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Functional limit theorems for random walks perturbed by positive alpha-stable jumps

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Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables of zero mean and finite variance and $\eta_1, \eta_2, \dots$ positive i.i.d. random variables whose distribution belongs to the domain of attraction of an α-stable distribution, $\alpha \in (0,1)$. The two collections are assumed independent. We consider a Markov chain with jumps of two types. If the present position of the Markov chain is positive, then the jump ξ_k occurs; if the present position of the Markov chain is nonpositive, then its next position is η_j. We prove a functional limit theorem for this Markov chain under Donsker's scaling. The weak limit is a nonnegative process $(X(t))_{t\geq 0}$ satisfying a stochastic equation $\mathrm{d}X(t) = \mathrm{d}W(t) + \mathrm{d}U_\alpha(L_X^{(0)}(t))$, where W is a Brownian motion, U_α is an α-stable subordinator which is independent of W, and $L_X^{(0)}$ is a local time of X at 0. Also, we explain that X is a Feller Brownian motion with a 'jump-type' exit from 0.

Keywords: Feller Brownian motion; functional limit theorem; locally perturbed random walk

1. Introduction and main result

Let $x_0 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $X(n)$ denote the number of claims at time $n \in \mathbb{N}_0$ in a discrete time single server queuing model. The sequence $X := (X(n))_{n\in\mathbb{N}_0}$ satisfies the classical Lindley recursion

$$X(0) = x_0, \quad X(n) = (X(n-1) + \theta_n)^+, \quad n \in \mathbb{N},$$

where, as usual, $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$, and a random variable θ_n represents 'arrival minus departure at step n', see, for instance, Section III.6 in [1] or Section 9.2 in [21]. On the other hand, the sequence X can be obtained as an action of the Skorokhod map Ψ on the random walk $S_\theta := (S_\theta(n))_{n\in\mathbb{N}_0}$ defined by $S_\theta(n) := x_0 + \theta_1 + \ldots + \theta_n$ for $n \in \mathbb{N}_0$, namely,

$$X = \Psi(S_\theta),$$

see, for instance, Section 9.3.1 in [20]. The scaling limit of $\Psi(S_\theta)$ is Skorokhod's reflection of the scaling limit of S_θ, provided that the latter is well-defined. Assume, for instance, that $\theta_1, \theta_2, \dots$ are independent identically distributed random variables of zero mean and finite variance. Then Donsker's scaling limit of X is a reflected Brownian motion. The results of this type are well-known and can be interpreted from different viewpoints, see, for example, Chapter VI in [2], Sections 1.9, 1.10 and 3.3 in [6], Sections 8.7 and 8.8 in [21]. Usually, the continuous mapping theorem is a main technical tool of the corresponding proofs.

Consider a slightly different model in which $X(n)$ is the number of goods at time n in a storage. If $X(n-1) + \theta_n < 0$ (that is, the request at time n cannot be satisfied), then the storage is refilled with a random amount of goods or several random batches of goods. The purpose of the present paper is to prove functional limit theorems with Donsker's scaling for this and similar models, which only differ by the way of reflection upon crossing 0.

Let ξ, ξ_1, $\xi_2,\dots$ be i.i.d. real-valued random variables and η, η_1, $\eta_2,\dots$ positive i.i.d. random variables; the two collections being independent. Given $x_0\geq 0$, define the random sequences $\tilde S:=(\tilde S(n))_{n\in\mathbb{N}_0}$, $\hat S:=(\hat S(n))_{n\in\mathbb{N}_0}$ and $\grave S:=(\grave S(n))_{n\in\mathbb{N}_0}$ recursively as follows: for $n\in\mathbb{N}_0$,

$$\tilde S(0)=0,\quad \tilde S(n+1)=\begin{cases}\tilde S(n)+\xi_{n+1}, & \tilde S(n)>0,\\ \eta_{n+1}, & \tilde S(n)\leq 0;\end{cases}\tag{1.1}$$

$$\hat S(0)=0,\quad \hat S(n+1)=\begin{cases}\hat S(n)+\xi_{n+1}, & \hat S(n)>0 \text{ and } \hat S(n)+\xi_{n+1}>0,\\ 0, & \hat S(n)>0 \text{ and } \hat S(n)+\xi_{n+1}\leq 0,\\ \eta_{n+1}, & \hat S(n)=0\end{cases}$$

and

$$\grave S(0)=0,\quad \grave S(n+1)=\begin{cases}\grave S(n)+\xi_{n+1}, & \grave S(n)>0,\\ \grave S(n)+\eta_{n+1}, & \grave S(n)\leq 0.\end{cases}$$

Denote by $D:=D[0,\infty)$ the Skorokhod space of right-continuous functions defined on $[0,\infty)$ with finite limits from the left. We assume that the space D is endowed with the J_1-topology and write $\Rightarrow$ for weak convergence in this space. Also, on several occasions, we denote by $\Rightarrow$ weak convergence in D^k for $k\geq 2$ equipped with the product J_1-topology. In the latter case the topology and the space will be specified. Comprehensive information concerning the J_1-topology can be found in the books [3, 11]. As usual, $\lfloor x\rfloor$ denotes the integer part of $x\in\mathbb{R}$ and $\circ$ denotes composition of (random) functions. Recall that the Euler gamma function Γ is given by $\Gamma(x):=\int_0^\infty e^{-y}y^{x-1}\mathrm{d}y$ for $x>0$.

Here are our main results.

Theorem 1.1. *Assume that* $\mathbb{E}\xi=0$, $\sigma^2:=\mathrm{Var}\,\xi\in(0,\infty)$ *and that*

$$\mathbb{P}\{\eta>x\}\sim x^{-\alpha}\ell(x),\quad x\to\infty\tag{1.2}$$

for some $\alpha\in(0,1)$ *and* ℓ *slowly varying at* ∞. *Then*

$$\Big(\frac{\tilde S(\lfloor vt\rfloor)}{\sigma v^{1/2}}\Big)_{t\geq 0}\ \Rightarrow\ (W_\alpha(t))_{t\geq 0},\quad v\to\infty,\tag{1.3}$$

where the limit process $W_\alpha:=(W_\alpha(t))_{t\geq 0}$ *is given by*

$$W_\alpha(t)=W(t)+U_\alpha\circ U_\alpha^{\leftarrow}\circ M(t),\quad t\geq 0.$$

Here $W:=(W(t))_{t\geq 0}$ *is a standard Brownian motion;*

$$M(t)=-\min_{s\in[0,\,t]}W(s),\quad t\geq 0,$$

$U_\alpha:=(U_\alpha(t))_{t\geq 0}$ *is a drift-free* α*-stable subordinator independent of* W *with*

$$\mathbb{E}\exp(-zU_\alpha(t))=\exp(-\Gamma(1-\alpha)tz^\alpha),\quad t,z\geq 0,\tag{1.4}$$

and $U_\alpha^{\leftarrow} := (U_\alpha^{\leftarrow}(t))_{t\geq 0}$ *is an inverse* α*-stable subordinator defined by*

$$U_\alpha^{\leftarrow}(t) = \inf\{s \geq 0 : U_\alpha(s) > t\}, \quad t \geq 0.$$

As usual, $\stackrel{\mathbb{P}}{\to}$ denotes convergence in probability.

Theorem 1.2. *Let* $x \geq 0$ *be given and, for each* $v > 0$, $\tilde{S}_v := (\tilde{S}_v(n))_{n\in\mathbb{N}_0}$ *a Markov chain having the same transition probabilities as* $\tilde{S}$. *Assume that*

$$\frac{\tilde{S}_v(0)}{v^{1/2}} \stackrel{\mathbb{P}}{\to} x, \quad v \to \infty.$$

Then, under the assumptions of Theorem 1.1,

$$\Big(\frac{\tilde{S}_v(\lfloor vt \rfloor)}{\sigma v^{1/2}}\Big)_{t\geq 0} \Rightarrow (W_\alpha(x,t))_{t\geq 0}, \quad v \to \infty, \tag{1.5}$$

where the limit process $W_\alpha^{(x)} := (W_\alpha(x,t))_{t\geq 0}$ *is given by*

$$W_\alpha(x,t) = x + W(t) + U_\alpha \circ U_\alpha^{\leftarrow} \circ ((-x + M(t))^+), \quad t \geq 0 \tag{1.6}$$

with the same W, M, U_α *and* $U_\alpha^{\leftarrow}$ *as in Theorem 1.1.*

***Remark* 1.1.** Plainly, Theorem 1.2 is more general than Theorem 1.1. We do not prove Theorem 1.2 and restrict ourselves by the following observation. The excursion of $\tilde{S}_v$ is divided into two parts: one part before the first jump over 0 and the other part. It is clear that the limit of the first part is equal to $x + W$ stopped at 0. The second part is amenable to Theorem 1.1.

The limit process $W_\alpha^{(x)}$ is rather non-standard. This statement is justified in Section 4. As an appetizer, we only mention here that $W_\alpha^{(x)}$ is a *Feller Brownian motion* with a 'jump-type' exit from 0.

Theorem 1.3 states that the result of Theorem 1.1 continues to hold, with the sequence $\tilde{S}$ replaced by either $\hat{S}$ or $\grave{S}$.

Theorem 1.3. *Under the assumptions of Theorem 1.1,*

$$\Big(\frac{\hat{S}(\lfloor vt \rfloor)}{\sigma v^{1/2}}\Big)_{v\geq 0} \Rightarrow (W_\alpha(t))_{t\geq 0}, \quad v \to \infty$$

and

$$\Big(\frac{\grave{S}(\lfloor vt \rfloor)}{\sigma v^{1/2}}\Big)_{t\geq 0} \Rightarrow (W_\alpha(t))_{t\geq 0}, \quad v \to \infty.$$

We note in passing that the case when η has a finite mean is much easier to deal with. The scaling limit for $\tilde{S}$, $\hat{S}$ and $\grave{S}$ is then a reflected Brownian motion, see [14, 17].

Some ideas of the present work borrow heavily from Itô's excursion theory as presented in the book [5]. To make the link visible, observe that the excursions between consecutive crossings of zero of the Markov chain under consideration coincide with the excursions of a random walk driven by ξ. Since

ξ has a finite second moment, these excursions should be close in some sense to those of a Brownian motion. Thus, the limit process has to behave like a Brownian motion in the upper half-plane. The distribution of the jumps $\eta_1, \eta_2, \dots$ into the positive halfline belongs to the domain of attraction of a stable distribution on $[0, \infty)$. In particular, the sum of these jumps, properly scaled, converges weakly to a stable subordinator. The additional contribution to the limit process is, roughly speaking, made by the composition of the sum of jumps and the number of crossings of 0 up to time n. Even though neither of the composed processes converges weakly under Donsker's scaling, their composition does indeed exhibit growth at the 'magic' square-root rate.

In the article [16] a functional limit theorem similar to ours is proved in a much simpler situation where ξ and η are integer-valued and ξ is bounded from below by -1. As far as we know there are no other functional limit theorems that would make an explicit link between random walks with a random-jump reflection at 0 and $W_\alpha^{(x)}$. Also relevant to the present work are the papers [13, 23, 24] and references therein, in which functional limit theorems are obtained for processes merged together from certain excursions.

The authors of [16] invoke a representation arising in a generalized Skorokhod reflection problem (see [15] for more details concerning this problem) as a principal tool. The approach of the cited paper fails in the present setting of real-valued random variables ξ and η, for no reduction to the generalized Skorokhod reflection problem seems to be possible. As a remedy, we offer a novel argument which forms the main achievement of the paper.

The remainder of the paper is structured as follows. Theorems 1.1 and 1.3 are proved in Sections 2 and 3, respectively. Properties of the limit process are studied in Section 4. Finally, the Appendix collects three auxiliary results concerning the J_1-convergence of deterministic functions.

2. Proof of Theorem 1.1

Let $X_1, X_2, \dots$ be independent copies of a real-valued random variable X. Throughout the paper, we adopt generic notation $S_X := (S_X(n))_{n\in\mathbb{N}}$ for a random walk with increments X_k, that is,

$$S_X(n) := X_1 + \ldots + X_n, \quad n \in \mathbb{N}.$$

Also, we put

$$\nu_X(t) := \inf\{k \in \mathbb{N} : S_X(k) > t\}, \quad t \geq 0,$$

so that $(\nu_X(t))_{t\geq 0}$ is the first-passage time process for $(S_X(n))_{n\in\mathbb{N}}$.

By Donsker's theorem,

$$\Big(\frac{S_\xi(\lfloor vt \rfloor)}{\sigma v^{1/2}}\Big)_{t\geq 0} \Rightarrow (W(t))_{t\geq 0}, \quad v \to \infty.$$

Condition (1.2) ensures

$$\Big(\frac{S_\eta(\lfloor vt \rfloor)}{a(v)}\Big)_{t\geq 0} \Rightarrow (U_\alpha(t))_{t\geq 0}, \quad v \to \infty, \tag{2.1}$$

where $a : [0, \infty) \to (0, \infty)$ is any function satisfying $\lim_{v\to\infty} v\mathbb{P}\{\eta > a(v)\} = 1$. Using (2.1) in combination with a standard inversion technique à la Feller (see Theorem 7 in [9]) we conclude that finite-dimensional distributions of $(\mathbb{P}\{\eta > v\}\nu_\eta(vt))_{t\geq 0}$ converge weakly to those of $(U_\alpha^{\leftarrow}(t))_{t\geq 0}$

as $v \to \infty$. The random process $t \mapsto \nu_\eta(vt)$ is almost surely (a.s.) nondecreasing, and the limit process $U_\alpha^{\leftarrow}$ is a.s. continuous. Hence, according to Remark 2.1 in [22], the weak convergence of finite-dimensional distributions is equivalent to

$$(\mathbb{P}\{\eta > v\}\nu_\eta(vt))_{t\geq 0} \;\Rightarrow\; (U_\alpha^{\leftarrow}(t))_{t\geq 0}, \quad v \to \infty.$$

With these at hand, an application of Theorem 3.6 in [19] yields

$$\Big(\frac{S_\eta(\nu_\eta(vt))}{v}\Big)_{t\geq 0} \;\Rightarrow\; (U_\alpha \circ U_\alpha^{\leftarrow}(t))_{t\geq 0}, \quad v \to \infty. \tag{2.2}$$

Invoking the continuous mapping theorem, independence of $(\xi_k)_{k\in\mathbb{N}}$ and $(\eta_j)_{j\in\mathbb{N}}$ and a.s. continuity of W we infer

$$\Big(\frac{S_\xi(\lfloor vt\rfloor)}{\sigma v^{1/2}}, \frac{-\min_{s\in[0,t]} S_\xi(\lfloor vs\rfloor)}{\sigma v^{1/2}}, \frac{S_\eta(\nu_\eta(vt))}{v}\Big)_{t\geq 0} \;\Rightarrow\; (W(t), M(t), U_\alpha \circ U_\alpha^{\leftarrow}(t))_{t\geq 0} \tag{2.3}$$

in D^3 in the product J_1-topology, where W and U_α are assumed independent.

2.1. Passage to an equivalent model

Recall that the starting point of $\tilde{S}$ is x_0. Assume that $x_0 = 0$. The case $x_0 > 0$ will be discussed at the end of proof.

Consider a possible realization of the first five elements of the sequence $\tilde{S}$: $\tilde{S}(0) = 0$, $\tilde{S}(1) = \eta_1 > 0$, $\tilde{S}(2) = \eta_1 + \xi_2 \leq 0$, $\tilde{S}(3) = \eta_3 > 0$, $\tilde{S}(4) = \eta_3 + \xi_4 \leq 0$, $\tilde{S}(5) = \eta_5$. We observe that the variables ξ_1, η_2, ξ_3 and η_4 are missing in this realization. More generally, for a given $k \in \mathbb{N}$ any particular realization involves either ξ_k or η_k but not both. Thus, the presence of missing variables is an intrinsic feature of the model. We prefer to work with an equivalent model whose construction uses the whole collection $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ with no gaps. To this end, we define a new sequence $S^* := (S^*(n))_{n\in\mathbb{N}_0}$ and an auxiliary sequence $(T(n))_{n\in\mathbb{N}_0}$ by

$$S^*(0) := 0, \quad T(0) := 1$$

and, for $n \in \mathbb{N}$,

$$S^*(n+1) := \begin{cases} S^*(n) + \xi_{n+1-T(n)}, & S^*(n) > 0, \\ \eta_{T(n)}, & S^*(n) \leq 0 \end{cases}$$

and

$$T(n+1) := 1 + \#\{1 \leq k \leq n : S^*(k) \leq 0\}.$$

The sequences S^* and $\tilde{S}$ have the same distribution. In view of this Theorem 1.1 follows if we can prove a counterpart of (1.3):

$$\Big(\frac{S^*(\lfloor vt\rfloor)}{\sigma v^{1/2}}\Big)_{t\geq 0} \;\Rightarrow\; (W_\alpha(t))_{t\geq 0}, \quad v \to \infty. \tag{2.4}$$

Put $\Theta_1 = 0$,

$$\Theta_i := \inf\{l > \Theta_{i-1} : S^*(l) \leq 0\}, \quad i \geq 2 \tag{2.5}$$

and then, for $i \in \mathbb{N}$,

$$\begin{aligned} \gamma_i &:= -S^*(\Theta_i), \\ \zeta_i &:= S^*(\Theta_i + 1) + \gamma_i = \eta_i + \gamma_i. \end{aligned} \tag{2.6}$$

The random variable Θ_i is the time of the i-th visit of S^* to $(-\infty, 0]$. The random variable γ_i is the *$(i-1)$-th overshoot of S^* into* $(-\infty, 0]$. Since $\gamma_i \geq 0$ a.s., we conclude that $\zeta_i \geq \eta_i$ a.s. and that $\zeta_i = \eta_i$ a.s. if and only if $\gamma_i = 0$ a.s. This simpler situation of zero overshoot into $(-\infty, 0]$ occurs in the setting of [16], where ξ is an integer-valued random variable with $\xi \geq -1$ a.s., and η is a positive integer-valued random variable. Observe that

$$S^*(n) = S_\xi(n - T(n)) + S_\zeta(T(n)), \quad n \in \mathbb{N}_0. \tag{2.7}$$

For $n \in \mathbb{N}_0$, put

$$m(n) := -\min_{0 \leq k \leq n} S_\xi(k)$$

and

$$R(n) := S_\xi(n) + S_\zeta \circ \nu_\zeta \circ m(n). \tag{2.8}$$

Now we explain how the rest of the proof is organized. We start by proving in Lemma 2.1 that the sequence $R := (R(n))_{n \in \mathbb{N}_0}$ can be obtained from S^* by a time-change. Then we show in Lemma 2.2 that the sum of the first n overshoots into $(-\infty, 0]$ is negligible in comparison to $S_\eta(n)$ as $n \to \infty$. Lemma 2.3 states that relation (2.4) holds with R replacing S^*. Finally, Lemma 2.5 makes it clear that the time-change defined in Lemma 2.1 is close to the identity mapping. Combining all these auxiliary results we arrive at (2.4).

2.2. Convergence of the time-changed version of S^*

In this section we show that the scaling limit of R is W_α. Put

$$\lambda(n) := \inf\{k \in \mathbb{N} : k - T(k) \geq n,\ S^*(k) > 0\}, \quad n \in \mathbb{N}_0$$

and note that, for $n \in \mathbb{N}$, $\lambda(n) < \infty$ a.s. because $\zeta > 0$ a.s.

Lemma 2.1. *With probability* 1

$$S^*(\lambda(n)) = R(n), \quad n \in \mathbb{N}_0, \tag{2.9}$$

that is, the sequence R is obtained from S^* by the time-change.

Proof. The definition of λ ensures that

$$\lambda(n) - T(\lambda(n)) = n, \quad n \in \mathbb{N}_0, \tag{2.10}$$

whence

$$S_\xi(k - T(k))|_{k = \lambda(n)} = S_\xi(n), \quad n \in \mathbb{N}_0. \tag{2.11}$$

Thus, it remains to check that

$$T \circ \lambda(n) = \nu_\zeta \circ m(n), \quad n \in \mathbb{N}_0 \quad \text{a.s.} \tag{2.12}$$

Fix $n \in \mathbb{N}$ and consider the events $A_n := \{S^*(\lambda(n) - 1) \le 0\}$ and $A_n^c = \{S^*(\lambda(n) - 1) > 0\}$. We shall show that
(I) on A_n: (2.12) holds;
(II) on A_n^c: $T \circ \lambda(n) - T \circ \lambda(n-1) = 0$;
(III) on A_n^c: $\nu_\zeta \circ m(n) \le T \circ \lambda(n)$.

These will guarantee that (2.12) holds by an induction in n. Indeed, $T \circ \lambda(0) = \nu_\zeta \circ m(0) = 1$, that is, (2.12) holds for $n = 0$. Assume that (2.12) holds for $n = k - 1$. The validity of (2.12) a.s. on A_k follows directly from (I). To prove that (2.12) holds true a.s. on A_k^c, use (II), (III) and the induction assumption. These yield

$$\nu_\zeta \circ m(k) \le T \circ \lambda(k) = T \circ \lambda(k-1) = \nu_\zeta \circ m(k-1),$$

and the claim follows, for $\nu_\zeta \circ m$ is a.s. nondecreasing.

Before going further we state as the claims two properties of the model.

Claim 2.1.1. *For $k \in \mathbb{N}_0$,*

$$\{T(k+1) > T(k)\} = \{S^*(k) \le 0\}.$$

This is obvious, no proof is needed.

Claim 2.1.2. *For $k \in \mathbb{N}$ such that $S^*(k) \le 0$, put*

$$\begin{aligned} g(k) &:= \inf\{l \in [2,k] : S^*(l) \le 0,\ S^*(l-1) > 0\}, \\ d(k) &:= \inf\{r \ge k : \quad S^*(r) \le 0,\ S^*(r+1) > 0\}. \end{aligned}$$

Then, for $k \in \mathbb{N}$,

$$\Big\{S^*(k) \le 0\Big\} \subset \Big\{-S_\xi(i - T(i)) = m(i - T(i)) \text{ for } i \in [g(k), d(k)+1]\Big\}. \tag{2.13}$$

Proof. Put $l = g(k)$ and $r = d(k)$. As $S^*(l-1) > 0$ it follows that $T(l) = T(l-1)$. Using representation (2.7) we infer

$$S_\xi(l - T(l)) \le -S_\zeta(T(l)) \quad \text{and} \quad S_\xi(l - 1 - T(l)) > -S_\zeta(T(l)).$$

Since S_η is a.s. nondecreasing, this implies that the minimum of S_ξ on the interval $[0, l - T(l)]$ is achieved at $l - T(l)$. Finally, observe that the function $i \mapsto i - T(i)$ is constant on $[l, r+1]$, because $T(i+1) = T(i) + 1$ for $i \in [l, r]$. ■

With the claims at hand we now prove (I), (II) and (III).

Proof of (I). Fix $\omega \in A_n$ and consider the number of elements of the sequence $(\eta_k)_{k \in \mathbb{N}}$ used in the construction of $(S^*(j))_{1 \le j \le \lambda(n)}$. In view of (2.7) this number is $T \circ \lambda(n)$. Further, note that $S^* \circ \lambda(n) > 0$, that is, in the notation of Claim 2.1.2,

$$d(\lambda(n) - 1) = \lambda(n) - 1.$$

Recalling (2.10) and using Claim 2.1.2 with $k = \lambda(n) - 1$ we conclude that

$$-S_\xi(\lambda(n) - T(\lambda(n))) = m(\lambda(n) - T(\lambda(n))) = m(n).$$

Since $S^*(\lambda(n)) > 0$ and $S^*(\lambda(n) - 1) \le 0$, we infer

$$S_\zeta \circ T(\lambda(n)) > -S_\xi(\lambda(n) - T(\lambda(n))) = m(n),$$
$$S_\zeta \circ T(\lambda(n) - 1) \le -S_\xi(\lambda(n) - 1 - T(\lambda(n) - 1)) = m(n).$$

This implies that the number of elements of the sequence $(\zeta_k)_{k\in\mathbb{N}}$ used in the construction of $(S^*(j))_{1\le j\le\lambda(n)}$ is equal to $\nu_\zeta \circ m(n)$, because it is the minimal number which makes S_ζ greater than $m(n)$.
PROOF OF (II). Fix $\omega \in A_n^c$. It follows from Claim 2.1.1 that $\lambda(n-1) = \lambda(n) - 1$. Also, Claim 2.1.1 guarantees that $T(\lambda(n) - 1) = T \circ \lambda(n)$. Hence,

$$T \circ \lambda(n-1) = T(\lambda(n) - 1) = T \circ \lambda(n).$$

PROOF OF (III). The subsequent argument works for both A_n^c and A_n. Since $S^*(\lambda(n)) > 0$ a.s. and according to (2.7),

$$S_\zeta \circ T \circ \lambda(n) = S^* \circ \lambda(n) - S_\xi(\lambda(n) - T \circ \lambda(n))$$
$$> m(\lambda(n) - T \circ \lambda(n)) = m(n),$$

applying ν_ζ to both sides of the last inequality yields

$$T \circ \lambda(n) \ge \nu_\zeta \circ m(n).$$

The proof of Lemma 2.1 is complete. ■

Recall that

$$\zeta_i = \eta_i + \text{'the } (i-1)\text{th overshoot into } (-\infty, 0]', \quad i \in \mathbb{N}.$$

The next result shows that the standard random walks S_η and S_ζ behave similarly which particularly means that the contribution of the sum of the overshoots is negligible in comparison to S_η.

Lemma 2.2. *Under the assumptions of Theorem 1.1,*

$$\frac{S_\zeta(n)}{S_\eta(n)} \overset{\mathbb{P}}{\to} 1, \ n \to \infty. \tag{2.14}$$

Proof. Put $\tau_0 := 0$ and, for $i \in \mathbb{N}$,

$$\tau_{i+1} := \inf\{k > \tau_i \ : \ S_\xi(k) < S_\xi(\tau_i)\} \quad \text{and} \quad \chi_i := S_\xi(\tau_{i-1}) - S_\xi(\tau_i).$$

The elements of the sequences $(\tau_i)_{i\in\mathbb{N}}$ and $(\chi_i)_{i\in\mathbb{N}}$ are called *descending ladder epochs* and *descending ladder heights* of S_ξ, respectively. By construction $\chi > 0$ a.s. Further, the assumptions $\mathbb{E}\xi = 0$ and $\operatorname{Var}\xi < \infty$ entail $\mu := \mathbb{E}\chi < \infty$, see, for instance, formula (4b) in [7]. Recall $(\gamma_i)_{i\in\mathbb{N}}$ from (2.6) and note that (except for γ_1 which is 0 a.s.) these are independent copies of a random variable γ with

$$\gamma \overset{\mathrm{d}}{=} S_\chi(\nu_\chi(\eta)) - \eta.$$

We start by showing that

$$\lim_{x\to\infty} \frac{\mathbb{P}\{\gamma > x\}}{\mathbb{P}\{\eta > x\}} = 0. \tag{2.15}$$

Denote by F_χ the distribution function of χ and U_χ the renewal function for $(S_\chi(n))_{n\in\mathbb{N}_0}$, that is, $U_\chi(x) := \sum_{n\geq 0} \mathbb{P}\{S_\chi(n) \leq x\}$ for $x \in \mathbb{R}$. Then

$$\mathbb{P}\{S_\chi(\nu_\chi(z)) - z > x\} = \int_{[0,\,z]} \big(1 - F_\chi(z + x - y)\big)\,\mathrm{d}U_\chi(y), \quad z, x \geq 0.$$

Further, for any $A > 0$,

$$\begin{aligned} \mathbb{P}\{\gamma > x\} &= \mathbb{E}\int_{[0,\,\eta]} \big(1 - F_\chi(\eta + x - y)\big)\,\mathrm{d}U_\chi(y) \\ &= \mathbb{E}\int_{[0,\,\eta]} (1 - F_\chi(\eta + x - y))\mathrm{d}U_\chi(y)(\mathbb{1}_{\{\eta\leq Ax\}} + \mathbb{1}_{\{\eta > Ax\}}) \\ &\leq (1 - F_\chi(x))\mathbb{E}U_\chi(\eta)\,\mathbb{1}_{\{\eta\leq Ax\}} + \mathbb{P}\{\eta > Ax\} \end{aligned} \tag{2.16}$$

having utilized monotonicity of F_χ for the inequality. By the elementary renewal theorem, $\lim_{x\to\infty} x^{-1}U_\chi(x) = \mu^{-1}$ and thereupon

$$\frac{\mathbb{E}U_\chi(\eta)\,\mathbb{1}_{\{\eta\leq Ax\}}}{\mathbb{P}\{\eta > x\}} \sim \frac{\mathbb{E}\eta\,\mathbb{1}_{\{\eta\leq Ax\}}}{\mu\mathbb{P}\{\eta > x\}}, \quad x \to \infty.$$

Recalling (1.2) and invoking Karamata's theorem (Theorem 1.6.4 in [4]) we infer

$$\frac{\mathbb{E}\eta\,\mathbb{1}_{\{\eta\leq Ax\}}}{\mu\mathbb{P}\{\eta > x\}} \sim \frac{\alpha A^{1-\alpha}}{(1-\alpha)\mu} x, \quad x \to \infty.$$

This yields

$$\lim_{x\to\infty} \frac{(1 - F_\chi(x))\mathbb{E}U_\chi(\eta)\,\mathbb{1}_{\{\eta\leq Ax\}}}{\mathbb{P}\{\eta > x\}} = 0, \tag{2.17}$$

because $\mathbb{E}\chi < \infty$ entails $\lim_{x\to\infty}(1 - F_\chi(x))x = 0$.

It follows from (2.16) and (2.17) that, for any $A > 0$,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\gamma > x\}}{\mathbb{P}\{\eta > x\}} \leq \lim_{x\to\infty} \frac{\mathbb{P}\{\eta > Ax\}}{\mathbb{P}\{\eta > x\}} = A^{-\alpha}.$$

Since $A > 0$ is arbitrary, we arrive at (2.15). Thus, we have proved that given $\varepsilon > 0$ there exists $x_0 > 0$ such that

$$\mathbb{P}\{\gamma > x\} \leq \varepsilon\mathbb{P}\{\eta > x\}$$

whenever $x \geq x_0$. Let $\hat{\eta}$ be a random variable with distribution

$$\mathbb{P}\{\hat{\eta} > x\} = \begin{cases} 1, & x < x_0, \\ \varepsilon\mathbb{P}\{\eta > x\}, & x \geq x_0. \end{cases}$$

Then $\mathbb{P}\{\gamma > x\} \le \mathbb{P}\{\hat{\eta} > x\}$ for $x \ge 0$ and, as a consequence, for each $n \in \mathbb{N}$,

$$\mathbb{P}\{\gamma_1 + \ldots + \gamma_n > x\} \le \mathbb{P}\{\hat{\eta}_1 + \ldots + \hat{\eta}_n > x\}, \quad x \ge 0, \tag{2.18}$$

where $\hat{\eta}_1, \hat{\eta}_2, \ldots$ are independent copies of $\hat{\eta}$. Since $\mathbb{P}\{\hat{\eta} > x\} \sim \varepsilon x^{-\alpha}\ell(x)$ as $x \to \infty$, we conclude that a counterpart of (2.1) holds for $(S_{\tilde{\eta}}(n))_{n \in \mathbb{N}_0}$. Its specialization to $v = 1$ reads

$$\frac{S_{\tilde{\eta}}(n)}{a_n} \stackrel{\mathrm{d}}{\to} \varepsilon^{1/\alpha} U_\alpha(1), \quad n \to \infty. \tag{2.19}$$

Since $\varepsilon > 0$ is arbitrary, we deduce from (2.18), (2.19) and one-dimensional version of (2.1) that

$$\frac{S_\gamma(n)}{S_\eta(n)} \stackrel{\mathbb{P}}{\to} 0, \quad n \to \infty.$$

This completes the proof of Lemma 2.2. ■

Lemma 2.3. *Under the assumptions of Theorem 1.1,*

$$\Big(\frac{S_\xi(\lfloor vt \rfloor)}{\sigma v^{1/2}}, \frac{R(\lfloor vt \rfloor)}{\sigma v^{1/2}}\Big)_{t \ge 0} \Rightarrow (W(t), W_\alpha(t))_{t \ge 0}, \quad v \to \infty. \tag{2.20}$$

Proof. Recall the definition of ζ from (2.6) and note that the process $(S_\zeta(\lfloor t \rfloor) - S_\eta(\lfloor t \rfloor))_{t \ge 0}$ is a.s. nondecreasing. Hence, for all $T > 0$,

$$\sup_{t \in [0,T]} \Big| \frac{S_\zeta(\lfloor vt \rfloor)}{a(v)} - \frac{S_\eta(\lfloor vt \rfloor)}{a(v)} \Big| \le \Big| \frac{S_\zeta(\lfloor vt \rfloor)}{S_\eta(\lfloor vT \rfloor)} - 1 \Big| \cdot \frac{S_\eta(\lfloor vT \rfloor)}{a(v)} \stackrel{\mathbb{P}}{\to} 0, \quad v \to \infty, \tag{2.21}$$

where the limit relation is a consequence of Lemma 2.2 and one-dimensional version of (2.1). This together with (2.3) yields

$$\Big(\frac{S_\xi(\lfloor vt \rfloor)}{\sigma v^{1/2}}, \frac{-\min_{s \in [0,t]} S_\xi(\lfloor vs \rfloor)}{\sigma v^{1/2}}, \frac{S_\zeta(\lfloor vt \rfloor)}{a(v)}\Big)_{t \ge 0} \Rightarrow (W(t), M(t), U_\alpha(t))_{t \ge 0}, \quad v \to \infty$$

in the product J_1-topology. An appeal to Theorem 3.6 in [19] enables us to conclude that, as $v \to \infty$,

$$\Big(\frac{S_\xi(\lfloor vt \rfloor)}{\sigma v^{1/2}}, \frac{-\min_{s \in [0,t]} S_\xi(\lfloor vs \rfloor)}{\sigma v^{1/2}}, \frac{S_\zeta(\nu_\zeta(\sigma v^{1/2} t))}{\sigma v^{1/2}}\Big)_{t \ge 0} \Rightarrow (W(t), M(t), U_\alpha \circ U_\alpha^{\leftarrow}(t))_{t \ge 0}$$

in the J_1-topology.

Our next step is to prove that

$$\Big(\frac{S_\xi(\lfloor vt \rfloor)}{\sigma v^{1/2}}, \frac{S_\zeta(\nu_\zeta(-\min_{s \in [0,t]} S_\xi(\lfloor vs \rfloor)))}{\sigma v^{1/2}}\Big)_{t \ge 0} \Rightarrow (W(t), U_\alpha \circ U_\alpha^{\leftarrow} \circ M(t))_{t \ge 0}, \quad v \to \infty \tag{2.22}$$

in the product J_1-topology. To this end, we intend to invoke Lemma 2.4 given next. Being of principal importance for the proof of Lemma 2.3, Lemma 2.4 should also be useful as far as other problems involving compositions are concerned, not necessarily related to the setting of the present paper. The proof of this lemma is postponed to the Appendix.

For $f \in D$, denote by $\mathrm{Disc}(f) := \{a : \ f(a-) \ne f(a)\}$ the set of discontinuities of f.

Lemma 2.4. *For $n \in \mathbb{N}_0$, let $x_n,\ y_n \in D$, y_n be nondecreasing and y_0 continuous. Assume that* $\lim_{n\to\infty} x_n = x_0$ *and* $\lim_{n\to\infty} y_n = y_0$ *in the J_1-topology in D and that if, for some $t \geq 0$, $y_0(t) \in$* $\mathrm{Disc}(x_0)$*, then $\#\{u \geq 0 : y_0(u) = y_0(t)\} = 1$. Then*

$$\lim_{n\to\infty} x_n \circ y_n = x_0 \circ y_0 \tag{2.23}$$

in the J_1-topology in D.

It is known, see, for instance, Lemma 11.17 in [18], that, for any fixed $a \geq 0$,

$$\mathbb{P}\big\{\#\{u \geq 0 : M(u) = a\} = 1\big\} = 1. \tag{2.24}$$

Since $U_\alpha \circ U_\alpha^{\leftarrow}$ and M are independent processes, and the set $\mathrm{Disc}(U_\alpha \circ U_\alpha^{\leftarrow})$ of discontinuities of $U_\alpha \circ U_\alpha^{\leftarrow}$ is a.s. countable, we conclude with the help of (2.24) that

$$\mathbb{P}\Big\{\#\{u : M(u) = a\} = 1 \text{ for } a \in \mathrm{Disc}(U_\alpha \circ U_\alpha^{\leftarrow})\Big\} = 1. \tag{2.25}$$

Finally, we note that, for each $v \geq 0$, the process $t \mapsto -\min_{s\in[0,\,t]} S_\xi(\lfloor vs \rfloor)$ is a.s. nondecreasing, and the process M is a.s. nondecreasing and continuous. Thus, we have checked that Lemma 2.4 applies to the processes discussed above or rather their versions whose existence is secured by Skorokhod's representation theorem. As a result, we obtain (2.22) and thereupon (2.20) because the summation operation (with two summands) is continuous whenever one of the summands is a continuous function, see, for instance, Theorem 4.1 in [20]. ■

2.3. Convergence of the scaled S^*

In Lemma 2.5 given next we do not assume that the sequences $(\xi_i)_{i\in\mathbb{N}}$ and $(\eta_j)_{j\in\mathbb{N}}$ are independent, nor that the distribution of η belongs to the domain of attraction of a stable distribution.

Lemma 2.5. *Assume that $\mathbb{E}|\xi| < \infty$ and $\mathbb{E}\eta = \infty$. Then, for all $T > 0$,*

$$\lim_{v\to\infty} \sup_{t\in[0,\,T]} \Big| \frac{\lambda(\lfloor vt \rfloor)}{v} - t \Big| = 0 \quad \text{a.s.} \tag{2.26}$$

Proof. The sequence $(\lambda(n))_{n\in\mathbb{N}_0}$ is a.s. nondecreasing and the limit candidate in (2.26), the identity function, is continuous. Hence, it suffices to prove that

$$\lim_{n\to\infty} \frac{\lambda(n)}{n} = 1 \quad \text{a.s.}$$

It follows from (2.10) that

$$\frac{\lambda(n)}{n} = \left(1 - \frac{T(\lambda(n))}{\lambda(n)}\right)^{-1}.$$

As a consequence, it is enough to check that

$$\lim_{n\to\infty} \frac{T(n)}{n} = 0 \quad \text{a.s.} \tag{2.27}$$

Observe that, for $\delta > 0$,

$$\begin{aligned}\Big\{T(n) \le n\delta\Big\} &\supset \Big\{|\xi_1| + \cdots + |\xi_{n-n\delta}| + \zeta_1 + \cdots + \zeta_{[\delta n]} > 0\Big\} \\ &\supset \Big\{-|\xi_1| - \cdots - |\xi_n| + \zeta_1 + \cdots + \zeta_{[\delta n]} > 0\Big\} \\ &\supset \Big\{-|\xi_1| - \cdots - |\xi_n| + \eta_1 + \cdots + \eta_{[\delta n]} > 0\Big\}.\end{aligned} \tag{2.28}$$

Thus,

$$\begin{aligned}\Big\{\lim_{n\to\infty} \frac{T(n)}{n} = 0\Big\} &= \Big\{\forall \delta > 0\ \exists n_0\ \forall n \ge n_0\ :\quad T(n) \le n\delta\Big\} \\ &\supset \Big\{\forall \delta > 0\ \exists n_0\ \forall n \ge n_0\ :\quad -|\xi_1| - \cdots - |\xi_n| + \eta_1 + \cdots + \eta_{[\delta n]} > 0\Big\} \\ &\supset \Big\{\forall \delta > 0\quad \lim_{n\to\infty} \frac{\eta_1 + \cdots + \eta_{[\delta n]}}{|\xi_1| + \cdots + |\xi_n|} = +\infty\Big\}.\end{aligned} \tag{2.29}$$

Since $\mathbb{E}|\xi| < \infty$ and $\mathbb{E}\eta = \infty$, the probability of the event on the right hand side of (2.29) is equal to 1 by the strong law of large numbers. ∎

Recall that

$$R(vt) = S^*(\lambda(vt)) = S^*\Big(\frac{\lambda(vt)}{v} v\Big), \quad t \ge 0,\ v > 0.$$

The time-change $t \mapsto v^{-1}\lambda(vt)$ is discontinuous and nondecreasing (rather than strictly increasing). Hence, negligibility of the distance in D between $(v^{-1/2}S^*(\lfloor vt \rfloor))_{t\ge 0}$ and $(v^{-1/2}R(vt))_{t\ge 0}$ as $v \to \infty$ cannot be deduced from the definition of the J_1-topology. Lemma 2.6 is designed to deal with this technicality. Its proof is deferred to the Appendix.

Lemma 2.6. *For $n \in \mathbb{N}_0$, let λ_n, $f_n \in D$, λ_n be nonnegative and nondecreasing. Assume that, for all $T > 0$,*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\lambda_n(t) - t| = 0 \tag{2.30}$$

and

$$\lim_{n\to\infty} f_n \circ \lambda_n = f_0 \tag{2.31}$$

in the J_1-topology in D. For $n \in \mathbb{N}$, denote by $(t_k^{(n)})_{k\in\mathbb{N}}$ elements of the set $\mathrm{Disc}(\lambda_n)$ and, for $k \in \mathbb{N}$, put $u_k^{(n)} := \lambda_n(t_k^{(n)}-)$ and $v_k^{(n)} := \lambda_n(t_k^{(n)})$. If, in addition to (2.30) and (2.31), for all $T > 0$,

$$\lim_{n\to\infty} \sup_{k\ge 1} \sup_{s\in[u_k^{(n)}, v_k^{(n)})\cap[0,T]} |f_n(s) - f_n(u_k^{(n)}-)| = 0, \tag{2.32}$$

then

$$\lim_{n\to\infty} f_n = f_0$$

in the J_1-topology in D.

For later needs, put

$$Y_n(t) := \frac{\max_{1\le i\le \lfloor nt\rfloor} |\xi_i|}{n^{1/2}}, \quad n\in\mathbb{N},\ t\ge 0.$$

The assumption $\mathbb{E}\xi^2<\infty$ ensures that

$$Y_n \ \Rightarrow\ \mathbf{0}, \quad n\to\infty \tag{2.33}$$

in D, where $\mathbf{0}(t):=0$ for $t\ge 0$.

According to the Skorokhod representation theorem in conjunction with Lemmas 2.3 and 2.5, for any sequence $(v_n)_{n\in\mathbb{N}}$ which diverges to $+\infty$ as $n\to\infty$, there exists a probability space which accommodates random elements $((S_\xi^{(n)}, R^{(n)}, \lambda^{(n)}, \hat{Y}_n))_{n\in\mathbb{N}}$ satisfying, for $n\in\mathbb{N}$,

$$\left(\frac{S_\xi^{(n)}(v_nt)}{v_n^{1/2}}, \frac{R^{(n)}(v_nt)}{v_n^{1/2}}, \frac{\lambda^{(n)}(v_nt)}{v_n}, \hat{Y}_n\right)_{t\ge 0} \overset{\mathrm{d}}{=} \left(\frac{S_\xi(v_nt)}{v_n^{1/2}}, \frac{R(v_nt)}{v_n^{1/2}}, \frac{\lambda(v_nt)}{v_n}, Y_n\right)_{t\ge 0}$$

and

$$\lim_{n\to\infty}\left(\frac{S_\xi^{(n)}(v_nt)}{\sigma v_n^{1/2}}, \frac{R^{(n)}(v_nt)}{\sigma v_n^{1/2}}, \frac{\lambda^{(n)}(v_nt)}{v_n}, \hat{Y}_n\right) = \left(\hat{W}(t), \hat{W}_\alpha(t), t, 0\right) \quad \text{a.s. in } D, \tag{2.34}$$

where $(\hat{W}, \hat{W}_\alpha)$ is a copy of the process (W, W_α). Fix any ω such that (2.34) holds. The aforementioned new probability space also accommodates copies $(S^{*(n)})_{n\in\mathbb{N}}$ of S^*. Representation (2.9) from Lemma 2.1 holds for these copies. We are going to apply Lemma 2.6 with

$$f_n(t) := \frac{R^{(n)}(v_nt)}{\sigma v_n^{1/2}} \text{ and } \lambda_n(t) := \frac{\lambda^{(n)}(v_nt)}{v_n}, \quad n\in\mathbb{N},\ t\ge 0.$$

For this particular choice, condition (2.32) is justified by the convergence of the fourth coordinate in (2.34) and the fact that the supremum in (2.32) does not exceed $\hat{Y}_n(T)$. By Lemma 2.6,

$$\lim_{n\to\infty}\frac{S^{*(n)}(v_nt)}{\sigma v_n^{1/2}} = \hat{W}_\alpha(t) \quad \text{in } D$$

for the chosen ω and thereupon a.s. This completes the proof of Theorem 1.1 in the case $x_0=0$.

Assume now that $x_0>0$. Then the excursions $(S^*(n))_{1\le n\le\Theta_1}, (S^*(n))_{\Theta_1+1\le n\le\Theta_2},\dots$ are still independent. The only minor complication is that the distribution of the first excursion $(S^*(n))_{1\le n\le\Theta_1}$ is different from the distribution of all the other excursions $(S^*(n))_{\Theta_1+1\le n\le\Theta_2}, (S^*(n))_{\Theta_2+1\le n\le\Theta_3},\dots$ As $v\to\infty$, the weak limit for $(v^{-1/2}S^*(\lfloor vt\rfloor))_{t\ge 0}$ is the same as that for $(v^{-1/2}S^*(\lfloor vt\rfloor))_{t\ge(\Theta_1+1)/v}$, hence the proof given for the case $x_0=0$ applies, after renumbering the excursions. The proof of Theorem 1.1 is now complete.

3. Proof of Theorem 1.3

PROOF FOR $\hat{S}$. By the definitions of $\hat{S}$ and $\tilde{S}$,

$$\max_{0\le k\le n}|\hat{S}(k)-\tilde{S}(k)| \le \max_{0\le k\le n}|\xi_k|, \quad n\in\mathbb{N} \quad \text{a.s.}$$

In view of (2.33), the result follows from Theorem 1.1 and Slutsky's lemma.
PROOF FOR $\grave{S}$. We shall use the notation similar to that introduced in the proof of Theorem 1.1. For $n \in \mathbb{N}_0$, put

$$\grave{T}(n) := \#\{k \le n : \grave{S}(k) \le 0\}, \qquad \grave{\lambda}(n) := \inf\{k \in \mathbb{N} : k - \grave{T}(k) \ge n,\ \grave{S}(k) > 0\},$$

and

$$\grave{R}(n) := S_\xi(n) + S_\eta \circ \nu_\eta \circ m(n).$$

Note that, unlike R, the definition of $\grave{R}$ includes η rather than ζ. Lemmas 2.1 and 2.3 remain true with $\grave{S}$, $\grave{R}$ and $\grave{T}$ replacing $\tilde{S}$, R and T, respectively. Lemma 2.5, with $\grave{\lambda}$ instead of λ, follows by a similar reasoning. We only note here that the middle part of (2.28) is obsolete, for there are no ζ's in this model. The final piece of the proof of Theorem 1.1 applies in the present setting, upon replacing S, R, T and λ with $\grave{S}$, $\grave{R}$, $\grave{T}$ and $\grave{\lambda}$, respectively.

***Remark* 3.1.** The only major difference between $\grave{S}$ and S^* is that S^* stays below 0 for shorter time periods than $\grave{S}$. But the proof for $\grave{S}$ is not affected by this fact. Also, observe that when analysing $\grave{S}$ we do not have to control the overshoots into $(-\infty, 0]$.

4. Properties of the limit process

In this section we discuss several properties of the limit process $W_\alpha^{(x)} = (W_\alpha(x,t))_{t\ge 0}$ arising in Theorem 1.2 such as self-similarity, properties of excursions and a Markov property. We explain that $W_\alpha^{(x)}$ admits a representation as the solution to a stochastic equation with reflection. Alternatively, it can be thought of as a Feller Brownian motion on $[0,\infty)$ with a 'jump-type' exit from 0.

Let $a > 0$. We start by noting that the distribution tail of $a\eta$ satisfies a counterpart of (1.2), with ℓ replaced by $a^{-\alpha}\ell$. Since the slowly varying function ℓ from (1.2) does not pop up in the limit process $W_\alpha^{(x)}$, limit relations (1.3) and (1.5) remain valid upon replacing $(\eta_n)_{n\in\mathbb{N}}$ with $(a\eta_n)_{n\in\mathbb{N}}$. Further, observe that the distribution of $W_\alpha^{(x)}$ does not change when replacing in (1.6) the process U_α with any other drift-free α-stable subordinator (without killing) V_α, say. Indeed, the distribution of V_α coincides with the distribution of $(U_\alpha(ct))_{t\ge 0}$ for some $c > 0$. An inverse α-stable subordinator $V_\alpha^{\leftarrow}$ has the same distribution as $(c^{-1}U_\alpha^{\leftarrow}(t))_{t\ge 0}$. Finally, the composition of $(U_\alpha(ct))_{t\ge 0}$ and $(c^{-1}U_\alpha^{\leftarrow}(t))_{t\ge 0}$ is $(U_\alpha \circ U_\alpha^{\leftarrow}(t))_{t\ge 0}$ (for all ω).

For all $x \ge 0$, the processes $W_\alpha^{(x)}$ are homogeneous Markov processes. Furthermore, these are Feller processes, see Theorem 3.11 in Chapter II of [5] and for $x \ne y$, $W_\alpha^{(x)}$ and $W_\alpha^{(y)}$ have the same transition probabilities.

It follows from the definition that $W_\alpha^{(x)}$ behaves like the Brownian motion W until it hits 0. Thus, $W_\alpha^{(x)}$ is a Feller Brownian motion, that is, a Markov extension of a Brownian motion after hitting 0.

***Remark* 4.1.** In the theory of Markov processes one usually considers a process Y, say under the collection of measures $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot|Y(0) = x)$ for $x \ge 0$. For our needs it is more convenient to work with the collection of processes $W_\alpha^{(x)}$, indexed by the initial starting point $x \ge 0$, under a single probability measure $\mathbb{P}$. We hope this does not lead to a confusion.

By Theorem 1.1,

$$\begin{array}{ccc} (v^{-1/2}\tilde{S}(vct))_{t\geq 0} & \Rightarrow & (W_\alpha(ct))_{t\geq 0} \\ \| & & \\ (c^{1/2}(vc)^{-1/2}\tilde{S}(vct))_{t\geq 0} & \Rightarrow & (c^{1/2}W_\alpha(t))_{t\geq 0} \end{array}$$

that is, the process W_α is self-similar with exponent $1/2$. Using this in combination with $1/2$ self-similarity of a Brownian motion started at x and stopped upon hitting 0 and the strong Markov property of $W_\alpha^{(x)}$ we conclude that, for any $c>0$ and $x\geq 0$, the process $(W_\alpha^{(x)}(ct))_{t\geq 0}$ has the same distribution as $\left(c^{1/2}W_\alpha^{(c^{-1/2}x)}(t)\right)_{t\geq 0}$.

Now we describe the process $W_\alpha^{(x)}$ from the resolvent point of view. To this end, define the resolvent

$$U^\lambda f(x) := \mathbb{E}\int_0^\infty e^{-\lambda s} f(W_\alpha^{(x)}(s))\mathrm{d}s, \quad x\geq 0.$$

Denote by $V^\lambda f(x)$ the resolvent of a Brownian motion on $[0,\infty)$ killed at 0. It is known that

$$V^\lambda f(x) = \int_0^\infty v^\lambda(x,y) f(y)\mathrm{d}y, \quad x>0,$$

where $v^\lambda(x,y) := \frac{1}{\sqrt{2\lambda}}(e^{-\sqrt{2\lambda}|x-y|} - e^{-\sqrt{2\lambda}|x+y|})$ for $x,y>0$, see p. 56 in [5]. Invoking the general theory of Markov processes one can show that

$$U^\lambda f(x) = V^\lambda f(x) + \mathbb{E}_x e^{-\lambda\sigma_0} U^\lambda f(0) = V^\lambda f(x) + e^{-x\sqrt{2\lambda}} U^\lambda f(0), \quad x>0,$$

where σ_0 is the first hitting time of 0, see p. 57 in [5]. Note that this formula holds true for any Markov extension of a Brownian motion after hitting 0. It follows from Theorem 3.11 in Chapter II of [5] that

$$U^\lambda f(0) = \Delta_\lambda^{-1}\int_0^\infty V^\lambda f(x)\frac{\mathrm{d}x}{x^{1+\alpha}}, \tag{4.1}$$

where

$$\Delta_\lambda = \int_0^\infty (1-e^{-x\sqrt{2\lambda}})\frac{\mathrm{d}x}{x^{1+\alpha}} = \frac{(2\lambda)^{\alpha/2}\Gamma(1-\alpha)}{\alpha}.$$

The last equality is obtained with the help of integration by parts.

***Remark* 4.2.** The book [5] only focused on the case $\lambda=1$. However, the case $\lambda\neq 1$ is analogous. Note that the value of the norming constant Δ_λ can be derived from the equality $U^\lambda 1(x) = \lambda^{-1}$, $x\geq 0$.

***Remark* 4.3.** Equation (4.1) entails that the entrance law for $W_\alpha^{(x)}$ is given by

$$\frac{\alpha}{2^{\alpha/2}\Gamma(1-\alpha)}\int_0^\infty P_t^0(x,\mathrm{d}y)\frac{\mathrm{d}x}{x^{1+\alpha}},$$

see Chapter V, §2 in [5], where

$$P_t^0(x,\mathrm{d}y)=(2\pi t)^{-1/2}(e^{-\frac{(x-y)^2}{2t}}-e^{-\frac{(x+y)^2}{2t}})\mathrm{d}y$$

is the transition kernel of the semigroup for a Brownian motion killed at 0.

Summarizing, we conclude that the resolvent kernel $r^\lambda(x,y)$ of W_α is given by

$$r^\lambda(x,y)=v^\lambda(x,y)+\Delta_\lambda^{-1}\int_0^\infty v^\lambda(z,y)\frac{\mathrm{d}z}{z^{1+\alpha}},\quad x,y>0.$$

Now we are going to point out the distributions of $(W(t),-\min_{s\in[0,\,t]}W(s))$ and $U_\alpha\circ U_\alpha^{\leftarrow}(t)$. According to Problem 1 on p. 27 in [10],

$$\mathbb{P}\{W(t)\in\mathrm{d}a,\ \max_{s\in[0,\,t]}W(s)\in\mathrm{d}b\}=\left(\frac{2}{\pi t^3}\right)^{\frac12}(2b-a)e^{(2b-a)^2/2t}\mathrm{d}a\mathrm{d}b,\quad t>0,\ 0\le b,\ b\ge a.$$

As a consequence,

$$\mathbb{P}\{W(t)\in\mathrm{d}a,\ -\min_{s\in[0,\,t]}W(s)\in\mathrm{d}b\}=\left(\frac{2}{\pi t^3}\right)^{\frac12}(2b+a)e^{(2b+a)^2/2t}\mathrm{d}a\mathrm{d}b,\quad t>0,\ 0\le b,\ b+a\ge 0.$$

Notice that $U_\alpha(U_\alpha^{\leftarrow}(t))-t$ is the overshoot of the process U_α at $t>0$. It follows from the Dynkin-Lamperti asymptotics (see, for instance, p. 135 in [12]) and self-similarity of $U_\alpha\circ U_\alpha^{\leftarrow}$ with exponent 1 (which is a consequence of (2.2)) that

$$\mathbb{P}\Big\{\frac{U_\alpha\circ U_\alpha^{\leftarrow}(t)}{t}\in\mathrm{d}x\Big\}=\frac{\sin(\pi\alpha)}{\pi}\frac{\mathbb{1}_{(1,\infty)}(x)}{(x-1)^\alpha x}\mathrm{d}x,\quad x>0,$$

whence

$$\mathbb{P}\{U_\alpha\circ U_\alpha^{\leftarrow}(t)\in\mathrm{d}x\}=\frac{t^\alpha\sin(\pi\alpha)}{\pi}\frac{\mathbb{1}_{(t,\infty)}(x)}{(x-t)^\alpha x}\mathrm{d}x,\quad x>0.$$

Absolute continuity of this distribution particularly implies that, for all $s>0$,

$$\mathbb{P}\{W_\alpha^{(x)}(s)=0\}=0.$$

Thus, the process $W_\alpha^{(x)}$ spends zero time at 0 with probability 1.

Even though the distributions of $(W(t),-\min_{s\in[0,\,t]}W(s))$ and $U_\alpha\circ U_\alpha^{\leftarrow}(t)$ are known explicitly we have been unable to find an explicit form of the transition density of $W_\alpha^{(x)}$.

According to [15], there exists a unique pair of nonnegative processes $(\hat W_\alpha^{(x)},L_\alpha^{(x)})$ satisfying a generalized Skorokhod reflection problem

$$\hat W_\alpha^{(x)}(t)=x+W(t)+U_\alpha(L_\alpha^{(x)}(t)),\quad t\ge 0. \tag{4.2}$$

Here, the unknown process $L_\alpha^{(x)}$ is a.s. continuous, nondecreasing and satisfies

$$L_\alpha^{(x)}(0)=0\ \text{and}\ \int_{[0,\,\infty)}\mathbb{1}_{\{\hat W_\alpha^{(x)}(s)>0\}}\mathrm{d}L_\alpha^{(x)}(s)=0. \tag{4.3}$$

Comparing (4.2) and (1.6) we conclude that $\hat{W}_\alpha^{(x)}(t) = W_\alpha^{(x)}(t)$ and $L_\alpha^{(x)}(t) = U_\alpha^{\leftarrow} \circ ((-x+M(t))^+)$ for $t \geq 0$.

It follows from (4.2) and (4.3) (or just from formula (1.6), or the Itô excursion theory together with (4.1)) that the increments of $W_\alpha^{(x)}$ coincide with those of W while $W_\alpha^{(x)}$ is positive. If $W_\alpha^{(x)}$ is discontinuous at t, then $W_\alpha(x,t-) = 0$ and $W_\alpha(x,t) = U_\alpha(L_\alpha^{(x)}(t)) - U_\alpha(L_\alpha^{(x)}(t)-)$, that is, jumps from 0 are governed by the process U_α and further controlled by an "inner" time given by $L_\alpha^{(x)}$. Further, if $(l(t_0), r(t_0))$ is an excursion interval of $W_\alpha^{(x)}$ that straddles a point $t_0 > 0$, then $W_\alpha(x, l(t_0)) > 0$ a.s. This implies that there is a 'jump-type' exit from 0 rather than a 'continuous' exit.

***Remark* 4.4.** An alert reader will notice that any right neighborhood $(r(t_0), r(t_0)+\varepsilon)$ contains an infinite number of excursions with probability 1. Thus, the picture is similar to the behavior of the Brownian motion excursions.

The process $L_\alpha^{(x)}$ is a continuous additive functional of the Markov process $W_\alpha^{(x)}$ whose points of increase are supported by the set $\{s \geq 0 \ : \ W_\alpha^{(x)}(s) = 0\}$, see p. 68–69 in [5]. Thus, $L_\alpha^{(x)}$ is the *Blumenthal-Getoor local time* up to a multiplicative constant. In particular, the process $L_\alpha^{(x)}$ is $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is a completion by sets of zero measure of the σ-algebra generated by $(W_\alpha^{(x)}(s))_{s\in[0,\,t]}$. This claim, which is not obvious, follows, for instance, from either of the following two representations for $L_\alpha^{(x)}$. The first one, in Theorem 4.1, is in terms of the number of jumps of $W_\alpha^{(x)}$. The other, in Theorem 4.2, is in terms of the number of large excursions of $W_\alpha^{(x)}$ up to time t.

Theorem 4.1. *For any $T > 0$, the convergence*

$$\lim_{\varepsilon\to 0+} \varepsilon^\alpha \big(\text{the number of jumps of } W_\alpha^{(x)} \text{ on } [0,\,t] \text{ which are not smaller than } \varepsilon\big)$$
$$= \lim_{\varepsilon\to 0+} \varepsilon^\alpha \sum_{s\in[0,\,t]} \mathbb{1}_{\{W_\alpha^{(x)}(s) - W_\alpha^{(x)}(s-) \geq \varepsilon\}} = L_\alpha^{(x)}(t)$$

is uniform in $t \in [0,\,T]$ with probability 1.

Theorem 4.2. *For any $T > 0$, the convergence*

$$\lim_{\varepsilon\to 0+} \varepsilon^{\alpha/2} \big(\text{the number of excursion intervals of } W_\alpha^{(x)} \text{ on } [0,\,t]$$
$$\text{whose lengths are not smaller than } \varepsilon\big) = \frac{\Gamma((1-\alpha)/2)}{(\pi 2^\alpha)^{1/2}} \varepsilon^{-\alpha/2} L_\alpha^{(x)}(t)$$

is uniform in $t \in [0,\,T]$ with probability 1.

Proof of Theorem 4.1. The proof of this result can be found in [5]. We recall its main steps because similar arguments are used in the proof of Theorem 4.2, and also for completeness.

Consider the Lévy-Itô representation of U_α

$$U_\alpha(t) = \int_{s\in[0,\,t]} \int_{[0,\,\infty)} u N(\mathrm{d}s, \mathrm{d}u), \quad t \geq 0.$$

Here, $N := \sum_k \delta_{(t_k,u_k)}$ is a Poisson random measure on $[0,\infty)\times(0,\infty]$ with intensity measure $\mathrm{LEB}\otimes\nu$; $\delta_{(t,x)}$ is the probability measure concentrated at (t,x); LEB is the Lebesgue measure on $[0,\infty)$, and ν is the Lévy measure given by

$$\nu(\mathrm{d}u) = \alpha u^{-1-\alpha}\,\mathbb{1}_{(0,\infty)}(u)\mathrm{d}u, \quad u\in\mathbb{R}. \tag{4.4}$$

For any $\varepsilon>0$,

$$\begin{aligned}&\text{the number of jumps of } W_\alpha^{(x)} \text{ on } [0,\,t] \text{ which are not smaller than } \varepsilon\\ &\quad= \text{the number of jumps of } U_\alpha \text{ on } [0,\,L_\alpha^{(x)}(t)] \text{ which are not smaller than } \varepsilon\\ &\quad= N([0,\,L_\alpha^{(x)}(t)]\times[\varepsilon,\infty)).\end{aligned} \tag{4.5}$$

By the strong law of large numbers for Poisson processes, for any fixed $t\geq 0$,

$$\lim_{\varepsilon\to 0+}\varepsilon^\alpha N([0,\,t]\times[\varepsilon,\infty)) = t \quad \text{a.s.} \tag{4.6}$$

because, for each $t>0$, the process $(N([0,\,t]\times[u^{-1},\infty)))_{u>0}$ is an inhomogeneous Poisson process of intensity $u\mapsto u^\alpha t$. As a consequence, relation (4.6) holds true with probability 1 for all rational $t\geq 0$. Since, for each $\varepsilon>0$, the process $(\varepsilon^\alpha N([0,\,t]\times[\varepsilon,\infty)))_{t\geq 0}$ is a.s. nondecreasing, and the limit function in (4.6) is continuous, we infer, for all $T>0$,

$$\lim_{\varepsilon\to 0+}\sup_{t\in[0,T]}\left|\varepsilon^\alpha N([0,\,t]\times[\varepsilon,\infty)) - t\right| = 0 \quad \text{a.s.}$$

This in combination with (4.5) and a.s. continuity of $L_\alpha^{(x)}$ completes the proof. ■

Proof of Theorem 4.2. Let $\theta_1,\theta_2,\ldots$ be independent copies of $\theta := \inf\{t\geq 0 \,:\, 1+W(t)=0\}$. By self-similarity of W and the fact that $-W$ has the same distribution as W,

$$u^2\theta \overset{\mathrm{d}}{=} \inf\{t\geq 0 \,:\, u+W(t)=0\} \overset{\mathrm{d}}{=} \inf\{t\geq 0 \,:\, -u+W(t)=0\}, \quad u>0,$$

where $\overset{\mathrm{d}}{=}$ denotes equality of distributions. Using this in combination with formula 2) on p. 25 in [10] we conclude that

$$\mathbb{P}\{u^2\theta\in\mathrm{d}z\} = \frac{u}{\sqrt{2\pi z^3}}e^{-u^2/2z}\,\mathbb{1}_{(0,\infty)}(z)\mathrm{d}z =: f(u,z)\mathrm{d}z, \quad z\in\mathbb{R},\ u>0.$$

We shall use the Poisson random measure N (or rather its atoms) defined in the proof of Theorem 4.1. Recall that N and W are independent. We proceed by noting that

$$\begin{aligned}&\big(\text{the number of excursion intervals of } W_\alpha^{(x)} \text{ starting in } [0,\,t]\\ &\qquad\text{whose lengths are not smaller than } \varepsilon\big)_{t\geq 0} \overset{\mathrm{d}}{=} \Big(\sum_{t_k\leq L_\alpha^{(x)}(t)}\sum_{u_k}\mathbb{1}_{\{u_k^2\theta_k\geq\varepsilon\}}\Big)_{t\geq 0}.\end{aligned}$$

Further,

$$\left(\sum_{t_k\le t}\sum_{u_k}\mathbb{1}_{\{u_k^2\theta_k\ge\varepsilon\}}\right)_{t\ge 0} \overset{\mathrm{d}}{=} \left(\int_{[0,\,t]}\int_{[\varepsilon,\infty)} M(\mathrm{d}s,\mathrm{d}v)\right)_{t\ge 0},$$

where M is a Poisson random measure on $[0,\infty)\times(0,\infty]$ with intensity measure LEB $\otimes\,\rho$, and ρ is a measure on $(0,\infty)$ defined by

$$\rho(\mathrm{d}z) = \int_{(0,\infty)} f(u,z)\nu(\mathrm{d}u)\mathrm{d}z, \quad z\in\mathbb{R}$$

with the Lévy measure ν defined in (4.4). In particular,

$$\nu([\varepsilon,\infty)) = \int_\varepsilon^\infty\int_0^\infty \frac{u}{\sqrt{2\pi z^3}}e^{-u^2/2z}\frac{\alpha}{u^{1+\alpha}}\mathrm{d}u\mathrm{d}z = \frac{\alpha}{2(\pi 2^\alpha)^{1/2}}\int_\varepsilon^\infty z^{-1-\frac{\alpha}{2}}\mathrm{d}z\int_0^\infty e^{-s}s^{-\frac{1+\alpha}{2}}\mathrm{d}s$$
$$= \frac{\Gamma((1-\alpha)/2)}{(\pi 2^\alpha)^{1/2}}\varepsilon^{-\alpha/2},$$

where the second equality follows by the change of variable $s = u^2/(2z)$.

The remaining part of the proof, which is similar to the corresponding part of the proof of Theorem 4.1, commences with checking the asymptotic relation: for any fixed $t\ge 0$,

$$\lim_{\varepsilon\to 0+}\varepsilon^{\alpha/2}M([0,\,t]\times[\varepsilon,\infty)) = \frac{\Gamma((1-\alpha)/2)}{(\pi 2^\alpha)^{1/2}}t \quad \text{a.s.}$$

Observe that the number of excursion intervals of $W_\alpha^{(x)}$ starting in $[0,\,t]$ whose lengths are not smaller than ε exceeds at most by one the number of excursion intervals of $W_\alpha^{(x)}$ belonging to $[0,\,t]$ whose lengths are not smaller than ε.

■

Appendix

In this section we collect a couple of technical results related to the J_1-convergence. We start with a classical characterization of the J_1-convergence which can be found in Proposition 6.5 of [8].

Proposition A.1. *For $n\in\mathbb{N}_0$, let $z_n\in D$. Then $\lim_{n\to\infty} z_n = z_0$ in the J_1-topology in D if, and only if, for any $u_0\ge 0$ and any sequence $(u_n)_{n\in\mathbb{N}}$ of nonnegative numbers satisfying $\lim_{n\to\infty} u_n = u_0$, the following conditions hold.*

- ***C.i*** *All limit points of $(z_n(u_n))_{n\in\mathbb{N}}$ are either $z_0(u_0)$ or $z_0(u_0-)$.*
- ***C.ii*** *If $\lim_{n\to\infty} z_n(u_n) = z_0(u_0)$, then $\lim_{n\to\infty} z_n(v_n) = z_0(u_0)$ for any sequence $(v_n)_{n\in\mathbb{N}}$ satisfying $v_n\ge u_n$ for $n\in\mathbb{N}$ and $\lim_{n\to\infty} v_n = u_0$.*
- ***C.iii*** *If $\lim_{n\to\infty} z_n(u_n) = z_0(u_0-)$, then $\lim_{n\to\infty} z_n(v_n) = z_0(u_0-)$ for any sequence $(v_n)_{n\in\mathbb{N}}$ satisfying $v_n\le u_n$ for $n\in\mathbb{N}$ and $\lim_{n\to\infty} v_n = u_0$.*

Proposition A.1 will now be essentially used for the proofs of Lemmas 2.4 and 2.6.

Proof of Lemma 2.4. Fix any $t_0 > 0$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence satisfying $\lim_{n\to\infty} t_n = t_0$. Since y_0 is continuous by assumption, the J_1-convergence $\lim_{n\to\infty} y_n = y_0$ is equivalent to locally uniform convergence. This entails $\lim_{n\to\infty} y_n(t_n) = y_0(t_0)$.

To prove (2.23) we intend to show that Conditions **C.i,ii,iii** of Proposition A.1 hold with $z_n = x_n \circ y_n$, $u_n = t_n$ and $u_0 = t_0$. While doing so, we use the other implication of Proposition A.1 with $z_n = x_n$, $u_n = y_n(t_n)$ and $u_0 = y_0(t_0)$, namely, the passage from the J_1-convergence $\lim_{n\to\infty} x_n = x_0$ to the corresponding Conditions **C.i,ii,iii**.

Condition **C.i**. In view of $\lim_{n\to\infty} x_n = x_0$, Condition **C.i** of Proposition A.1 tells us that the limit points of the sequence $(x_n \circ y_n(t_n))_{n\in\mathbb{N}} = (x_n(u_n))_{n\in\mathbb{N}}$ are either $x_0(u_0) = x_0 \circ y_0(t_0)$ or $x_0(u_0-) = x_0(y_0(t_0)-)$. Thus, it suffices to prove that either $x_0(y(t_0)-) = x_0 \circ y_0(t_0)$ or $x_0(y_0(t_0)-) = x_0 \circ y_0(t_0-)$. Indeed, if $y_0(t_0) \notin \mathrm{Disc}(x_0)$, then $x_0(y_0(t_0)-) = x_0 \circ y_0(t_0)$. If $y_0(t_0) \in \mathrm{Disc}(x_0)$, then using the assumptions that y_0 is nondecreasing and that $\#\{u \geq 0 : y_0(u) = y_0(t)\} = 1$ we infer $y_0(s) < y_0(t_0)$ for any $s < t_0$, whence $x_0(y_0(t_0)-) = x_0 \circ y_0(t_0-)$. It remains to note that, in view of right-continuity, Condition **C.i** obviously holds true for $t_0 = 0$.

Condition **C.ii**. Assume that $\lim_{n\to\infty} t_n = t_0$ and $\lim_{n\to\infty} x_n \circ y_n(t_n) = x_0 \circ y_0(t_0)$. Let $(s_n)_{n\in\mathbb{N}}$ be any sequence satisfying $s_n \geq t_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = t_0$. Since y_n is nondecreasing, we infer

$$v_n = y_n(s_n) \geq y_n(t_n) = u_n, \quad n \in \mathbb{N}.$$

It has already been mentioned that $\lim_{n\to\infty} y_n(s_n) = y_0(t_0)$ in view of continuity of y_0. Thus, using $\lim_{n\to\infty} x_n = x_0$ and invoking Condition **C.ii** of Proposition A.1 we conclude that

$$x_n \circ y_n(s_n) = x_n(v_n) \;\to\; x_0(u_0) = x_0 \circ y_0(t_0), \quad n \to \infty.$$

Condition **C.iii** can be checked analogously. ■

Proof of Lemma 2.6. Fix any $T > 0$. Let $(t_n)_{n\in\mathbb{N}_0}$ be any sequence satisfying $t_n \in [0, T)$ for $n \in \mathbb{N}_0$. Assume that the limit $\lim_{n\to\infty} f_n(t_n)$ exists. We shall use Proposition A.1.

For a function x, denote by $\mathrm{Range}(x)$ its range, that is, the set of all possible values of x. To verify Condition **C.i** of Proposition A.1 we have to show that $\lim_{n\to\infty} f_n(t_n) \in \{f(t_0-), f(t_0)\}$. Assume first that there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that $t_{n_k} \in \mathrm{Range}(\lambda_{n_k})$ for all $k \in \mathbb{N}$. Without loss of generality, we can and do assume that $t_n \in \mathrm{Range}(\lambda_n)$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, put

$$\mu_n := \lambda_n^{\leftarrow}(t_n)$$

($\lambda_n^{\leftarrow}$ is right-continuous generalized inverse of λ_n) and note that

$$\lim_{n\to\infty} f_n(t_n) = \lim_{n\to\infty} f_n \circ \lambda_n(\mu_n).$$

In view of (2.30) it follows that $\lim_{n\to\infty} \mu_n = t_0$. Formula (2.31) and Condition **C.i** of Proposition A.1 imply that $\lim_{n\to\infty} f_n \circ \lambda_n(\mu_n) \in \{f(t_0-), f(t_0)\}$, whence

$$\lim_{n\to\infty} f_n(t_n) \in \{f(t_0-), f(t_0)\}.$$

Assume now that $t_n \notin \mathrm{Range}(\lambda_n)$ for all $n \in \mathbb{N}$ (we do not need to investigate an intermediate situation in which $t_n \notin \mathrm{Range}(\lambda_n)$ for some n and $t_n \in \mathrm{Range}(\lambda_n)$ for the other n; indeed, passing to a subsequence we can ensure that exactly one of these alternatives prevails for all values of indices). Then, with the same μ_n as before,

$$u_n := \lambda_n(\mu_n-) \leq t_n < \lambda_n(\mu_n) =: v_n, \quad n \in \mathbb{N}. \tag{A.7}$$

For each fixed n, there are two possibilities: either $\lambda_n(\mu) < u_n$ for $\mu < \mu_n$ or $\lambda_n(\mu) = u_n$ for $\mu \in [\mu_n - \varepsilon_n, \mu_n]$ for some $\varepsilon_n > 0$. Assuming that the first possibility prevails for all $n \in \mathbb{N}$, we select a sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfying $\rho_n < \mu_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \rho_n = t_0$ and

$$\lim_{n\to\infty} |f_n(\lambda_n(\rho_n)) - f_n(u_n-)| = 0, \quad \lim_{n\to\infty} |\lambda_n(\rho_n) - u_n| = 0.$$

This is possible because

$$\lim_{\mu\to\mu_n-} f_n(\lambda_n(\mu)) = \lim_{t\to u_n-} f_n(t) = f_n(u_n-).$$

Using (A.7) in combination with (2.32) yields

$$|f_n(t_n) - f_n(\lambda_n(\rho_n))| \le |f_n(t_n) - f_n(u_n-)| + |f_n(u_n-) - f_n(\lambda_n(\rho_n))| \to 0, \quad n \to \infty.$$

Hence, $\lim_{n\to\infty} f_n(\lambda_n(\rho_n))$ exists and is equal to $\lim_{n\to\infty} f_n(t_n)$. According to (2.31) and Condition **C.i** of Proposition A.1 we have $\lim_{n\to\infty} f_n(t_n) \in \{f(t_0-), f(t_0)\}$.

Assume now that, for each n, there exists $\varepsilon_n > 0$ such that $\lambda_n(\mu) = u_n$ for $\mu \in [\mu_n - \varepsilon_n, \mu_n]$. Let $(\rho_n)_{n\in\mathbb{N}}$ be any sequence satisfying $\rho_n \in [\mu_n - \varepsilon_n, \mu_n)$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \rho_n = t_0$. As a consequence of $t_n \in [u_n, v_n)$ (see (A.7)), as $n \to \infty$,

$$|f_n(t_n) - f_n(\lambda_n(\rho_n))| = |f_n(t_n) - f_n(u_n)| \le |f_n(t_n) - f_n(u_n-)| + |f_n(u_n-) - f_n(u_n))| \to 0.$$

Hence, $\lim_{n\to\infty} f_n(\lambda_n(\rho_n))$ exists and is equal to $\lim_{n\to\infty} f_n(t_n)$. By the same argument as before we infer $\lim_{n\to\infty} f_n(t_n) \in \{f(t_0-), f(t_0)\}$.

Conditions **C.ii** and **C.iii** of Proposition A.1 can be verified similarly. ■

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