On L^p-convergence of the Biggins martingale with complex parameter

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Abstract

We prove necessary and sufficient conditions for the L^p-convergence, $p > 1$, of the Biggins martingale with complex parameter in the supercritical branching random walk. The results and their proofs are much more involved (especially in the case $p \in (1, 2)$) than those for the Biggins martingale with real parameter. Our conditions are ultimate in the case $p \geq 2$ only.

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1 Introduction

We start by recalling the definition of the branching random walk. Consider an individual, the ancestor, located at the origin of the real line at time $n = 0$. At time $n = 1$ the ancestor produces a random number J of offspring which are placed at points of the real line according to a random point process $\mathcal{M} = \sum_{i=1}^{J} \delta_{X_i}$ on $\mathbb{R}$ with intensity measure μ (particularly, $J = \mathcal{M}(\mathbb{R})$). The random variable J is allowed to be infinite with positive probability. The first generation formed by the offspring of the ancestor produces the second generation whose displacements with respect to their mothers are distributed according to independent copies of the same point process $\mathcal{M}$. The second generation produces the third one, and so on. All individuals act independently of each other.

More formally, let $\mathbb{V} = \cup_{n \in \mathbb{N}_0} \mathbb{N}^n$ be the set of all possible individuals. The ancestor is identified with the empty word $\varnothing$ and its position is $S(\varnothing) = 0$. On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $(\mathcal{M}(u))_{u \in \mathbb{V}}$ be a family of independent identically distributed (i.i.d.) copies of the point process $\mathcal{M}$. An individual $u = u_1 \dots u_n$ of the nth generation whose position on the real line is denoted by $S(u)$ produces at time $n + 1$ a random number $J(u)$ of offspring which are placed at random locations on $\mathbb{R}$ given by the positions of the point process

$$\sum_{i=1}^{J(u)} \delta_{S(u)+X_i(u)}$$

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where $\mathcal{M}(u)=\sum_{i=1}^{J(u)}\delta_{X_i(u)}$ and $J(u)$ is the number of points in $\mathcal{M}(u)$. The offspring of the individual u are enumerated by $ui=u_1\ldots u_n i$, where $i=1,\ldots,J(u)$ (if $J(u)<\infty$) or $i=1,2,\ldots$ (if $J(u)=\infty$), and the positions of the offspring are denoted by $S(ui)$. Note that no assumptions are imposed on the dependence structure of the random variables $J(u),X_1(u),X_2(u),\ldots$ for fixed $u\in\mathbb{V}$. The point process of the positions of the nth generation individuals will be denoted by $\mathcal{M}_n$ so that $\mathcal{M}_0=\delta_0$ and

$$\mathcal{M}_{n+1}=\sum_{|u|=n}\sum_{i=1}^{J(u)}\delta_{S(u)+X_i(u)},$$

where, by convention, $|u|=n$ means that the sum is taken over all individuals of the nth generation rather than over all $u\in\mathbb{N}^n$. The sequence of point processes $(\mathcal{M}_n)_{n\in\mathbb{N}_0}$ is then called a *branching random walk* (BRW).

Throughout the paper, we assume that the BRW is *supercritical*, that is $\mathbb{E}J>1$. In this case, the event $\mathcal{S}$ that the population survives has positive probability. Note that, provided that $J<\infty$ almost surely (a.s.), the sequence $(\mathcal{M}_n(\mathbb{R}))_{n\in\mathbb{N}_0}$ of generation sizes in the BRW forms a Galton–Watson process.

The Laplace transform of the intensity measure μ

$$m(\lambda):=\int_{\mathbb{R}}e^{-\lambda x}\mu(\mathrm{d}x)=\mathbb{E}\sum_{|u|=1}e^{-\lambda S(u)},\quad \lambda=\theta+\mathrm{i}\gamma\in\mathbb{C}$$

plays an important role in what follows. Throughout the paper we reserve the notation θ for the real part of λ and γ for the imaginary part of λ, and assume that

$$\mathcal{D}:=\{\lambda\in\mathbb{C}: m(\lambda)\text{ converges absolutely}\}=\{\theta\in\mathbb{R}: m(\theta)<\infty\}+\mathrm{i}\mathbb{R}\neq\oslash.$$

Further, we define the sets

$$\mathcal{D}_0:=\mathcal{D}\cap\{\lambda\in\mathbb{C}:|m(\lambda)|=0\},\quad \mathcal{D}_{\neq 0}=\mathcal{D}\setminus\mathcal{D}_0\,.$$

For $\lambda\in\mathcal{D}_{\neq 0}$ and $n\in\mathbb{N}_0$ set

$$Z_n(\lambda):=\frac{1}{m(\lambda)^n}\int_{\mathbb{R}}e^{-\lambda x}\mathcal{M}_n(\mathrm{d}x)=\frac{1}{m(\lambda)^n}\sum_{|u|=n}e^{-\lambda S(u)}.$$

Let $\mathcal{F}_0$ be the trivial σ-field and $\mathcal{F}_n$ the σ-field generated by the first n generations, that is, $\mathcal{F}_n:=\sigma(\mathcal{M}(u):u\in\cup_{k=0}^{n-1}\mathbb{N}^k)$. The sequence $(Z_n(\lambda),\mathcal{F}_n)_{n\in\mathbb{N}_0}$ forms a complex-valued martingale of mean one that we call the Biggins martingale with complex parameter. A non-exhaustive list of very recent articles investigating these objects includes [12, 13, 17, 18]. We would like to stress that the Biggins martingale with complex parameter has received much less attention than its counterpart with real parameter and similar martingale related to a branching Brownian motion. See [14, 22] for recent contributions in the latter case.

The purpose of this article is to provide necessary and sufficient conditions for the L^p-convergence of the martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ for $p>1$. Our main results, Theorems 3.1, 3.4 and 3.7, improve upon Theorem 1 in [7] and Theorem 5.1.1 in the unpublished thesis [19] which give sufficient conditions for the aforementioned convergence in the cases $p\in(1,2]$ and $p>2$, respectively. Necessary and sufficient conditions for the L^1-convergence of $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ are beyond our reach. Finding them seems to be a major open problem for the Biggins martingales with complex parameter. For the time being, our necessary and sufficient conditions for the L^p-convergence for p close to 1 can be used as (non-optimal) sufficient conditions for the L^1-convergence.

The rest of the paper is organized as follows. We give some preliminaries in Section 2. Our results are formulated in Section 3 and then proved in Section 4.

2 Preliminaries

Let $\lambda = \theta + \mathrm{i}\gamma \in \mathcal{D}$ be fixed. Keeping in mind the inequality $|m(\lambda)| \leq m(\theta)$ we distinguish three cases:

$$\text{(I) } |m(\lambda)| = m(\theta); \qquad \text{(II) } 0 < |m(\lambda)| < m(\theta); \qquad \text{(III) } |m(\lambda)| = 0.$$

Perhaps, it is not obvious that Case III can occur. To convince the reader we give an example of the BRW satisfying $m(\theta) < \infty$ and $m(\lambda) = 0$ for some $\lambda \in \mathbb{C}$. Let

$$\mu(\mathrm{d}x) = \frac{2}{\pi} \frac{e^{\theta x}(1 - \cos x)}{x^2} \mathbb{1}_{\mathbb{R}}(x)\mathrm{d}x.$$

Then $m(\theta) = 2$ and

$$m(\lambda) = \frac{4}{\pi} \int_{\mathbb{R}} e^{-\mathrm{i}\gamma x} x^{-2}(1 - \cos x)\mathrm{d}x = 2(1 - |\gamma|)\, \mathbb{1}_{(-1,1)}(\gamma).$$

In particular, $m(\lambda) = 0$ whenever $|\gamma| > 1$.

We do not touch Case III in this paper, just because the sequence $(Z_n^{(0)}(\lambda), \mathcal{F}_n)_{n \in \mathbb{N}_0}$ defined for $\lambda \in \mathcal{D}_0$ by

$$Z_n^{(0)}(\lambda) := \int_{\mathbb{R}} e^{-\lambda x} \mathcal{M}_n(\mathrm{d}x) = \sum_{|u|=n} e^{-\lambda S(u)}$$

does not form a martingale, for it is comprised of complex-valued martingale differences.

<u>Case I</u>: $|m(\lambda)| = m(\theta)$. Since $m(\lambda) = e^{\mathrm{i}\varphi} m(\theta)$ for some $\varphi \in [0, 2\pi)$ we infer

$$\mathbb{E} \sum_{|u|=1} e^{-\theta S(u)} \left(e^{-\mathrm{i}(\varphi + \gamma S(u))} - 1 \right) = 0$$

and thereupon $e^{-\mathrm{i}\gamma S(u)} = e^{\mathrm{i}(\varphi + 2\pi k)} = e^{\mathrm{i}\varphi}$ for integer k whenever $|u| = 1$. This gives $Z_n(\lambda) = Z_n(\theta)$ for $n \in \mathbb{N}$ a.s. Therefore, $(Z_n(\lambda), \mathcal{F}_n)_{n \in \mathbb{N}_0}$ is a nonnegative unit mean martingale.

Proposition 2.1 reminds a criterion for the L^p-convergence ($p > 1$) of the Biggins martingale with real parameter. The result is well-known and can be found in Theorem 2.1 of [20], Corollary 5 of [16], Theorem 3.1 of [4], and perhaps some other articles.

Proposition 2.1. *Let $p > 1$ and $m(\theta) < \infty$ for some $\theta \in \mathbb{R}$. Then the martingale $(Z_n(\theta))_{n \in \mathbb{N}_0}$ converges in L^p if, and only if,*

$$\mathbb{E}[Z_1(\theta)]^p < \infty \quad \text{and} \quad \frac{m(p\theta)}{m(\theta)^p} < 1.$$

Remark 2.2. When $\theta = 0$ and $m(0) < \infty$, the condition $m(0) < m(0)^p$ holds automatically because $m(0) > 1$ by supercriticality. Hence, the martingale $(Z_n(0))_{n \in \mathbb{N}_0}$ converges in L^p if, and only if, $\mathbb{E}[Z_1(0)]^p < \infty$. This result goes back to Corollary on p. 714 in [10].

Therefore, in Case I we conclude that the martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges in L^p if, and only if, the conditions of Proposition 2.1 hold true.

<u>Case II</u>: $0<|m(\lambda)|<m(\theta)$. From the preceding discussion it is clear that only this case gives us a truly complex-valued martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$, the object we shall concentrate on in what follows. In our analysis distinguishing the cases $p<2$ and $p\geq 2$ seems inevitable. To explain this point somewhat informally we restrict our attention to the case $\theta=0$ and note that the L^p-convergence of the martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ is regulated, among others, by the asymptotic behavior of $\mathbb{E}(\sum_{j=1}^n \xi_j^2)^{p/2}$ as $n\to\infty$ for ξ_1, $\xi_2,\ldots$ independent copies of the random variable $|Z_1(\lambda)-1|$ with finite pth moment. If $p\geq 2$, then $\mathbb{E}\xi_1^2<\infty$ and one expects that

$$\mathbb{E}\left(\sum_{j=1}^n \xi_j^2\right)^{p/2} \sim (\mathbb{E}\xi_1^2)^{p/2} n^{p/2}, \quad n\to\infty.$$

If $p\in(1,2)$ and $\mathbb{E}\xi_1^2=\infty$ the last asymptotic relation is no longer true, and one expects that in typical situations

$$0<\liminf_{n\to\infty} n^{-p/\alpha}\mathbb{E}\left(\sum_{j=1}^n \xi_j^2\right)^{p/2} \leq \limsup_{n\to\infty} n^{-p/\alpha}\mathbb{E}\left(\sum_{j=1}^n \xi_j^2\right)^{p/2}<\infty \qquad (1)$$

for some $\alpha\in(p,2)$. It seems that the α cannot be expressed in terms of moments.

Before closing the section we recall that according to the Kesten-Stigum theorem (see, for instance, Theorem 2.1 on p. 23 in [5]) we have $\lim_{n\to\infty} Z_n(0)=0$ a.s. whenever $m(0)<\infty$ and $\mathbb{E}Z_1(0)\log^+ Z_1(0)=\infty$. However, by the Seneta-Heyde theorem (see, for instance, Theorem 5.1 on p. 83 and Corollary 5.3 on p. 85 in [5]) there exists a positive slowly varying function ℓ with $\lim_{t\to\infty}\ell(t)=\infty$ such that

$$\lim_{n\to\infty} Z_n(0)\ell(m(0)^n)=\tilde{Z}_\infty(0)$$

for a random variable $\tilde{Z}_\infty(0)$ which is positive with positive probability.

3 Main results

We are ready to state a criterion for the L^p-convergence, $p\in(1,2)$. The cases $\theta=0$ and $\theta\neq 0$ are treated separately in Theorems 3.1 and 3.4, respectively.

Theorem 3.1. *Let $p\in(1,2)$, $\gamma\in\mathbb{R}\backslash\{0\}$, $\lambda=\mathrm{i}\gamma$ and $0<|m(\lambda)|<m(0)<\infty$. Assume that*

$$\mathbb{E}|Z_1(\lambda)|^2<\infty \qquad (2)$$

or

$$0<\liminf_{x\to\infty} x^\alpha\mathbb{P}\{|Z_1(\lambda)|>x\}\leq \limsup_{x\to\infty} x^\alpha \mathbb{P}\{|Z_1(\lambda)|>x\}<\infty \qquad (3)$$

for some $\alpha\in(1,2)$. If either $\mathbb{E}Z_1(0)\log^+ Z_1(0)=\infty$ and

$$A:=\sum_{n\geq 0}\frac{1}{\ell(m(0)^n)^{p/\alpha}}=\infty,$$

where ℓ is a slowly varying function appearing in the Seneta-Heyde theorem, and we take $\alpha=2$ when condition (2) holds, or $\mathbb{E}Z_1(0)\log^+ Z_1(0)<\infty$, then the martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges in L^p if, and only if,

$$p<\alpha \qquad (4)$$

and

$$\frac{m(0)}{|m(\lambda)|^\alpha} < 1. \tag{5}$$

If $\mathbb{E}Z_1(0)\log^+ Z_1(0) = \infty$ *and* $A < \infty$, *then the martingale* $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ *converges in* L^p *if, and only if, condition* (4) *holds and*

$$\frac{m(0)}{|m(\lambda)|^\alpha} \le 1. \tag{6}$$

Remark 3.2. A perusal of the proof of Theorem 3.1 reveals that conditions (3) and (4) can be safely replaced by the (seemingly) less restrictive condition (1), thereby extending the range of applicability of the result.

Remark 3.3. Let us note that irrespective of the $x\log x$ condition $Z_n(0)\ell(m(0)^n)$ converges a.s. to a random variable which is positive with positive probability. Here, the slowly varying function ℓ is identically one when $\mathbb{E}Z_1(0)\log^+ Z_1(0) < \infty$. In view of this we can reformulate Theorem 3.1 in a more succinct form: under assumptions (2) and (3) the martingale $(Z_n(\mathrm{i}\gamma))_{n\in\mathbb{N}_0}$ converges in L^p, $p \in (1,2)$ if, and only if, condition (4) holds and

$$\sum_{n\ge 0}\Big(\frac{m(0)}{|m(\mathrm{i}\gamma)|^\alpha}\Big)^n \frac{1}{\ell(m(0)^n)^{p/\alpha}} < \infty.$$

Theorem 3.4. *Let* $p \in (1,2)$, $\theta, \gamma \in \mathbb{R}\backslash\{0\}$, $\lambda = \theta + \mathrm{i}\gamma$ *and* $0 < |m(\lambda)| < m(\theta) < \infty$. *Assume that conditions* (2) *and* (3) *hold with the present* λ, *and that the martingale* $(Z_n(\alpha\theta))_{n\in\mathbb{N}_0}$ *is uniformly integrable (we take* $\alpha = 2$ *when condition* (2) *holds). Then the martingale* $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ *converges in* L^p *if, and only if, condition* (4) *holds and*

$$\frac{m(\alpha\theta)}{|m(\lambda)|^\alpha} < 1. \tag{7}$$

Remark 3.5. Necessary and sufficient conditions for the uniform integrability of the Biggins martingale with real parameter were obtained in increasing generality in [6], [21] and [2]. Simple sufficient conditions for the uniform integrability of the martingale $(Z_n(\alpha\theta))_{n\in\mathbb{N}_0}$ are $\mathbb{E}Z_1(\alpha\theta)\log^+ Z_1(\alpha\theta) < \infty$ and $-\alpha\theta\mathbb{E}\sum_{|u|=1} e^{-\alpha\theta S(u)}S(u) \in [-\infty, m(\alpha\theta)\log m(\alpha\theta))$.

Theorem 3.4 requires that the martingale $(Z_n(\alpha\theta))_{n\in\mathbb{N}_0}$ be uniformly integrable which is an unpleasant feature. The problem is that it seems that the other assumptions of Theorem 3.4 do not lead to any conclusions concerning the asymptotics of $\mathbb{E}[Z_n(\alpha\theta)]^{p/\alpha}$ as $n \to \infty$, when $(Z_n(\alpha\theta))_{n\in\mathbb{N}_0}$ is not uniformly integrable martingale. Although in the latter case there are several results (see [1, 9, 15]) concerning distributional convergence of $Z_n(\alpha\theta)a_n$ as $n \to \infty$ for appropriate constants (a_n), the assumptions imposed in the cited works are too restrictive for our purposes. Fortunately, there is (at least) one exception arising in the case $\theta = 0$ which allowed us to provide a more complete result in Theorem 3.1.

Necessary and sufficient conditions given in Theorems 3.1 and 3.4 like any other necessary and sufficient conditions are of mainly theoretical interest. For applications easily verifiable sufficient conditions are of greater use. Biggins in Theorem 1 of [7] shows that the conditions $\mathbb{E}[Z_1(\theta)]^\gamma < \infty$ for some $\gamma \in (1,2)$ and $m(p\theta)/|m(\lambda)|^p < 1$ for some $p \in (1,\gamma]$ are sufficient for the L^p-convergence of $(Z_n(\lambda)_{n\in\mathbb{N}_0}$. Albeit looking differently Proposition 3.6 given next is essentially equivalent to the Biggins conditions, the improvement being that we use a moment condition for $Z_1(\lambda)$ rather than for $Z_1(\theta)$.

Proposition 3.6. *Let $p \in (1,2)$, $\gamma \in \mathbb{R}\backslash\{0\}$, $\lambda = \theta + \mathrm{i}\gamma$ and $0 < |m(\lambda)| < m(\theta) < \infty$. The conditions*

$$\mathbb{E}|Z_1(\lambda)|^r < \infty \quad \text{and} \quad \frac{m(r\theta)}{|m(\lambda)|^r} < 1 \tag{8}$$

for some $r \in [p,2]$ are sufficient for the L^p-convergence of the martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$.

Now we formulate a criterion for the L^p-convergence, $p \geq 2$. In the sequel we use the standard notation

$$x \vee y = \max(x,y) \quad \text{ and } \quad x \wedge y = \min(x,y).$$

Theorem 3.7. *Let $p \geq 2$, $\theta, \gamma \in \mathbb{R}$, $\gamma \neq 0$, $\lambda = \theta + \mathrm{i}\gamma$ and $0 < |m(\lambda)| < m(\theta) < \infty$. If $\theta \neq 0$, the martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges in L^p if, and only if,*

$$\mathbb{E}|Z_1(\lambda)|^p < \infty, \tag{9}$$

$$\frac{m(2\theta)}{|m(\lambda)|^2} \vee \frac{m(p\theta)}{|m(\lambda)|^p} < 1 \tag{10}$$

and, when $p > 2$,

$$\mathbb{E}[Z_1(2\theta)]^{p/2} < \infty. \tag{11}$$

If $\theta = 0$, the martingale $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges in L^p if, and only if, conditions (9) and (11) hold, and

$$\frac{m(0)}{|m(\lambda)|^2} < 1.$$

4 Proofs

We first formulate a version of the Burkholder inequality for complex-valued martingales. Although we think the result is known, we have not been able to locate it in the literature.

Lemma 4.1. *Let $p > 1$ and $(X_n)_{n\in\mathbb{N}_0}$ be a complex-valued martingale with $X_0 = 0$. Then the martingale $(X_n)_{n\in\mathbb{N}_0}$ converges in L^p if, and only if, $\mathbb{E}(\sum_{n\geq 0} |X_{n+1} - X_n|^2)^{p/2} < \infty$. If one of these holds, then*

$$c_p \mathbb{E}\Big(\sum_{n\geq 0} |X_{n+1} - X_n|^2\Big)^{p/2} \leq \mathbb{E}|X|^p \leq C_p \mathbb{E}\Big(\sum_{n\geq 0} |X_{n+1} - X_n|^2\Big)^{p/2} \tag{12}$$

for appropriate positive and finite constants c_p and C_p, where X is the L^p- limit of $(X_n)_{n\in\mathbb{N}_0}$.

Proof. We only need to prove (12). According to Theorem 1 on p. 414 in [11] inequality (12) holds for real-valued martingales with constants c_p^* and C_p^* in place of c_p and C_p. We shall deduce (12) for complex-valued martingales from the cited theorem and the fact that $(\operatorname{Re} X_n)_{n\in\mathbb{N}_0}$ and $(\operatorname{Im} X_n)_{n\in\mathbb{N}_0}$ are real-valued martingales. From the elementary inequalities

$$(2^{r-1} \wedge 1)(a^r + b^r) \leq (a+b)^r \leq (2^{r-1} \vee 1)(a^r + b^r), \quad a, b \geq 0,\ r > 0$$

we obtain

$$(2^{p/2-1} \wedge 1)(|\operatorname{Re} X|^p + |\operatorname{Im} X|^p) \leq |X|^p \leq (2^{p/2-1} \vee 1)(|\operatorname{Re} X|^p + |\operatorname{Im} X|^p).$$

Therefore,

$$\begin{aligned}
& \mathbb{E}|X|^p \\
\le\; & (2^{p/2-1}\vee 1)(\mathbb{E}|\mathrm{Re}\,X|^p+\mathbb{E}|\mathrm{Im}\,X|^p) \\
\le\; & (2^{p/2-1}\vee 1)C_p^*\left(\mathbb{E}\left[\Big(\sum_{n\ge 0}\big(\mathrm{Re}\,(X_{n+1}-X_n)\big)^2\Big)^{p/2}\right]+\mathbb{E}\left[\Big(\sum_{n\ge 0}\big(\mathrm{Im}\,(X_{n+1}-X_n)\big)^2\Big)^{p/2}\right]\right) \\
\le\; & \frac{2^{p/2-1}\vee 1}{2^{p/2-1}\wedge 1}\,C_p^*\mathbb{E}\Big(\sum_{n\ge 0}\big|X_{n+1}-X_n\big|^2\Big)^{p/2}
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}|X|^p \\
\ge\; & (2^{p/2-1}\wedge 1)(\mathbb{E}|\mathrm{Re}\,X|^p+\mathbb{E}|\mathrm{Im}\,X|^p) \\
\ge\; & (2^{p/2-1}\wedge 1)c_p^*\left(\mathbb{E}\left[\Big(\sum_{n\ge 0}\big(\mathrm{Re}\,(X_{n+1}-X_n)\big)^2\Big)^{p/2}\right]+\mathbb{E}\left[\Big(\sum_{n\ge 0}\big(\mathrm{Im}\,(X_{n+1}-X_n)\big)^2\Big)^{p/2}\right]\right) \\
\ge\; & \frac{2^{p/2-1}\wedge 1}{2^{p/2-1}\vee 1}\,c_p^*\mathbb{E}\Big(\sum_{n\ge 0}\big|X_{n+1}-X_n\big|^2\Big)^{p/2}.
\end{aligned}$$

□

In Lemma 4.2 given next which is needed for the proof of Theorem 3.1 we use the notation introduced in the paragraph preceding Theorem 3.1.

Lemma 4.2. *Let $r\in(0,1)$, $m(0)\in(1,\infty)$ and $\mathbb{E}Z_1(0)\log^+ Z_1(0)=\infty$. Then $\mathbb{E}[\tilde Z_\infty(0)]^r<\infty$ and*

$$\mathbb{E}[Z_n(0)]^r\;\sim\;\frac{\mathbb{E}[\tilde Z_\infty(0)]^r}{\ell(m(0)^n)^r},\quad n\to\infty.$$

Proof. By Corollary 5.5 on p. 86 in [5], the function $x\mapsto\int_0^x\mathbb{P}\{\tilde Z_\infty(0)>y\}\mathrm{d}y$ slowly varies at ∞. This entails $\mathbb{E}[\tilde Z_\infty(0)]^r<\infty$.

From Theorem 5.1 on p. 83 in [5] (and its proof) and Corollary 5.3 on p. 85 in [5] we know that $m(0)^{-n}\ell(m(0)^n)\sim h_n(s_0)$ as $n\to\infty$, where $h_n(s)$ is the inverse function of $x\mapsto-\log\mathbb{E}e^{-x\mathcal{M}_n(\mathbb{R})}$ for $n\in\mathbb{N}$ and s_0 is a small enough positive number, and that $(\exp(-h_n(s_0)\mathcal{M}_n(\mathbb{R})))_{n\in\mathbb{N}}$ is a martingale with respect to the natural filtration which converges a.s. and in mean as $n\to\infty$ to $\exp(-\tilde Z_\infty(0))$. The first of these facts tells us that it suffices to prove that

$$\lim_{n\to\infty}\mathbb{E}[h_n(s_0)\mathcal{M}_n(\mathbb{R})]^r=\mathbb{E}[\tilde Z_\infty(0)]^r. \tag{13}$$

As a consequence of the second we infer that, for each $s\in(0,1)$, $(\exp(-sh_n(s_0)\mathcal{M}_n(\mathbb{R})))_{n\in\mathbb{N}}$ is a submartingale. In particular,

$$1-\mathbb{E}e^{-sh_n(s_0)\mathcal{M}_n(\mathbb{R})}\le 1-\mathbb{E}e^{-s\tilde Z_\infty(0)},\quad s\in(0,1). \tag{14}$$

To prove (13) we shall use the following formula which holds for any nonnegative random variable X and $a\in(0,1)$:

$$\mathbb{E}X^a=\frac{a}{\Gamma(1-a)}\int_0^\infty s^{-a-1}(1-\mathbb{E}e^{-sX})\mathrm{d}s, \tag{15}$$

where $\Gamma(\cdot)$ is the gamma function. This equality follows from $\mathbb{E}e^{-sX} = \mathbb{P}\{R > sX\}$ for $s \geq 0$, where R is an exponentially distributed random variable of unit mean which is independent of X.

With the help of $\lim_{n\to\infty} \mathbb{E}e^{-sh_n(s_0)\mathcal{M}_n(\mathbb{R})} = \mathbb{E}e^{-s\tilde{Z}_\infty(0)}$ for all $s \geq 0$, inequality (14) and the fact that $1 - \mathbb{E}e^{-sh_n(s_0)\mathcal{M}_n(\mathbb{R})} \leq 1$ for $s \geq 1$, we obtain

$$\begin{aligned}\mathbb{E}[h_n(s_0)\mathcal{M}_n(\mathbb{R})]^r &= \frac{r}{\Gamma(1-r)}\int_0^\infty s^{-r-1}(1-\mathbb{E}e^{-sh_n(s_0)\mathcal{M}_n(\mathbb{R})})\mathrm{d}s \\ &\to \frac{r}{\Gamma(1-r)}\int_0^\infty s^{-r-1}(1-\mathbb{E}e^{-s\tilde{Z}_\infty(0)})\mathrm{d}s = \mathbb{E}[\tilde{Z}_\infty(0)]^r\end{aligned}$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem. □

For any $u \in \mathbb{V}$ and $\lambda \in \mathcal{D}_{\neq 0}$, set

$$Z_1^{(u)}(\lambda) := \frac{1}{m(\lambda)}\sum_{|v|=1} e^{-\lambda(S(uv)-S(u)} \quad \text{and} \quad Y_u(\lambda) := \frac{e^{-\lambda S(u)}}{m(\lambda)^{|u|}}.$$

Thus, $Z_1^{(u)}(\lambda)$ is the analogue of $Z_1(\lambda)$, but based on the progeny of individual u rather than the progeny of the initial ancestor $\varnothing$. Observe that, for the individuals u with $|u| = n$ for some $n \in \mathbb{N}$, the Y_u are $\mathcal{F}_n$–measurable, whereas the $Z_1^{(u)}(\lambda)$ are independent of $\mathcal{F}_n$.

Lemma 4.3. *Let $p \in (1, 2)$, $\gamma \in \mathbb{R}\backslash\{0\}$, $\lambda = \theta + \mathrm{i}\gamma$ and $0 < |m(\lambda)| < m(\theta) < \infty$. Assume that (3) holds for $\alpha \in (p, 2)$ and, when $\theta \neq 0$, that $m(\alpha\theta) < \infty$ and the martingale $(Z_n(\alpha\theta))_{n\in\mathbb{N}_0}$ is uniformly integrable. Then there exist positive constants c and C such that for each $n \in \mathbb{N}$,*

$$c\Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha}\mathbb{E}[Z_n(0)]^{p/\alpha} \leq \mathbb{E}\Big(\sum_{|u|=n}|Y_u(\lambda)|^2|Z_1^{(u)}(\lambda)-1|^2\Big)^{p/2} \leq C\Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha}\mathbb{E}[Z_n(0)]^{p/\alpha} \tag{16}$$

when $\theta = 0$, and

$$c\Big(\frac{m(\alpha\theta)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha} \leq \mathbb{E}\Big(\sum_{|u|=n}|Y_u(\lambda)|^2|Z_1^{(u)}(\lambda)-1|^2\Big)^{p/2} \leq C\Big(\frac{m(\alpha\theta)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha} \tag{17}$$

when $\theta \neq 0$.

Proof. Denote by $\xi_1, \xi_2, \ldots$ independent random variables which are distributed as $|Z_1(\lambda) - 1|$ and independent of $\mathcal{F}_n$. Further, let $\eta_1, \eta_2, \ldots$ be i.i.d. positive random variables with

$$\mathbb{P}\{\eta_1 > x\} \sim bx^{-\alpha}, \quad x \to \infty$$

for some $b > 0$ and the same α as in (3). It is clear that $\varphi(s) := \mathbb{E}e^{-s\eta_1^2}$ satisfies

$$-\log\varphi(s) \sim b\Gamma(1-\alpha/2)s^{\alpha/2}, \quad s \to 0+. \tag{18}$$

Let $\eta_{\alpha/2}$ be a positive $(\alpha/2)$-stable random variable with the Laplace transform

$$\Psi(s) := \mathbb{E}\exp(-s\eta_{\alpha/2}) = \exp(-b\Gamma(1-\alpha/2)s^{\alpha/2}), \quad s \geq 0.$$

<u>Case</u> $\theta = 0$. Set

$$\Psi_k(s) := \mathbb{E}\exp\Big(\frac{\eta_1^2 + \ldots + \eta_k^2}{k^{2/\alpha}}\Big) = [\varphi(sk^{-2/\alpha})]^k, \quad s \geq 0,\ k \in \mathbb{N}.$$

It is easily seen that

$$\lim_{k\to\infty} \Psi_k(s) = \Psi(s), \quad s \geq 0. \tag{19}$$

We intend to show that

$$\lim_{k\to\infty} \mathbb{E}\bigg(\frac{\eta_1^2 + \ldots + \eta_k^2}{k^{2/\alpha}}\bigg)^{p/2} = \mathbb{E}[\eta_{\alpha/2}]^{p/2} < \infty. \tag{20}$$

According to formula (15) relation (20) is equivalent to

$$\lim_{k\to\infty} \int_0^\infty s^{-p/2-1}(1-\Psi_k(s))\mathrm{d}s = \int_0^\infty s^{-p/2-1}(1-\Psi(s))\mathrm{d}s. \tag{21}$$

With (19) at hand we shall prove (21) with the help of Lebesgue's dominated convergence theorem. In view of (18), for $s_0 > 0$ small enough there exists $r > 0$ such that $-\log\varphi(s) \leq rs^{\alpha/2}$ whenever $s \in [0, s_0]$. Hence, for such s

$$1 - \Psi_k(s) \leq -\log\Psi_k(s) \leq rs^{\alpha/2},$$

and $\int_0^{s_0} s^{-p/2-1+\alpha/2}\mathrm{d}s = \frac{2}{\alpha-p}s_0^{(\alpha-p)/2} < \infty$ because $p < \alpha$. For $s \geq s_0$ we use the crude estimate $1 - \Psi_n(s) \leq 1$ which suffices in view of $\int_{s_0}^\infty s^{-p/2-1}\mathrm{d}s = (2/p)s_0^{-p/2} < \infty$. The proof of (21) is complete.

As a consequence of (21) and (3) we obtain

$$c \leq \mathbb{E}\Big(\frac{\xi_1^2 + \ldots + \xi_k^2}{k^{2/\alpha}}\Big)^{p/2} \leq C$$

for all $k \in \mathbb{N}$ and appropriate $c, C > 0$, whence

$$\mathbb{E}\bigg(\sum_{j=1}^{\mathcal{M}_n(\mathbb{R})} \xi_j^2\bigg)^{p/2} = \mathbb{E}[\mathcal{M}_n(\mathbb{R})]^{p/\alpha}\mathbb{E}\bigg(\bigg(\frac{\sum_{j=1}^{\mathcal{M}_n(\mathbb{R})}\xi_j^2}{\mathcal{M}_n(\mathbb{R})^{2/\alpha}}\,\mathbb{1}_{\{\mathcal{M}_n(\mathbb{R})\geq 1\}}\bigg)^{p/2}\bigg|\mathcal{F}_n\bigg) \geq c\mathbb{E}[\mathcal{M}_n(\mathbb{R})]^{p/\alpha}.$$

Arguing similarly for the the upper bound we arrive at

$$c\mathbb{E}[\mathcal{M}_n(\mathbb{R})]^{p/\alpha} \leq \mathbb{E}\bigg(\sum_{j=1}^{\mathcal{M}_n(\mathbb{R})} \xi_j^2\bigg)^{p/2} \leq C\mathbb{E}[\mathcal{M}_n(\mathbb{R})]^{p/\alpha} \tag{22}$$

which is equivalent to (16).

<u>Case</u> $\theta \neq 0$. Like in the previous part of the proof, inequality (17) follows if we can show that

$$\lim_{n\to\infty} \mathbb{E}\bigg(\frac{\sum_{|u|=n} e^{-2\theta S(u)}|Z_1^{(u)}(\lambda) - 1|^2}{m(\alpha\theta)^{2n/\alpha}}\bigg)^{p/2} = \mathbb{E}[\eta_{\alpha/2} Z_\infty(\alpha\theta)^{2/\alpha}]^{p/2} < \infty \tag{23}$$

assuming that $|Z_1(\lambda) - 1|$ has the same distribution as η_1. Here, $Z_\infty(\alpha\theta)$ is the a.s. and L_1-limit of the uniformly integrable martingale $(Z_n(\alpha\theta))_{n\in\mathbb{N}_0}$. Furthermore, $Z_\infty(\alpha\theta)$ is assumed independent of $\eta_{\alpha/2}$. By (15), relation (23) is equivalent to

$$\lim_{n\to\infty} \int_0^\infty s^{-p/2-1}(1-\Phi_n(s))\mathrm{d}s = \int_0^\infty s^{-p/2-1}(1-\Phi(s))\mathrm{d}s, \tag{24}$$

where

$$\Phi_n(s) := \mathbb{E}\exp\Big(-s\frac{\sum_{|u|=n} e^{-2\theta S(u)}|Z_1^{(u)}(\lambda)-1|^2}{m(\alpha\theta)^{2n/\alpha}}\Big), \quad s\geq 0,\ n\in\mathbb{N}$$

and

$$\Phi(s) := \mathbb{E}\exp(-s\eta_{\alpha/2}Z_\infty(\alpha\theta)^{2/\alpha}) = \mathbb{E}\exp(-b\Gamma(1-\alpha/2)s^{\alpha/2}Z_\infty(\alpha\theta)).$$

By Theorem 3 in [8],

$$\frac{\sup_{|u|=n} e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}} = \Big(\frac{\sup_{|u|=n} e^{-\alpha\theta S(u)}}{m(\alpha\theta)^n}\Big)^{2/\alpha} \ \to\ 0 \quad \text{a.s.}$$

as $n\to\infty$. This in combination with (18) yields, for $s\geq 0$,

$$\begin{aligned} &-\log\mathbb{E}\Big(\exp\Big(-s\frac{\sum_{|u|=n} e^{-2\theta S(u)}|Z_1^{(u)}(\lambda)-1|^2}{m(\alpha\theta)^{2n/\alpha}}\Big)\Big|\mathcal{F}_n\Big)\\ =\ & \sum_{|u|=n} -\log\varphi\Big(s\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\Big)\\ \sim\ & b\Gamma(1-\alpha/2)s^{\alpha/2}Z_n(\alpha\theta)\ \to\ b\Gamma(1-\alpha/2)s^{\alpha/2}Z_\infty(\alpha\theta) \quad \text{a.s.} \end{aligned}$$

as $n\to\infty$, thereby proving that

$$\lim_{n\to\infty}\Phi_n(s) = \Phi(s), \quad s\geq 0.$$

To justify (24) we shall use Lebesgue's dominated convergence theorem. As a consequence of (18), given $s_0>0$ small enough there exist positive constants B_1 and B_2 such that

$$\frac{s^{\alpha/2}}{1-\varphi(s)}\leq B_1 \quad\text{and}\quad \frac{1-\varphi(sx)}{1-\varphi(s)}\leq B_2x^{\alpha/2}$$

whenever $s\in(0,s_0]$ and $sx\in(0,s_0]$. Therefore, for $s\in(0,s_0]$ and $S(u)$ with $|u|=n$,

$$\begin{aligned} \frac{1-\varphi\Big(s\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\Big)}{1-\varphi(s)} &= \frac{1-\varphi\Big(s\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\Big)}{1-\varphi(s)}\mathbb{1}_{\{\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\leq\frac{s_0}{s}\}} + \frac{1-\varphi\Big(s\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\Big)}{1-\varphi(s)}\mathbb{1}_{\{\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}>\frac{s_0}{s}\}}\\ &\leq \Big(B_2+\frac{B_1}{s_0^{\alpha/2}}\Big)\frac{e^{-\alpha\theta S(u)}}{m(\alpha\theta)^n} \quad \text{a.s.} \end{aligned}$$

This yields, for each $n\in\mathbb{N}$ and $s\in(0,s_0]$,

$$\begin{aligned} 1-\Phi_n(s) &= \mathbb{E}\Big(1-\prod_{|u|=n}\varphi\Big(s\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\Big)\Big)\leq(1-\varphi(s))\mathbb{E}\sum_{|u|=n}\frac{1-\varphi\Big(s\frac{e^{-2\theta S(u)}}{m(\alpha\theta)^{2n/\alpha}}\Big)}{1-\varphi(s)}\\ &\leq \Big(B_2+\frac{B_1}{s_0^{\alpha/2}}\Big)(1-\varphi(s))\mathbb{E}\sum_{|u|=n}\frac{e^{-\alpha\theta S(u)}}{m(\alpha\theta)^n} = \Big(B_2+\frac{B_1}{s_0^{\alpha/2}}\Big)(1-\varphi(s)). \end{aligned}$$

The so obtained majorant is appropriate because

$$\int_0^{s_0} s^{-p/2-1}(1-\varphi(s))\mathrm{d}s<\infty$$

as a consequence of $\mathbb{E}[\eta_1]^p<\infty$ (recall that $p<\alpha$). When $s>s_0$ the crude bound $1-\Phi_n(s)\leq 1$ suffices, for $\int_{s_0}^\infty s^{-p/2-1}\mathrm{d}s<\infty$. The proof of Lemma 4.3 is complete. □

For the proof of Theorem 3.7 we shall need a version of Lemma 3.3 in [3].

Lemma 4.4. *Assume that $m(p\theta) \geq m(\theta)^p$ and $\mathbb{E}[Z_1(\theta)]^p < \infty$ for some $p > 1$ and $\theta \in \mathbb{R}\backslash\{0\}$. Then*

$$\mathbb{E}[Z_n(\theta)]^p = O\Big(n^c\Big(\frac{m(p\theta)}{m(\theta)^p}\Big)^n\Big), \quad n \to \infty$$

for a finite nonnegative constant c (explicitly known).

We are now ready to prove our main results.

Proof of Theorem 3.1. Necessity of (4) and (5) or (6). Set $R := \sum_{n\geq 0} |Z_{n+1}(\lambda) - Z_n(\lambda)|^2$ and assume that $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges in L^p, $p \in (1,2)$. Then $\mathbb{E}R^{p/2} < \infty$ by Lemma 4.1. In particular, this entails $\mathbb{E}|Z_1(\lambda)|^p < \infty$ thereby showing the necessity of (4).

Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers which satisfies $a := \sum_{n\geq 0} a_n < \infty$. Since the function $x \mapsto x^{p/2}$ is concave on $[0,\infty)$ we infer

$$\begin{aligned}
R^{p/2} &= a^{p/2}\Big(\sum_{n\geq 0}(a_n/a)(1/a_n)|Z_{n+1}(\lambda) - Z_n(\lambda)|^2\Big)^{p/2} \\
&\geq a^{p/2}\sum_{n\geq 0}(a_n/a)(1/a_n)^{p/2}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p \\
&= a^{p/2-1}\sum_{n\geq 0} a_n^{1-p/2}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p.
\end{aligned}$$

Given $\mathcal{F}_n$, the random variable $Z_{n+1}(\lambda) - Z_n(\lambda)$, being a weighted sum of i.i.d. complex-valued zero-mean random variables, is the terminal value of a martingale. Hence, Lemma 4.1 applies and gives

$$\begin{aligned}
C_p\mathbb{E}\Big(\sum_{|u|=n}|Y_u(\lambda)|^2|Z_1^{(u)}(\lambda) - 1|^2\Big)^{p/2} &\geq \mathbb{E}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p \\
&\geq c_p\mathbb{E}\Big(\sum_{|u|=n}|Y_u(\lambda)|^2|Z_1^{(u)}(\lambda) - 1|^2\Big)^{p/2} =: c_p A_n \quad (25)
\end{aligned}$$

(the left-hand inequality is not needed here and will be used later).

Assume that condition (2) holds. Then $\alpha = 2$ by our convention. Using once again concavity of $x \mapsto x^{p/2}$ on $[0,\infty)$ we obtain

$$A_n \geq \mathbb{E}|Z_1(\lambda) - 1|^p\mathbb{E}\Big(\sum_{|u|=n}|Y_u(\lambda)|^2\Big)^{p/2} = \Big(\frac{m(0)}{|m(\lambda)|^2}\Big)^{np/2}\mathbb{E}|Z_1(\lambda) - 1|^p\mathbb{E}[Z_n(0)]^{p/2}$$

and thereupon

$$\infty > \mathbb{E}R^{p/2} \geq a^{p/2-1}c_p\mathbb{E}|Z_1(\lambda) - 1|^p\sum_{n\geq 0} a_n^{1-p/2}\Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha}\mathbb{E}[Z_n(0)]^{p/\alpha}. \quad (26)$$

Assume now that condition (3) holds. Then $\alpha \in (p,2)$ (recall (4)). According to (16),

$$A_n \geq c\Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha}\mathbb{E}[Z_n(0)]^{p/\alpha}$$

and thereupon

$$\infty > \mathbb{E}R^{p/2} \geq a^{p/2-1} c c_p \sum_{n\geq 0} a_n^{1-p/2} \Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha} \mathbb{E}[Z_n(0)]^{p/\alpha}.$$

Observe that the series on the right-hand side is the same as in (26). Further, we have to consider two cases.
<u>Case</u> $\mathbb{E}Z_1(0)\log^+ Z_1(0) < \infty$. According to the Kesten-Stigum theorem, already mentioned in Section 3, $Z_n(0)$ converges a.s. and in mean as $n \to \infty$ to a random variable $Z_\infty(0)$. Therefore, $\lim_{n\to\infty} \mathbb{E}[Z_n(0)]^{p/\alpha} = \mathbb{E}[Z_\infty(0)]^{p/\alpha} \in (0,\infty)$, and the necessity of (5) follows upon setting

$$a_n = \Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^n, \quad n \in \mathbb{N}_0. \tag{27}$$

<u>Case</u> $\mathbb{E}Z_1(0)\log^+ Z_1(0) = \infty$. By Lemma 4.2, we have

$$\mathbb{E}[Z_n(0)]^{p/\alpha} \ \sim \ \frac{\mathbb{E}[\tilde{Z}_\infty(0)]^{p/\alpha}}{\ell(m(0)^n)^{p/\alpha}}, \quad n \to \infty$$

for some positive slowly varying ℓ with $\lim_{t\to\infty} \ell(t) = \infty$. Assume that $\sum_{n\geq 0} \ell(m(0)^n)^{-p/\alpha}$ is a divergent series. Then choosing a_n as in (27) we see that condition (5) is necessary. If the series $\sum_{n\geq 0} \ell(m(0)^n)^{-p/\alpha}$ converges then choosing any sequence $(a_n)_{n\in\mathbb{N}_0}$ with the property $\lim_{n\to\infty} e^{bn} a_n = \infty$ for any $b > 0$, we conclude that condition (6) is necessary.
Sufficiency of (4) and (5) or (6). By Lemma 4.1, it suffices to show that $\mathbb{E}R^{p/2} < \infty$. Using subadditivity of $x \mapsto x^{p/2}$ on $[0,\infty)$ we obtain

$$\mathbb{E}R^{p/2} \leq \sum_{n\geq 0} \mathbb{E}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p.$$

Further, in view of (25)

$$\mathbb{E}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p \leq C_p \mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^2 |Z_1^{(u)}(\lambda) - 1|^2\Big)^{p/2}.$$

Assume first that condition (2) holds, so that $\alpha = 2$. Using conditional Jensen's inequality yields

$$\begin{aligned}\mathbb{E}\Big(\Big(\sum_{|u|=n} |Y_u(\lambda)|^2 |Z_1^{(u)}(\lambda) - 1|^2\Big)^{p/2}\Big|\mathcal{F}_n\Big) \ &\leq \ \mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^2 |Z_1^{(u)}(\lambda) - 1|^2\Big|\mathcal{F}_n\Big)^{p/2} \\ &= \ \big[\mathbb{E}|Z_1(\lambda) - 1|^2\big]^{p/2} \Big(\frac{m(0)}{|m(\lambda)|^2}\Big)^{np/2} Z_n(0)^{p/2} \text{ a.s.,}\end{aligned}$$

whence

$$\mathbb{E}R^{p/2} \leq C_p \big[\mathbb{E}|Z_1(\lambda) - 1|^2\big]^{p/2} \sum_{n\geq 0} \Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha} \mathbb{E}[Z_n(0)]^{p/\alpha}.$$

Assume now that condition (3) holds which together with (4) ensures that $\alpha \in (p, 2)$. In view of (16)

$$A_n \leq C\Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha} \mathbb{E}[Z_n(0)]^{p/\alpha}$$

which entails

$$\mathbb{E}R^{p/2} \leq C_p C \sum_{n\geq 0} \Big(\frac{m(0)}{|m(\lambda)|^\alpha}\Big)^{np/\alpha} \mathbb{E}[Z_n(0)]^{p/\alpha}.$$

Arguing as in the proof of necessity we conclude the following. If either $\mathbb{E}Z_1(0)\log^+ Z_1(0) = \infty$ and $A = \infty$, or $\mathbb{E}Z_1(0)\log^+ Z_1(0) < \infty$, then condition (5) is sufficient, whereas if $\mathbb{E}Z_1(0)\log^+ Z_1(0) = \infty$ and $A < \infty$, then condition (6) is sufficient. The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.4. The proof is a simpler counterpart of the proof of Theorem 3.1 which uses inequality (17) rather than (16). We omit details. □

Proof of Proposition 3.6. We have for r satisfying (8)

$$\begin{aligned}\mathbb{E}R^{p/2} &\leq C_p \mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^2 |Z_1^{(u)}(\lambda) - 1|^2\Big)^{p/2} \\ &\leq C_p \mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^r |Z_1^{(u)}(\lambda) - 1|^r\Big)^{p/r} \leq C_p [\mathbb{E}|Z_1(\lambda) - 1|^r]^{p/r} \sum_{n\geq 0} \Big(\frac{m(r\theta)}{|m(\lambda)|^r}\Big)^{np/r} < \infty\end{aligned}$$

which proves the result in view of Lemma 4.1. The first inequality was obtained in the proof of sufficiency in Theorem 3.1. The second and third are consequences of subadditivity of $x \mapsto x^{r/2}$ and Jensen's inequality, respectively. □

Proof of Theorem 3.7. NECESSITY OF (9), (10) AND (11). Assume that $(Z_n(\lambda))_{n\in\mathbb{N}_0}$ converges in L^p and recall the notation $R = \sum_{n\geq 0} |Z_{n+1}(\lambda) - Z_n(\lambda)|^2$. Then $\mathbb{E}R^{p/2} < \infty$ by Lemma 4.1. Recalling that $p \geq 2$ and using superadditivity of $x \mapsto x^{p/2}$ on $[0, \infty)$ we further infer

$$\sum_{n\geq 0} \mathbb{E}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p \leq \mathbb{E}R^{p/2} < \infty. \tag{28}$$

On the one hand, we obtain for A_n defined in (25),

$$A_n \geq \mathbb{E} \sum_{|u|=n} |Y_u(\lambda)|^p |Z_1^{(u)}(\lambda) - 1|^p = \mathbb{E}|Z_1(\lambda) - 1|^p \Big(\frac{m(p\theta)}{|m(\lambda)|^p}\Big)^n$$

having utilized the aforementioned superadditivity. In view of (28) this proves the necessity of (9) for $p \geq 2$ and $m(p\theta) < |m(\lambda)|^p$. On the other hand, we conclude that

$$\begin{aligned}A_n &\geq \mathbb{E}\Big[\mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^2 |Z_1^{(u)}(\lambda) - 1|^2 \Big| \mathcal{F}_n\Big)^{p/2}\Big] = \Big(\mathbb{E}|Z_1(\lambda) - 1|^2\Big)^{p/2} \mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^2\Big)^{p/2} \\ &\geq \Big(\mathbb{E}|Z_1(\lambda) - 1|^2 \Big(\frac{m(2\theta)}{|m(\lambda)|^2}\Big)^n\Big)^{p/2},\end{aligned}$$

where the first and second inequalities are consequences of conditional and usual Jensen's inequality, respectively. This proves the necessity of $m(2\theta) < |m(\lambda)|^2$. Using the last chain of inequalities with $n = 1$ we observe that

$$\mathbb{E}\Big(\sum_{|u|=1} |Y_u(\lambda)|^2\Big)^{p/2} = \frac{1}{|m(\lambda)|^p} \mathbb{E}\Big(\sum_{|u|=1} e^{-2\theta S(u)}\Big)^{p/2} < \infty$$

which in combination with the already checked finiteness of $m(2\theta)$ proves the necessity of (11). Finally, if $\theta = 0$, then conditions $m(0) > 1$ and $m(0) < |m(\lambda)|^2$ imply that $|m(\lambda)| > 1$. Therefore, $m(0) < |m(\lambda)|^p$ is a consequence of $m(0) < |m(\lambda)|^2$.

Sufficiency of (9), (10) and (11). By Lemma 4.1, it suffices to check that $\mathbb{E}R^{p/2} < \infty$. Using the triangle inequality in $L_{p/2}$ yields

$$\mathbb{E}R^{p/2} \leq \Big(\sum_{n\geq 0} \big[\mathbb{E}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p\big]^{2/p} \Big)^{p/2}.$$

To show that the right-hand side is finite, we write

$$\begin{aligned} C_p^{-1}\mathbb{E}|Z_{n+1}(\lambda) - Z_n(\lambda)|^p &\leq \mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^2 |Z_1^{(u)}(\lambda) - 1|^2 \Big)^{p/2} \\ &= \mathbb{E}\Big(\sum_{|v|=n} |Y_v(\lambda)|^2 \sum_{|u|=n} \frac{|Y_u(\lambda)|^2}{\sum_{|v|=n}|Y_v(\lambda)|^2} |Z_1^{(u)}(\lambda) - 1|^2 \Big)^{p/2} \\ &\leq \mathbb{E}|Z_1(\lambda) - 1|^p \mathbb{E}\Big(\sum_{|u|=n} |Y_u(\lambda)|^2 \Big)^{p/2} \\ &= \mathbb{E}|Z_1(\lambda) - 1|^p \Big(\frac{m(2\theta)}{|m(\lambda)|^2} \Big)^{np/2} \mathbb{E}[Z_n(2\theta)]^{p/2}. \end{aligned} \tag{29}$$

We have used (25) for the first inequality and convexity of $x \mapsto x^{p/2}$ on $[0, \infty)$ for the second. Now we have to analyze the asymptotic behavior of $\mathbb{E}[Z_n(2\theta)]^{p/2}$ as $n \to \infty$. While doing so, distinguishing two cases seems inevitable.

Case $p = 2$. The right-hand side of (29) is equal to $\mathbb{E}|Z_1(\lambda) - 1|^2 \Big(\frac{m(2\theta)}{|m(\lambda)|^2} \Big)^n$. Therefore, conditions $\mathbb{E}|Z_1(\lambda)|^2 < \infty$ and $m(2\theta) < |m(\lambda)|^2$ ensure $\mathbb{E}R < \infty$.

Case $p > 2$.

Subcase $m(p\theta) < m(2\theta)^{p/2}$, $\theta \in \mathbb{R}$. In view of the present assumption on m and (11) we have $\sup_{n\geq 0} \mathbb{E}[Z_n(2\theta)]^{p/2} < \infty$ by Proposition 2.1. Hence, the right-hand side of (29) is $O\Big(\Big(\frac{m(2\theta)}{|m(\lambda)|^2} \Big)^{np/2} \Big)$. This in combination with $m(2\theta) < |m(\lambda)|^2$ proves $\mathbb{E}R^{p/2} < \infty$. If $\theta = 0$, this completes the proof of sufficiency because the complementary case considered below which reads $m(0) \geq m(0)^{p/2}$ is impossible in view of $m(0) \in (1, \infty)$.

Subcase $m(p\theta) \geq m(2\theta)^{p/2}$, $\theta \in \mathbb{R}\backslash\{0\}$. In view of the present assumption on m and (11) we can apply Lemma 4.4 with 2θ and $p/2$ replacing θ and p to obtain

$$\mathbb{E}[Z_n(2\theta)]^{p/2} = O\Big(n^c \Big(\frac{m(p\theta)}{m(2\theta)^{p/2}} \Big)^n \Big), \quad n \to \infty$$

for appropriate finite constant c. Hence, the right-hand side of (29) is $O\Big(n^c \Big(\frac{m(p\theta)}{|m(\lambda)|^p} \Big)^n \Big)$ which proves $\mathbb{E}R^{p/2} < \infty$ because $m(p\theta) < |m(\lambda)|^p$.

The proof of Theorem 3.7 is complete. □

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