measure ν is non-zero. In order to show that the Lévy measure ν is non-zero, we adopt the analytic argument invented in [14]. To this end, we take a nondecreasing function $\varphi \in C^2(\mathbb{R}_>)$ such that $\varphi(t) = 0$ for $t < 1/2$ and $\varphi(t) = 1$ for $t > 1$. We further set $\varphi(0) := 0$ and $\varphi(z) := \varphi(|z|)$ if $z \in \mathbb{C} \setminus \{0\}$. For $s \in \mathbb{C}$, define $\kappa(s)$ by

$$\kappa(s) := \int |z|^{-s} \bigg(\mathbb{E}\bigg[\varphi\bigg(\sum_{|u|=1} zL(u)[Z]_u \bigg) - \sum_{|u|=1} \varphi(zL(u)[Z]_u) \bigg] \bigg) \, \varrho(\mathrm{d}z)$$

whenever the absolute value of the integrand is ϱ-integrable. Using the linearity of the ϱ-integral, we may write this integral as the difference of two ϱ-integrals. Straightforward estimates now show that both these integrals are finite if $\mathrm{Re}(s) \in (1,\alpha)$ since the latter entails $m(\mathrm{Re}(s)\theta) < \infty$. For s in the strip $1 < \mathrm{Re}(s) < \alpha$, using Fubini's theorem and the invariance of the Haar measure ϱ, we may rewrite $\kappa(s)$ in the form

$$\begin{aligned}\kappa(s) &= \int |z|^{-s}\varphi(z) \, \varrho(\mathrm{d}z) \bigg(\mathbb{E}\bigg[|Z|^s - \sum_{|u|=1} |L(u)[Z]_u|^s \bigg] \bigg) \\ &= \int |z|^{-s}\varphi(z) \, \varrho(\mathrm{d}z) \cdot \Big(1 - \frac{m(s\theta)}{|m(\lambda)|^s} \Big) \cdot \mathbb{E}[|Z|^s]. \end{aligned} \tag{6.24}$$

Lemma 6.5. *For any $s > 1$ there exists a finite constant C_s such that, for any $x, y \in \mathbb{C}$, we have*

$$\int |z|^{-s} |\varphi(zy) - \varphi(zx)| \, \varrho(\mathrm{d}z) \leq C_s \big| |y|^s - |x|^s \big|$$

and $\sup_{s \in I} C_s < \infty$ for every closed interval $I \subset (1,\infty)$.

Proof. Without loss of generality we assume that $|x| \leq |y|$. Then $|\varphi(zy) - \varphi(zx)|$ may only be positive when $|zy| > 1/2$ and $|zx| < 1$. We shall consider the three cases $|zx| < 1/2 < |zy| < 1$, $|zx| < 1 < |zy|$ and $1/2 < |zx| < |zy| < 1$ separately. In the second case, we conclude that the integral in focus does not exceed

$$\int_{\{|x| < \frac{1}{|z|} < |y|\}} |z|^{-s} |\varphi(zy) - \varphi(zx)| \, \varrho(\mathrm{d}z) \leq \int_{\{|x| < \frac{1}{|z|} < |y|\}} |z|^{-s} \, \varrho(\mathrm{d}z) = \frac{1}{s}(|y|^s - |x|^s).$$

Analogously, in the first case, we obtain the upper bound

$$\int_{\{|x| < \frac{1}{2|z|} < |y|\}} |z|^{-s} |\varphi(zy) - \varphi(zx)| \, \varrho(\mathrm{d}z) \leq \frac{1}{s2^s}(|y|^s - |x|^s).$$

It remains to get the bound in the third case. Since the derivative of φ (as a function on $\mathbb{R}_>$) is bounded so that $\|\varphi'\|_\infty = \sup_{x>0} \varphi'(x) < \infty$ we have the upper bound

$$\begin{aligned}\int_{\{|y| < \frac{1}{|z|} < 2|x|\}} |z|^{-s} |\varphi(zx) - \varphi(zy)| \, \varrho(\mathrm{d}z) &\leq \|\varphi'\|_\infty \int_{\{|y| < \frac{1}{|z|} < 2|x|\}} |z|^{-s} (|zy| - |zx|) \, \varrho(\mathrm{d}z) \\ &= \frac{\|\varphi'\|_\infty}{s-1} (|y| - |x|)(|2x|^{s-1} - |y|^{s-1}) \leq \frac{\|\varphi'\|_\infty}{s-1} 2^{s-1} (|y|^s - |x|^s). \end{aligned}$$

The latter follows from the elementary inequality

$$(t-1)(2^{s-1}-t^{s-1}) \leq 2^{s-1}(t-1) \leq 2^{s-1}(t^s-1),$$

that we use for $t = \frac{|y|}{|x|} \in (1,2)$. The claim concerning the local boundedness of $s \mapsto C_s$ is now obvious. □

Lemma 6.6. *Suppose that* (2.19) *and* (2.20) *hold. Then* κ *is well-defined and holomorphic on the strip* $1 < \mathrm{Re}(s) < \alpha + \epsilon$ *for some* $\epsilon > 0$.

Proof. Let $s \in (1,\gamma)$ where $\gamma \in (\alpha, 2]$ is as in (2.19). First, we show that

$$\mathbb{E}\bigg[\bigg|\bigg|\sum_{|u|=1} L(u)[Z]_u\bigg|^s - \max_{|u|=1} |L(u)[Z]_u|^s\bigg|\bigg] < \infty, \tag{6.25}$$

$$\mathbb{E}\bigg[\sum_{|u|=1} |L(u)[Z]_u|^s - \max_{|u|=1} |L(u)[Z]_u|^s\bigg] < \infty, \quad \text{and} \tag{6.26}$$

$$\mathbb{E}\bigg[\int |z|^{-s}\bigg(\sum_{|u|=1} h(|zL(u)[Z]_u|) - h(\max\nolimits_{|u|=1} |zL(u)[Z]_u|)\bigg)\,\varrho(\mathrm{d}z)\bigg] < \infty \tag{6.27}$$

where $h = \mathbb{1}_{(1,\infty)}$ or $h = \varphi$ for φ defined in the paragraph preceding Lemma 6.5. In order to prove (6.25), we first observe that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg|\bigg|\sum_{|u|=1} L(u)[Z]_u\bigg|^s - \bigg(\sum_{|u|=1} |L(u)[Z]_u|^2\bigg)^{s/2}\bigg|\bigg] \\
&= \mathbb{E}\bigg[\bigg|\bigg(\sum_{|u|=1} |L(u)[Z]_u|^2 + \sum_{\substack{|u|=|v|=1\\ u\neq v}} \mathrm{Re}(L(u)[Z]_u\overline{L(v)[Z]_v})\bigg)^{s/2} - \bigg(\sum_{|u|=1} |L(u)[Z]_u|^2\bigg)^{s/2}\bigg|\bigg] \\
&\leq \mathbb{E}\bigg[\bigg(\sum_{\substack{|u|=|v|=1\\ u\neq v}} |L(u)[Z]_u L(v)[Z]_v|\bigg)^{s/2}\bigg] \\
&= \mathbb{E}\bigg[\mathbb{E}_1\bigg[\bigg(\sum_{\substack{|u|=|v|=1\\ u\neq v}} |L(u)[Z]_u L(v)[Z]_v|\bigg)^{s/2}\bigg]\bigg] \\
&\leq (\mathbb{E}[|Z|])^s \cdot \mathbb{E}\bigg[\bigg(\sum_{\substack{|u|=|v|=1\\ u\neq v}} |L(u)||L(v)|\bigg)^{s/2}\bigg] \\
&\leq (\mathbb{E}[|Z|])^s \cdot \mathbb{E}\bigg[\bigg(\sum_{|u|=1} |L(u)|\bigg)^{s}\bigg] < \infty,
\end{aligned}$$

where we used the subadditivity on $[0,\infty)$ of the function $t \mapsto t^{s/2}$ for the first inequality, Jensen's inequality for the second, and (2.19) to conclude the finiteness. This in combination with the inequality

$$0 \leq \bigg(\sum_{|u|=1} |L(u)[Z]_u|^2\bigg)^{s/2} - \max_{|u|=1} |L(u)[Z]_u|^s \leq \sum_{|u|=1} |L(u)[Z]_u|^s - \max_{|u|=1} |L(u)[Z]_u|^s,$$

which follows from the aforementioned subadditivity, shows that (6.25) is a consequence of (6.26).

For $h = \mathbb{1}_{(1,\infty)}$ or $h = \varphi$ and positive x_j, we have

$$(1 - h(\max_j x_j)) \leq \prod_j (1 - h(x_j)) + \Big(\sum_{j \neq k} \mathbb{1}_{\{x_j > 1/2,\ x_k > 1/2\}}\Big) \wedge 1.$$

Further, there exists some finite $C \geq 1$ such that $x - 1 + e^{-x} \leq C(x \wedge x^2)$ for all $x \geq 0$ and $\mathbb{P}(|Z| > t) \leq Ct^{-\alpha}$, $\mathbb{P}(|Z| > t) \leq Ct^{-1}$ for all $t > 0$ (the latter follows from Markov's inequality). Using these facts, we infer

$$\begin{aligned}
&\int |z|^{-s} \Big(\mathbb{E}_1\Big[\sum_{|u|=1} h(|zL(u)[Z]_u|) - h(\max_{|u|=1} |zL(u)[Z]_u|)\Big]\Big)\, \varrho(\mathrm{d}z) \\
&\quad \leq \int |z|^{-s} \Big(\sum_{|u|=1} \mathbb{E}_1[h(|zL(u)[Z]_u|)] - 1 + e^{-\sum_{|u|=1} \mathbb{E}_1[h(|zL(u)[Z]_u|)]}\Big)\, \varrho(\mathrm{d}z) \\
&\qquad + \int |z|^{-s} \Big(\Big(\sum_{\substack{|u|=|v|=1 \\ u \neq v}} \mathbb{P}_1(|zL(u)[Z]_u| > 1/2,\ |zL(v)[Z]_v| > 1/2)\Big) \wedge 1\Big)\, \varrho(\mathrm{d}z) \\
&\quad \leq C \int |z|^{-s} \Big(\sum_{|u|=1} \mathbb{E}_1[h(|zL(u)[Z]_u|)]\Big) \wedge \Big(\sum_{|u|=1} \mathbb{E}_1[h(|zL(u)[Z]_u|)]\Big)^2\, \varrho(\mathrm{d}z) \\
&\qquad + 4^\alpha C^2 \int |z|^{-s} \Big(\Big(\sum_{\substack{|u|=|v|=1 \\ u \neq v}} |zL(u)|^\alpha |zL(v)|^\alpha\Big) \wedge 1\Big)\, \varrho(\mathrm{d}z) \\
&\quad \leq 4C^3 \int |z|^{-s} \Big(\sum_{|u|=1} |L(u)||z|\Big) \wedge \Big(\sum_{|u|=1} |L(u)||z|\Big)^2\, \varrho(\mathrm{d}z) \\
&\qquad + 4^\alpha C^2 \Big(\sum_{\substack{|u|=|v|=1 \\ u \neq v}} |L(u)|^\alpha |L(v)|^\alpha\Big)^{s/2\alpha} \int |z|^{-s}(|z|^{2\alpha} \wedge 1)\, \varrho(\mathrm{d}z) \\
&\quad \leq \Big(4C^3 \int |z|^{-s}(|z| \wedge |z|^2)\, \varrho(\mathrm{d}z) + 4^\alpha C^2 \int |z|^{-s}(|z|^{2\alpha} \wedge 1)\, \varrho(\mathrm{d}z)\Big) \\
&\qquad \cdot \Big(\sum_{|u=1|} |L(u)|\Big)^s
\end{aligned} \tag{6.28}$$

where in the last step we have used that $(\sum_{|u|=1} |L(u)|^\alpha)^{s/\alpha} \leq (\sum_{|u|=1} |L(u)|)^s$. Notice that the last two ϱ-integrals in (6.28) are finite since $1 < s < 2$. Assumption (2.19) entails $\mathbb{E}[(\sum_{|u|=1} |L(u)|)^s] < \infty$ which proves (6.27). Choosing $h = \mathbb{1}_{(1,\infty)}$ and taking the expectation in (6.28), we conclude that (6.26) holds. In particular, for $s \in \mathbb{C}$ with $1 < s_1 = \mathrm{Re}(s) < \gamma$, we infer

$$\begin{aligned}
&\int \bigg| |z|^{-s} \bigg(\mathbb{E}[\varphi(zZ)] - \sum_{|u|=1} \mathbb{E}[\varphi(zL(u)[Z]_u)] \bigg) \bigg| \, \varrho(\mathrm{d}z) \\
&= \int |z|^{-s_1} \bigg| \mathbb{E}[\varphi(zZ)] - \sum_{|u|=1} \mathbb{E}[\varphi(zL(u)[Z]_u)] \bigg| \, \varrho(\mathrm{d}z) \\
&\leq \int |z|^{-s_1} \bigg| \mathbb{E}\Big[\varphi\Big(z \sum_{|u|=1} L(u)[Z]_u\Big)\Big] - \mathbb{E}\Big[\varphi(\max_{|u|=1} |zL(u)[Z]_u|)\Big] \bigg| \, \varrho(\mathrm{d}z) \\
&\quad + \int |z|^{-s_1} \bigg(\sum_{|u|=1} \mathbb{E}[\varphi(zL(u)[Z]_u)] - \mathbb{E}\Big[\varphi(\max_{|u|=1} |zL(u)[Z]_u|)\Big] \bigg) \, \varrho(\mathrm{d}z) \\
&\leq C_{s_1} \cdot \mathbb{E}\bigg[\bigg| \bigg| \sum_{|u|=1} L(u)[Z]_u \bigg|^{s_1} - \max_{|u|=1} |L(u)[Z]_u|^{s_1} \bigg|\bigg] \\
&\quad + \int |z|^{-s_1} \bigg(\sum_{|u|=1} \mathbb{E}[\varphi(zL(u)[Z]_u)] - \mathbb{E}\Big[\varphi(\max_{|u|=1} |zL(u)[Z]_u|)\Big] \bigg) \, \varrho(\mathrm{d}z) < \infty
\end{aligned}$$

by Lemma 6.5, (6.25) and (6.27). Therefore, κ is well defined on the strip $1 < \mathrm{Re}(s) < \gamma$. Moreover, for any closed triangle Δ in $1 < \mathrm{Re}(s) < \gamma$, by the above calculation, we can apply Fubini's theorem and the holomorphy of $s \mapsto |z|^{-s}$ in $\mathrm{Re}(s) > 1$ to conclude that

$$\begin{aligned}
&\int_{\partial\Delta} \kappa(s) \, \mathrm{d}s \\
&= \iint_{\partial\Delta} |z|^{s} \, \mathrm{d}s \bigg(\mathbb{E}\bigg[\varphi\bigg(\bigg| \sum_{|u|=1} zL(u)[Z]_u \bigg| \bigg) - \sum_{|u|=1} \varphi(|zL(u)[Z]_u|) \bigg]\bigg) \, \varrho(\mathrm{d}z) = 0,
\end{aligned}$$

which implies that κ is holomorphic on the strip $1 < \mathrm{Re}(s) < \gamma$ by Morera's theorem. □

Theorem 6.7. *If* (2.19) *and* (2.20) *hold, then the Lévy measure* ν *is non-zero.*

Proof. For $s \in \mathbb{C}$ with $0 < \mathrm{Re}(s) < \alpha$, we define the holomorphic function $F(s) := \mathbb{E}[|Z|^s]$. The functions κ and F are related by the identity, valid for $1 < \mathrm{Re}(s) < \alpha$,

$$F(s) = \frac{\kappa(s)}{1 - \frac{m(s\theta)}{|m(\lambda)|^s}} \cdot \bigg(\int |z|^{-s} \varphi(|z|) \, \varrho(\mathrm{d}z) \bigg)^{-1}, \tag{6.29}$$

which is a direct consequence of (6.24). According to Lemma 6.6, κ possesses a holomorphic extension to some neighborhood of α. Assuming that $\nu = 0$ we show that F has such an extension as well. The latter statement will lead to a contradiction. Indeed, if $\nu = 0$, then $\int \varphi(|z|) \, \nu(\mathrm{d}z) = 0$, which together with (6.13) shows that $\kappa'(\alpha) = 0$. Since also $\kappa(\alpha) = 0$ by (6.14), we infer that the numerator in (6.29) has a 0 of at least second order at α, while the denominator has a zero of at most second order at α by virtue of

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \frac{m(s\theta)}{|m(\lambda)|^s} = \mathbb{E}\bigg[\sum_{|u|=1} \log^2 \Big(\frac{e^{-\theta S(u)}}{|m(\lambda)|} \Big) \Big(\frac{e^{-\theta S(u)}}{|m(\lambda)|} \Big)^s \bigg] > 0$$

for all $1 < s < \alpha + \epsilon$, in particular for $s = \alpha$. Hence, F does possess a holomorphic extension to some neighborhood of α. We conclude from Landau's theorem (cf. [40, Theorems 5a and 5b in Chap. II]) that

$$\mathbb{E}[|Z|^{\alpha+\delta}] < \infty, \tag{6.30}$$

for some $\delta > 0$.

By $\mathcal{I}_n$ let us denote an increasing family of subtrees of the Harris-Ulam tree $\mathcal{I}$ such that $\mathcal{I}_1 = \{\varnothing\}$, $|\mathcal{I}_n| = n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{I}_n = \mathcal{I}$. By a classical diagonal argument such a family exists. Write u_n for the unique vertex from the set $\mathcal{I}_n \setminus \mathcal{I}_{n-1}$, $n \in \mathbb{N}$. Next, we define

$$M_n := \sum_{k=1}^{n} L(u_k)([Z_1]_{u_k} - 1) \text{ and } M_n^{(t)} = \sum_{k=1}^{n} L(u_k)\mathbb{1}_{\{L(u_k|_j)\leq t \text{ for all } j\leq |u_k|\}}([Z_1]_{u_k} - 1),$$

and observe that they constitute martingales with respect to the filtration $(\mathcal{H}_n)_{n\in\mathbb{N}_0}$ where $\mathcal{H}_n := \sigma(\mathcal{Z}(u_k) : k = 1, \ldots, n)$ for $n \in \mathbb{N}_0$. We claim that M_n converges a. s. to $Z - 1$. To see this, recall from the proof of [29, Theorem 2.1] that on the set $\{\max_{v\in\mathcal{G}} |L(v)| \leq t\}$, we have

$$Z_n - 1 = Z_n^{(t)} - 1 = \sum_{|u|\leq n-1} L(u)\mathbb{1}_{\{L(u|_j)\leq t \text{ for } j<n\}}([Z_1]_u - 1).$$

Further, the martingale $Z_n^{(t)} - 1$ converges a. s. and in L^γ to some finite limit $Z^{(t)} - 1$ which equals $Z - 1$ on $\{\max_{v\in\mathcal{G}} |L(v)| \leq t\}$. We prove that $Z^{(t)} - 1$ is also the limit in L^γ of the martingale $(M_n^{(t)})_{n\geq 0}$. Indeed, we write

$$\begin{aligned}
&\mathbb{E}[|Z^{(t)} - 1 - M_n^{(t)}|^\gamma] \\
&= \mathbb{E}\Big[\Big|\sum_{k=0}^{\infty}\sum_{|u|=k} L(u)\mathbb{1}_{\{L(u|_j)\leq t \text{ for } j\leq k\}}([Z_1]_u - 1) \\
&\qquad - \sum_{u\in\mathcal{I}_n\cap\mathcal{G}} L(u)\mathbb{1}_{\{L(u|_j)\leq t \text{ for } j\leq |u|\}}([Z_1]_u - 1)\Big|^\gamma\Big] \\
&= \mathbb{E}\Big[\Big|\sum_{k=0}^{\infty}\sum_{\substack{|u|=k,\\ u\notin\mathcal{I}_n}} L(u)\mathbb{1}_{\{L(u|_j)\leq t \text{ for } j\leq k\}}([Z_1]_u - 1)\Big|^\gamma\Big].
\end{aligned}$$

Notice that given $\mathcal{F}_k$ the sum $\sum_{|u|=k, u\notin\mathcal{I}_n} L(u)\mathbb{1}_{\{L(u|_j)\leq t \text{ for } j\leq k\}}([Z_1]_u - 1)$ is a weighted sum of centered random variables and can be considered a martingale increment. Two applications of Lemma A.1 yield

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{k=0}^{\infty}\sum_{\substack{|u|=k,\\ u\notin\mathcal{I}_n}} L(u)\mathbb{1}_{\{L(u|_j)\leq t \text{ for } j\leq k\}}([Z_1]_u - 1)\Big|^\gamma\Big] \\
&\leq 4\mathbb{E}\Big[\sum_{k=0}^{\infty}\Big|\sum_{\substack{|u|=k,\\ u\notin\mathcal{I}_n}} L(u)\mathbb{1}_{\{L(u|_j)\leq t \text{ for } j\leq k\}}([Z_1]_u - 1)\Big|^\gamma\Big] \\
&\leq 16 \cdot \mathbb{E}[|Z_1 - 1|^\gamma] \cdot \mathbb{E}\Big[\sum_{k=0}^{\infty}\sum_{\substack{|u|=k,\\ u\notin\mathcal{I}_n}} |L(u)|^\gamma \mathbb{1}_{\{L(u|_j)\leq t \text{ for } j\leq k\}}\Big].
\end{aligned}$$

The above expectations are finite by (2.19) and the arguments from the proof of Theorem 2.1 in [29]. By the dominated convergence theorem, we infer that $\lim_{n\to\infty} \mathbb{E}[|Z^{(t)} - 1 - M_n^{(t)}|^\gamma] = 0$. Since $\mathbb{P}(\max_{v\in\mathcal{G}} |L(v)| > t) \leq t^{-\alpha}$, we conclude that $M_n \to Z - 1$ a. s. In view of this and (6.30),

$$\mathbb{E}\bigg[\bigg(\sum_{v\in\mathcal{G}} |L(v)|^2 |[Z_1]_v - 1|^2\bigg)^{(\alpha+\delta)/2}\bigg] < \infty \tag{6.31}$$

by the complex version of Burkholder's inequality. On the other hand, from [33, Theorem 1.5] we have

$$\mathbb{P}(\max_{u\in\mathcal{G}} |L(u)| > t) > ct^{-\alpha} \tag{6.32}$$

for some $c > 0$ and all sufficiently large t. Pick $t_0 > 0$ such that $\mathbb{P}(|Z_1 - 1| > t_0) \geq \frac{1}{2}$. Denote by $\mathcal{N}_t$ the set of individuals u that are the first in their ancestral line with the property that $|L(u)| > t$, i.e.,

$$\mathcal{N}_t = \{u \in \mathcal{I} : |L(u)| > t, \text{ and } L(u|_k) \leq t \text{ for all } k < |u|\}. \tag{6.33}$$

Then $\mathcal{N}_t$ is an optional line in the sense of Jagers [26, Section 4]. Denote by $\mathcal{F}_{\mathcal{N}_t}$ the σ-field that contains the information of all reproduction point processes of all individuals that are neither in $\mathcal{N}_t$ nor a descendent of a member of $\mathcal{N}_t$, see again Jagers [26, Section 4] for a precise definition. Then, by the strong Markov branching property [26, Theorem 4.14] (the σ-field $\mathcal{F}_{\mathcal{N}_t}$ was introduced for a proper application of this result) and (6.32), we infer

$$\begin{aligned}\mathbb{P}\bigg(\sum_{u\in\mathcal{G}} |L(u)|^2 |[Z_1]_u - 1|^2 > t_0^2 t^2\bigg) &\geq \mathbb{P}\bigg(\sum_{u\in\mathcal{N}_t} |[Z_1]_u - 1|^2 > t_0^2\bigg) \\ &\geq \mathbb{P}(\mathcal{N}_t \neq \emptyset) \cdot \mathbb{P}(|Z_1 - 1| > t_0) \geq \frac{c}{2} t^{-\alpha}\end{aligned}$$

for all sufficiently large t. This contradicts to (6.31), thereby proving that ν is non-zero. □

Appendix A. Auxiliary results

In this section, we gather auxiliary facts needed in the proofs of our main results.

A.1. **Inequalities for complex random variables.**

Throughout the paper, we need the complex analogues of known inequalities for real-valued random variables.

The Topchiĭ-Vatutin inequality for complex martingales. We begin with an extension of the Topchiĭ-Vatutin inequality [39, Theorem 2] to complex-valued martingales.

Lemma A.1. *Let $f : [0,\infty) \to [0,\infty)$ be a nondecreasing convex function with $f(0) = 0$ such that $g(x) := f(\sqrt{x})$ is concave on $(0,\infty)$. Let $(M_n)_{n\in\mathbb{N}_0}$ be a complex-valued martingale with $M_0 = 0$ a. s. and set $D_n := M_n - M_{n-1}$ for $n \in \mathbb{N}$. If $\mathbb{E}[f(|D_k|)] < \infty$ for $k = 1, \ldots, n$, then*

$$\mathbb{E}[f(|M_n|)] \leq 4 \sum_{k=1}^{n} \mathbb{E}[f(|D_k|)]. \tag{A.1}$$

Further, if $f(x) > 0$ for some $x > 0$ and $\sum_{k=1}^\infty \mathbb{E}[f(|D_k|)] < \infty$, then $M_n \to M_\infty$ a. s. for some random variable M_∞ and (A.1) holds for $n = \infty$.

Proof. We first observe that

$$f(|z+w|)+f(|z-w|) \leq 2(f(|z|)+f(|w|)) \quad \text{for all } z,w \in \mathbb{C}. \tag{A.2}$$

To see this, note that g is subadditive as a concave function with $g(0)=0$, whence

$$g(x^2+y^2) \leq g(x^2)+g(y^2) = f(x)+f(y) \quad \text{for all } x,y \geq 0.$$

Put $x=|u+v|/2$ and $s=|u-v|/2$ for $u,v \in \mathbb{C}$ and observe that

$$\Big|\frac{u+v}{2}\Big|^2+\Big|\frac{u-v}{2}\Big|^2 = \frac{|u|^2+|v|^2}{2}.$$

This together with the concavity of g gives

$$\frac{1}{2}\big(f(|u|)+f(|v|)\big) = \frac{1}{2}\big(g(|u|^2)+g(|v|^2)\big) \leq g\Big(\frac{|u|^2+|v|^2}{2}\Big) \leq f\Big(\frac{|u+v|}{2}\Big)+f\Big(\frac{|u-v|}{2}\Big).$$

Multiply this inequality by 2 and set $u=z+w$ and $v=z-w$ for $z,w \in \mathbb{C}$ to infer (A.2).

The remainder of the proof closely follows the proof of Theorem 2 in [39]. For $k=1,\ldots,n$ assume that $\mathbb{E}[f(|D_k|)]<\infty$ and denote by D_k^* a random variable such that D_k and D_k^* are i.i.d. conditionally given M_{k-1}. Then

$$\begin{aligned}\mathbb{E}[f(|M_{k-1}+D_k-D_k^*|)] &= \mathbb{E}[\mathbb{E}[f(|M_{k-1}+D_k-D_k^*|) \mid M_{k-1}]] \\ &= \mathbb{E}[\mathbb{E}[f(|M_{k-1}-(D_k-D_k^*)|) \mid M_{k-1}]] \\ &= \mathbb{E}[f[|M_{k-1}-(D_k-D_k^*)|]].\end{aligned}$$

An appeal to (A.2) thus yields

$$\mathbb{E}[f(|M_{k-1}+D_k-D_k^*|)] \leq \mathbb{E}[f(|M_{k-1}|)]+\mathbb{E}[f(|D_k-D_k^*|)]. \tag{A.3}$$

Another application of (A.2) yields

$$\mathbb{E}[f(|D_k-D_k^*|)] \leq 4\mathbb{E}[f(|D_k|)]. \tag{A.4}$$

Further,

$$\begin{aligned}|M_{k-1}+D_k| &= |M_{k-1}+D_k-\mathbb{E}[D_k^*|M_{k-1},D_k]| \\ &= |\mathbb{E}[M_{k-1}+D_k-D_k^* \mid M_{k-1},D_k]| \\ &\leq \mathbb{E}[|M_{k-1}+D_k-D_k^*| \mid M_{k-1},D_k]\end{aligned}$$

by Jensen's inequality for conditional expectation in $\mathbb{R}^2$, which is applicable because the function $x \mapsto |x|$ is convex on $\mathbb{R}^2$. Hence,

$$\mathbb{E}[f(|M_{k-1}+D_k|)] \leq \mathbb{E}[f(\mathbb{E}[|M_{k-1}+D_k-D_k^*||M_{k-1},D_k])] \leq \mathbb{E}[f(|M_{k-1}+D_k-D_k^*|)] \tag{A.5}$$

by the monotonicity of f and Jensen's inequality. Combining (A.3), (A.4) and (A.5), we arrive at

$$\mathbb{E}[f(|M_k|)] \leq \mathbb{E}f(|M_{k-1}|)+4\mathbb{E}f(|D_k|).$$

The claimed inequality follows recursively.

Finally, suppose that $f(x)>0$ for some $x>0$ and that $\sum_{k=1}\mathbb{E}[f(|D_k|)]<\infty$. Then $\sup_{n\in\mathbb{N}}\mathbb{E}[f(|M_n|)] \leq \sum_{k=1}\mathbb{E}[f(|D_k|)]<\infty$. Since $f(0)=0$ and f is convex, we conclude that f grows at least linearly fast and, thus, $\sup_{n\in\mathbb{N}}\mathbb{E}[|M_n|]<\infty$. By the martingale convergence theorem, $M_n \to M_\infty$ a.s. for some random variable

M_∞. Further, f is continuous as it is convex and non-decreasing. Therefore, Fatou's lemma implies

$$\mathbb{E}[f(|M_\infty|)] = \mathbb{E}[\liminf_{n\to\infty} f(|M_\infty|)] \leq \liminf_{n\to\infty} \mathbb{E}[f(|M_n|)] \leq \sum_{k=1}^{\infty} \mathbb{E}[f(|D_k|)] < \infty.$$

□

Tail bounds for sums of weighted i.i.d. complex random variables. Actually, we shall need the following corollary of Lemma A.1, which is closely related to Lemma 2.1 in [8] (see also formulae (2.3) and (2.10) in [30]) but deals with complex-valued rather than real-valued random variables.

Corollary A.2. *Let $c_1, \ldots, c_n$ be complex numbers satisfying $\sum_{k=1}^n |c_k| = 1$. Further, let $Y_1, \ldots, Y_n$ be independent copies of a complex-valued random variable Y with $\mathbb{E}[Y] = 0$ and $\mathbb{E}[|Y|] < \infty$. Then, for $\varepsilon \in (0,1)$ and with $c := \max_{k=1,\ldots,n} |c_k|$,*

$$\mathbb{P}\bigg(\Big|\sum_{k=1}^n c_k Y_k\Big| > \varepsilon\bigg) \leq \frac{8}{\varepsilon^2}\bigg(\int_0^{1/c} cx\mathbb{P}(|Y| > x)\,\mathrm{d}x + \int_{1/c}^{\infty} \mathbb{P}(|Y| > x)\,\mathrm{d}x\bigg).$$

Proof. We use Lemma A.1 with $f(x) = x^2\mathbb{1}_{[0,1]}(x) + (2x-1)\mathbb{1}_{(1,\infty)}(x)$ for $x \geq 0$. Clearly, f is convex such that g defined by $g(x) = f(\sqrt{x}) = x\mathbb{1}_{[0,1]}(x) + (2x^{1/2} - 1)\mathbb{1}_{(1,\infty)}(x)$ for $x \geq 0$ is concave. Furthermore, f is differentiable on $[0,\infty)$ with nondecreasing and continuous derivative $f'(x) = 2x\mathbb{1}_{[0,1]}(x) + 2\mathbb{1}_{(1,\infty)}(x)$ for $x \geq 0$. For $\varepsilon \in (0,1)$, by Markov's inequality,

$$\begin{aligned}
\mathbb{P}\bigg(\Big|\sum_{k=1}^n c_k Y_k\Big| > \varepsilon\bigg) &\leq \frac{1}{f(\varepsilon)}\mathbb{E}\bigg[f\bigg(\Big|\sum_{k=1}^n c_k Y_k\Big|\bigg)\bigg] \leq \frac{4}{\varepsilon^2}\sum_{k=1}^n \mathbb{E}[f(|c_k||Y_k|)] \\
&= \frac{4}{\varepsilon^2}\sum_{k=1}^n |c_k| \int_0^\infty f'(|c_k|x)\,\mathbb{P}(|Y| > x)\,\mathrm{d}x \\
&\leq \frac{4}{\varepsilon^2}\sum_{k=1}^n |c_k| \int_0^\infty f'(cx)\mathbb{P}(|Y| > x)\,\mathrm{d}x \\
&= \frac{8}{\varepsilon^2}\bigg(\int_0^{1/c} cx\mathbb{P}(|Y| > x)\,\mathrm{d}x + \int_{1/c}^\infty \mathbb{P}(|Y| > x)\,\mathrm{d}x\bigg),
\end{aligned}$$

where we have used Lemma A.1 and $f(\varepsilon) = \varepsilon^2$ for $\varepsilon \in (0,1)$ for the second inequality, and monotonicity of f' for the third. Integration by parts gives the first equality. □

A.2. **The minimal position.** In this section, we collect some known results concerning the minimal position in a branching random walk in what is called the boundary case, see [11].

Proposition A.3. *Let $((V(u))_{|u|=n})_{n\in\mathbb{N}_0}$ be a branching random walk such that the positions in the first generation $V(u)$, $|u| = 1$ satisfy the assumptions* (2.11), (2.14) *and* (2.15). *Then the sequence of distributions of $n^{3/2}\sup_{|u|=n} e^{-V(u)}$, $n \in \mathbb{N}$ is tight. In particular,*

$$n^{1/2} \sup_{|u|=n} e^{-V(u)} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty. \tag{A.6}$$

Notice that, under some extra non-lattice assumption, [1, Theorem 1.1] gives the stronger statement

$$\lim_{n\to\infty} \mathbb{P}\Big(\min_{|u|=n} V(u) - \frac{3}{2}\log n \geq x\Big) = \mathbb{E}[e^{-C^* e^x D_\infty}] \tag{A.7}$$

for all $x \in \mathbb{R}$ where C^* is a positive constant and, as before, D_∞ is the limit of the derivative martingale defined in (2.12). For our proposition, we do not need the full strength of (A.7), which allows us to work without the lattice assumption.

Proof of Proposition A.3. We need to estimate $\min_{|v|=n} V_n(u)$ from below. Recall that, for $u \in \mathcal{G}$, we write $\underline{V}(u) := \min_{k=0,\ldots,|u|} V(u|_k)$. With this notation, we have

$$\begin{aligned}\mathbb{P}\Big[\min_{|v|=n} V_n(u) < -x\Big] &\leq \mathbb{P}\Big(\min_{u\in\mathcal{G}} V(u) \leq -x\Big) + \mathbb{P}\Big(\min_{\substack{|u|=n,\\ \underline{V}(u)\geq -x}} V_n(u) < -x\Big)\\ &\leq C(1+x)e^{-x}\end{aligned}$$

by inequality (4.12) from [34] and [1, Corollary 3.4]. The latter does not require a non-lattice assumption. □

A.3. **Asymptotic cancellation.** Let $L_u = (L_u(v))_{v\in\mathbb{N}}$, $u \in \mathbb{V}$ denote a family of i.i.d. copies of a sequence $(L(v))_{v\in\mathbb{N}} = (L_\varnothing(v))_{v\in\mathbb{N}}$ of complex-valued random variables satisfying

$$\#\{v \in \mathbb{N} : L(v) \neq 0\} < \infty \quad \text{a. s.} \tag{A.8}$$

Define $L(\varnothing) := 1$ and, recursively,

$$L(uv) := L(u) \cdot L_u(v)$$

for $u \in \mathbb{V}$ and $v \in \mathbb{N}$. Further, we let

$$Z_n := \sum_{|u|=n} L(u) \quad \text{and} \quad W_n := \sum_{|u|=n} |L(u)|$$

where summation over $|u| = n$ here means summation over all $u \in \mathbb{N}^n$ with $L(u) \neq 0$. Finally, we set $\tilde{W}_1 := \sum_{|v|=1} |L(v)| \log_-(|L(v)|)$. We extend the shift-operator notation introduced in Section 3.2 to the present context, so if $X = \Psi((L_v)_{v\in\mathbb{V}})$ is a function of the whole family $(L_v)_{v\in\mathbb{V}}$ and $u \in \mathbb{V}$, then $[X]_u := \Psi((L_{uv})_{v\in\mathbb{V}})$.

Lemma A.4. *Assume that* $\mathbb{E}[W_1] = 1$ *and that* $a := \mathbb{E}[Z_1] \in \mathbb{C}$ *satisfies* $|a| < 1$.
(a) If $\mathbb{E}[\sum_{|v|=1} |L(v)|^p] < 1$ *and* $\mathbb{E}[W_1^p] < \infty$ *for some* $p > 1$*, then* $Z_n \to 0$ *a. s. and in* $L^{p\wedge 2}$.
(b) Suppose that one of the following two conditions holds.
(i) $W_n \to W$ *in* L^1*;*
(ii) $\mathbb{E}[\sum_{|v|=1} |L(v)| \log(|L(v)|)] = 0$, $\mathbb{E}[W_1 \log_+^2(W_1)] < \infty$, $\mathbb{E}[\tilde{W}_1 \log_+(\tilde{W}_1)] < \infty$,

$$\mathbb{E}\Big[\sum_{|v|=1} |L(v)| \log^2(|L(v)|)\Big] < \infty.$$

Then $Z_n \to 0$ *in probability if (i) holds and* $n^{1/2} Z_n \to 0$ *in probability if (ii) holds.*

Proof. (a) We can assume without loss of generality that $p \in (1, 2]$. According to [20, Corollary 5] or [31, Theorem 2.1] the martingale W_n converges a. s. and in L^p to some limit W. Let

$$q := \max\{|a|^p, \mathbb{E}[\textstyle\sum_{|v|=1} |L(v)|^p]\} < 1.$$

For $k = \lfloor n/2 \rfloor$, we have

$$\begin{aligned}
\mathbb{E}[|Z_n|^p] &\leq 2^{p-1}\mathbb{E}[|Z_n - a^k Z_{n-k}|^p] + 2^{p-1}\mathbb{E}[|a^k Z_{n-k}|^p] \\
&\leq 2^{p-1}\mathbb{E}\bigg[\mathbb{E}\bigg[\bigg|\sum_{|v|=n-k} L(v)([Z_k]_v - a^k)\bigg|^p \,\bigg|\, \mathcal{F}_{n-k}\bigg]\bigg] + 2^{p-1}|a|^{kp}\mathbb{E}[W_{n-k}^p] \\
&\leq 2^{p+1}\mathbb{E}\bigg[\sum_{|v|=n-k} |L(v)|^p \mathbb{E}[|Z_k - a^k|^p]\bigg] + 2^{p-1}q^k\mathbb{E}[W^p] \\
&\leq 2^{p+1}\mathbb{E}\bigg[\sum_{|v|=n-k} |L(v)|^p\bigg] 2^{p-1}(\mathbb{E}[|Z_k|^p] + |a|^{kp}) + 2^{p-1}q^k\mathbb{E}[W^p] \\
&\leq \big(2^{p+1}(\mathbb{E}[W^p] + 1) + \mathbb{E}[W^p]\big)2^{p-1}q^k
\end{aligned}$$

where we have repeatedly used that $|z+w|^p \leq 2^{p-1}(|z|^p + |w|^p)$ and Lemma A.1 for the third inequality. The bound decays exponentially as $n \to \infty$ giving $Z_n \to 0$ in L^p and also $Z_n \to 0$ a. s. by virtue of the Borel-Cantelli lemma and Markov's inequality.

(b) Let $\mathcal{S} := \{W_n > 0 \text{ for all } n \geq 0\}$ denote the survival set of the system. It is clear that the claimed convergence holds on the set of extinction $\mathcal{S}^c$. Therefore, in what follows we work under $\mathbb{P}^*(\cdot) := \mathbb{P}(\cdot|\mathcal{S})$.

We first assume that (i) holds. Then

$$W_n \to W \quad \mathbb{P}^*\text{-a. s.} \tag{A.9}$$

Eq. (A.9) in combination with (3.2) gives

$$\frac{\sup_{|v|=n} |L(v)|}{W_n} \to 0 \quad \mathbb{P}^*\text{-a. s.} \tag{A.10}$$

If, on the other hand, assumption (ii) is satisfied, then [2, Theorem 1.1] gives

$$\sqrt{n}W_n \to W^* \quad \text{in } \mathbb{P}^*\text{-probability} \tag{A.11}$$

for some random variable W^* satisfying $\mathbb{P}^*(W^* > 0) = 1$. As before, we need control over $\max_{|u|=n} |L(u)|$. A combination of (A.11) and Proposition A.3 gives

$$\frac{\sup_{|v|=n} |L(v)|}{\sum_{|v|=n} |L(v)|} = \frac{n^{3/2} \sup_{|v|=n} |L(v)|}{n^{1/2} W_n} \frac{1}{n} \overset{\mathbb{P}^*}{\to} 0 \quad \text{as } n \to \infty. \tag{A.12}$$

From now on, we treat both cases, (i) and (ii), simultaneously. In view of (A.9) and (A.11), it remains to prove that

$$\lim_{n\to\infty} \frac{Z_n}{W_n} = 0 \quad \text{in } \mathbb{P}^*\text{-probability.} \tag{A.13}$$

The last relation follows if we can show that, for any fixed positive integer $k < n$,

$$\lim_{n\to\infty} \frac{Z_n - a^k Z_{n-k}}{W_n} = 0 \quad \text{in } \mathbb{P}^*\text{-probability} \tag{A.14}$$

and that, for all $\varepsilon \in (0,1)$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}^*\bigg(\bigg|\frac{a^k Z_{n-k}}{W_n}\bigg| > \varepsilon\bigg) = 0. \tag{A.15}$$

Since, for any $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{W_{n-k}}{W_n} = 1 \quad \text{in } \mathbb{P}^*\text{-probability}, \tag{A.16}$$

for all k such that $|a|^k < \varepsilon/2$, we have

$$\mathbb{P}^*\bigg(\bigg|\frac{a^k Z_{n-k}}{W_n}\bigg| > \varepsilon\bigg) \leq \mathbb{P}^*\bigg(\bigg|\frac{a^k W_{n-k}}{W_n}\bigg| > \varepsilon\bigg) \to 0 \quad \text{as } n \to \infty.$$

Hence, (A.15) holds and it remains to check (A.14). In view of (A.16), relation (A.14) is equivalent to

$$\lim_{n\to\infty} \frac{Z_n - a^k Z_{n-k}}{W_{n-k}} = 0 \quad \text{in } \mathbb{P}^*\text{-probability}. \tag{A.17}$$

Setting $\mathcal{S}_j := \{W_j > 0\}$ for $j \in \mathbb{N}_0$, we have $\mathcal{S}_j \downarrow \mathcal{S}$ as $j \to \infty$. Thus, for $\varepsilon > 0$, we infer

$$\mathbb{P}\bigg(\bigg|\frac{Z_n - a^k Z_{n-k}}{W_{n-k}}\bigg| > \varepsilon, \mathcal{S}\bigg) \leq \mathbb{E}\bigg[\mathbb{1}_{\mathcal{S}_{n-k}} \mathbb{P}\bigg(\bigg|\frac{Z_n - a^k Z_{n-k}}{W_{n-k}}\bigg| > \varepsilon \bigg| \mathcal{F}_{n-k}\bigg)\bigg],$$

so that it suffices to show that the right-hand side converges to zero. To this end, we work on $\mathcal{S}_{n-k}$ without further notice. We use the representation

$$\frac{Z_n - a^k Z_{n-k}}{W_{n-k}} = \sum_{|v|=n-k} \frac{L(v)}{W_{n-k}}([Z_k]_v - a^k).$$

Given $\mathcal{F}_{n-k}$, the right-hand side is a weighted sum of i.i.d. centered complex-valued random variables which satisfies the assumptions of Corollary A.2 with $c_v = L(v)/W_{n-k}$, $|v| = n-k$ and $Y = Z_k - a^k$. Note that $\#\{c_v : c_v \neq 0, |v| = n-k\} < \infty$ a. s. in view of (A.8), that $\sum_{|v|=n-k} |c_v| = 1$ a. s. and that $\mathbb{E}[|Z_k - a^k|] \leq \mathbb{E}[|Z_k|] + |a|^k \leq \mathbb{E}[W_k] + |a|^k \leq 2$. With

$$c(n-k) := \frac{\sup_{|v|=n-k} |L(v)|}{\sum_{|v|=n-k} |L(v)|}$$

an application of Corollary A.2 yields

$$\mathbb{P}\bigg(\bigg|\frac{Z_n - a^k Z_{n-k}}{W_{n-k}}\bigg| > \varepsilon \big| \mathcal{F}_{n-k}\bigg) \leq \frac{8}{\varepsilon^2}\bigg(c(n-k)\int_0^{1/c(n-k)} x\,\mathbb{P}(|Z_k - a^k| > x)\,\mathrm{d}x + \int_{1/c(n-k)}^{\infty} \mathbb{P}(|Z_k - a^k| > x)\,\mathrm{d}x\bigg).$$

We claim that the right-hand side converges to zero in probability as $n \to \infty$. Indeed, according to (A.10) and (A.12), respectively, $c(n-k) \to 0$ in $\mathbb{P}^*$-probability. It remains to use the following simple fact. If h is a measurable function satisfying $\lim_{y\to 0} h(y) = 0$ and if $\lim_{n\to\infty} \tau_n = 0$ in probability, then $\lim_{n\to\infty} h(\tau_n) = 0$ in probability. For instance, apply this to $h(y) = y\int_0^{1/y} x\mathbb{P}(|Z_k - a^k| > x)\,\mathrm{d}x$ and $\tau_n = c(n-k)$. It follows from Markov's inequality and $\mathbb{E}[|Z_k - a^k|] < \infty$ that $\lim_{x\to\infty} x\mathbb{P}(|Z_k - a^k| > x) = 0$, which in turn implies $h(y) \to 0$ as $y \to 0$. The proof of (A.17), and hence of (A.14), is complete. $\square$

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References

[1] Elie Aïdékon. Convergence in law of the minimum of a branching random walk. *Ann. Probab.*, 41(3A):1362–1426, 2013.
[2] Elie Aïdékon and Zhan Shi. The Seneta-Heyde scaling for the branching random walk. *Ann. Probab.*, 42(3):959–993, 2014.
[3] G. Alsmeyer, A. Iksanov, S. Polotskiy, and U. Rösler. Exponential rate of L_p-convergence of intrinsic martingales in supercritical branching random walks. *Theory Stoch. Process.*, 15(2):1–18, 2009.
[4] Gerold Alsmeyer, J. D. Biggins, and Matthias Meiners. The functional equation of the smoothing transform. *Ann. Probab.*, 40(5):2069–2105, 2012.
[5] Gerold Alsmeyer and Alexander Iksanov. A log-type moment result for perpetuities and its application to martingales in supercritical branching random walks. *Electron. J. Probab.*, 14:no. 10, 289–312, 2009.
[6] J. D. Biggins. Martingale convergence in the branching random walk. *J. Appl. Probability*, 14(1):25–37, 1977.
[7] J. D. Biggins. Uniform convergence of martingales in the branching random walk. *Ann. Probab.*, 20(1):137–151, 1992.
[8] J. D. Biggins and A. E. Kyprianou. Seneta-Heyde norming in the branching random walk. *Ann. Probab.*, 25(1):337–360, 1997.
[9] John D. Biggins. Lindley-type equations in the branching random walk. *Stochastic Process. Appl.*, 75(1):105–133, 1998.
[10] John D. Biggins and Andreas E. Kyprianou. Measure change in multitype branching. *Advances in Applied Probability*, 36(2):544–581, 2004.
[11] John D. Biggins and Andreas E. Kyprianou. Fixed points of the smoothing transform: the boundary case. *Electron. J. Probab.*, 10:no. 17, 609–631 (electronic), 2005.
[12] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, Inc., New York-London-Sydney, 1968.
[13] Wolfgang J. Bühler. Ein zentraler Grenzwertsatz für Verzweigungsprozesse. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 11:139–141, 1969.
[14] Dariusz Buraczewski, Ewa Damek, Yves Guivarc'h, Andrzej Hulanicki, and Roman Urban. Tail-homogeneity of stationary measures for some multidimensional stochastic recursions. *Probab. Theory Related Fields*, 145(3-4):385–420, 2009.
[15] Dariusz Buraczewski, Ewa Damek, Sebastian Mentemeier, and Mariusz Mirek. Heavy tailed solutions of multivariate smoothing transforms. *Stochastic Process. Appl.*, 123(6):1947–1986, 2013.
[16] Lisa Hartung and Anton Klimovsky. The phase diagram of the complex branching Brownian motion energy model, 2017.
[17] Inge S. Helland. Central limit theorems for martingales with discrete or continuous time. *Scand. J. Statist.*, 9(2):79–94, 1982.
[18] C. C. Heyde. A rate of convergence result for the super-critical Galton-Watson process. *J. Appl. Probability*, 7:451–454, 1970.
[19] C. C. Heyde and B. M. Brown. An invariance principle and some convergence rate results for branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 20:271–278, 1971.
[20] Alexander Iksanov. Elementary fixed points of the BRW smoothing transforms with infinite number of summands. *Stochastic Process. Appl.*, 114(1):27–50, 2004.
[21] Alexander Iksanov and Zakhar Kabluchko. A central limit theorem and a law of the iterated logarithm for the Biggins martingale of the supercritical branching random walk. *J. Appl. Probab.*, 53(4):1178–1192, 2016.
[22] Alexander Iksanov, Konrad Kolesko, and Matthias Meiners. Stable-like fluctuations of Biggins' martingales, 2017. arXiv:1709.07362.

[23] Alexander Iksanov, Xingang Liang, and Quansheng Liu. On L_p-convergence of the Biggins martingale with complex parameter, 2018. Work in progress.

[24] Alexander Iksanov and Matthias Meiners. Exponential rate of almost-sure convergence of intrinsic martingales in supercritical branching random walks. *J. Appl. Probab.*, 47(2):513–525, 2010.

[25] O. M. Īksanov. On the rate of convergence of a regular martingale related to a branching random walk. *Ukraïn. Mat. Zh.*, 58(3):326–342, 2006.

[26] Peter Jagers. General branching processes as Markov fields. *Stochastic Process. Appl.*, 32(2):183–212, 1989.

[27] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[28] Konrad Kolesko. Tail homogeneity of invariant measures of multidimensional stochastic recursions in a critical case. *Probab. Theory Related Fields*, 156(3-4):593–612, 2013.

[29] Konrad Kolesko and Matthias Meiners. Convergence of complex martingales in the branching random walk: the boundary. *Electron. Comm. Probab.*, 18(22):1–14, 2017.

[30] Thomas G. Kurtz. Inequalities for the law of large numbers. *Ann. Math. Statist.*, 43:1874–1883, 1972.

[31] Quansheng Liu. On generalized multiplicative cascades. *Stochastic Process. Appl.*, 86(2):263–286, 2000.

[32] Russell Lyons. A simple path to Biggins' martingale convergence for branching random walk. In *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pages 217–221. Springer, New York, 1997.

[33] Thomas Madaule. The tail distribution of the derivative martingale and the global minimum of the branching random walk. pages 1–25, 2016. arXiv:1606.03211.

[34] Thomas Madaule. Convergence in law for the branching random walk seen from its tip. *J. Theoret. Probab.*, 30(1):27–63, 2017.

[35] Pascal Maillard and Michel Pain. 1-stable fluctuations in branching Brownian motion at critical temperature I: the derivative martingale, 2018. arXiv:1806.05152.

[36] Matthias Meiners and Sebastian Mentemeier. Solutions to complex smoothing equations. *Probab. Theory Related Fields*, 168(1-2):199–268, 2017.

[37] Zhan Shi. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer, Cham, 2015. Lecture notes from the 42nd Probability Summer School held in Saint Flour, 2012, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[38] Frank Spitzer. *Principles of random walk*. Springer-Verlag, New York-Heidelberg, second edition, 1976. Graduate Texts in Mathematics, Vol. 34.

[39] V. A. Vatutin and V. A. Topchiĭ. The maximum of critical Galton-Watson processes, and left-continuous random walks. *Teor. Veroyatnost. i Primenen.*, 42(1):21–34, 1997.

[40] D.V. Widder. *The Laplace transform*. Princeton mathematical series. Princeton university press, 1946.