A functional limit theorem without centering for general shot noise processes

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July 22, 2020

Abstract

We call a general shot noise process the convolution of a deterministic càdlàg function and a locally finite counting process concentrated on the nonnegative halfline. In the paper we provide sufficient conditions which ensure that a general shot noise process, properly normalized without centering, converges weakly in the Skorokhod space. We give several examples of particular counting processes that satisfy the sufficient conditions and formulate the corresponding limit theorems. The present work continues investigation initiated in Iksanov and Rashytov (2020), where a functional limit theorem with centering was proved in the situation that the limit process is a (Gaussian) process of Riemann-Liouville type.

1 Introduction and main result

Let $(S_k)_{k\in\mathbb{N}_0}$ ($\mathbb{N}_0 := \mathbb{N}\cup\{0\}$) be a not necessarily monotone sequence of nonnegative random variables. Define the counting process $N := (N(t))_{t\geq 0}$ by

$$N(t) := \sum_{k\geq 0} \mathbb{1}_{\{S_k\leq t\}}, \quad t\geq 0.$$

Throughout the paper we assume that $N(t) < \infty$ almost surely (a.s.) for $t \geq 0$.

Denote by $D := D[0,\infty)$ ($D(0,\infty)$) the Skorokhod space of right-continuous real-valued functions which are defined on $[0,\infty)$ ($(0,\infty)$) and have finite limits from the left at each positive point. For a function $h\in D$, define the random process $X := (X(t))_{t\geq 0}$ by

$$X(t) := \sum_{k\geq 0} h(t-S_k)\,\mathbb{1}_{\{S_k\leq t\}} = \int_{[0,\,t]} h(t-y)\mathrm{d}N(y), \quad t\geq 0.$$

We call X a *general shot noise process* because no assumptions are imposed other than $N(t) < \infty$ a.s. It is clear that $X\in D$ a.s.

In the article [8] sufficient conditions were found under which a general shot noise process, properly normalized and centered, converges weakly in the Skorokhod space to a Gaussian process of Riemann-Liouville type. We refer the reader to the cited work and to [3] for the motivation

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behind studying the general shot noise processes, their connection with (more complicated) random processes with immigration at random times, as well as a detailed review of existing literature. Note that mathematicians are still much interested in shot noise processes which is confirmed by the recent articles [12], [15] and [17]. In the present paper we are aimed at finding sufficient conditions, formulated in terms of the response function h and the counting process N, under which a general shot noise process, properly normalized (without centering), satisfies a functional limit theorem in the Skorokhod space. As an illustration, we point out particular counting processes which satisfy the aforementioned sufficient conditions.

To formulate our main result we need additional notation. For $\lambda > 0$, denote by $V_\lambda := (V_\lambda(u))_{u\geq 0}$ a random process which is a.s. nondecreasing, Hölder continuous with exponent λ and satisfies $V_\lambda(0) = 0$ a.s. In particular, for all $T > 0$, all $0 \leq x, y \leq T$ and some a.s. finite random variable M_T

$$|V_\lambda(x) - V_\lambda(y)| \leq M_T|x-y|^{\lambda\wedge 1}. \tag{1}$$

For $\gamma > -\lambda$, define the random process $Z_{\lambda,\gamma} := (Z_{\lambda,\gamma}(u))_{u\geq 0}$ by

$$Z_{\lambda,\gamma}(u) := \int_{[0,\,u]} (u-y)^\gamma \mathrm{d}V_\lambda(y), \quad u \geq 0, \tag{2}$$

where the integral exists as a Riemann-Stieltjes integral. Equivalently, for $\gamma > -\lambda$, $\gamma \neq 0$ the process $Z_{\lambda,\gamma}$ is given by a Lebesgue integral. Indeed, integrating by parts we obtain, for $\gamma > 0$,

$$Z_{\lambda,\gamma}(u) := \gamma \int_0^u (u-y)^{\gamma-1} V_\lambda(y)\mathrm{d}y, \quad u > 0, \quad Z_{\lambda,\gamma}(0) := 0, \tag{3}$$

and, for $-\lambda < \gamma < 0$, –

$$Z_{\lambda,\gamma}(u) := u^\gamma V_\lambda(u) + |\gamma| \int_0^u (V_\lambda(u) - V_\lambda(u-y)) y^{\gamma-1}\mathrm{d}y, \quad u > 0, \quad Z_{\lambda,\gamma}(0) := \lim_{u\to+0} Z_{\lambda,\gamma}(u). \tag{4}$$

We conclude with the help of (1) that $Z_{\lambda,\gamma}(0) = 0$ a.s. whenever $\gamma \in (-\lambda, 0)$. When $\gamma \geq 0$ the convergence of the integral in (2) and a.s. continuity of the process $Z_{\lambda,\gamma}$ are trivial. When $\gamma \in (-\lambda, 0)$ these two facts follow from Lemma 2.1 in [8].

Throughout the paper we assume that the spaces D and $D(0,\infty)$ are endowed with the J_1-topology and denote weak convergence in these spaces by $\overset{J_1}{\Longrightarrow}$. Comprehensive information concerning the J_1-topology can be found in the books [1, 9]. Below we use the notation $\mathbb{R}^+ := [0,\infty)$.

Theorem 1.1. *Let $\alpha, \lambda > 0$ and $h \in D$ be a nonnegative monotone function which is regularly varying at ∞ of index $\beta > -\min(\alpha, \lambda)$. Assume that*

$$a(t)N(t\cdot) \overset{J_1}{\Longrightarrow} V_\lambda(\cdot), \quad t \to \infty \tag{5}$$

on D, where a function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nonincreasing and regularly varying at ∞ of index $-\alpha$, and that, for all $q > 0$ and $0 < a < b < \infty$,

$$t^{-q} \sup_{u\in[a,\,b]} (N(ut) - N(ut-1)) \overset{\mathbb{P}}{\to} 0, \quad t \to \infty. \tag{6}$$

If h is nondecreasing, then

$$\frac{a(t)}{h(t)} X(t\cdot) \overset{J_1}{\Longrightarrow} Z_{\lambda,\beta}(\cdot), \quad t \to \infty \tag{7}$$

on D.

If h is nonincreasing, we additionally assume that, for all $x>0$, $t\geq 1$ and $k\in\mathbb{N}_0$,

$$\mathbb{P}\{a(t)(N((k+1)t)-N(kt))>x\}\leq f(x), \tag{8}$$

where a nonnegative function f is nonincreasing and satisfies

$$\lim_{x\to\infty}\sum_{j\geq 1}2^j f(x2^{jc})=0$$

for any $c>0$. Then limit relation (7) holds on $D(0,\infty)$.

Remark 1.2. *Since, for each $t>0$, the random function $u\mapsto N(tu)$ is a.s. nondecreasing and the process V_λ is a.s continuous, then, according to Remark 2.1 in [16], functional convergence (5) is equivalent to weak convergence of finite-dimensional distributions.*

The remainder of the article is structured as follows. In Section 2 we prove Theorem 1.1. In Section 3 we demonstrate that the sufficient conditions of Theorem 1.1 hold true for particular input sequences $(S_i)_{i\in\mathbb{N}_0}$. Finally, we collect technical auxiliary results in Section 4.

2 Proof of Theorem 1.1

We start with an auxiliary result. For $t,T>0$, put

$$A_t:=\Big\{(u,v)\in\mathbb{R}^2:0\leq v<u\leq T, u-v\geq 1/t\Big\}.$$

Lemma 2.1. *Assume that a function a satisfies the assumptions of Theorem 1.1 and that condition (8) holds. Then, for all $\delta\in(0,\alpha)$,*

$$\lim_{x\to\infty}\limsup_{t\to\infty}\mathbb{P}\Big\{\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}>x\Big\}=0.$$

Proof. By assumption, a is regularly varying at ∞. Hence, it is sufficient to prove the lemma in the case $T=1$ only. Using the fact that N is a.s. nondecreasing we obtain

$$\begin{aligned}
&\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}\\
\leq\;&\sup_{1/t\leq h\leq 1}\;\sup_{0\leq u\leq 1}\frac{a(t)(N(ut)-N((u-h)t))}{h^{\alpha-\delta}}\\
\leq\;&\sup_{j=1,\dots,\lceil\log_2 t\rceil}\;\sup_{2^{-j}\leq h\leq 2^{-j+1}}\;\sup_{0\leq u\leq 1}\frac{a(t)(N(ut)-N((u-h)t))}{h^{\alpha-\delta}}\\
\leq\;&\sup_{j=1,\dots,\lceil\log_2 t\rceil}\;\sup_{2^{-j}\leq h\leq 2^{-j+1}}\;\sup_{k=1,\dots,2^{j-1}}\;\sup_{(k-1)2^{-j+1}\leq u\leq k2^{-j+1}}\frac{a(t)(N(ut)-N((u-h)t))}{h^{\alpha-\delta}}\\
\leq\;&\sup_{j=1,\dots,\lceil\log_2 t\rceil}\;\sup_{k=1,\dots,2^{j-1}}\frac{a(t)(N(tk2^{-j+1})-N(t(k-2)2^{-j+1}))}{2^{-j(\alpha-\delta)}}.
\end{aligned}$$

Here, $\lceil x\rceil$ denotes the smallest integer which is larger than or equal to $x\in\mathbb{R}$. By Boole's inequality,

$$\mathbb{P}\Big\{\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}>x\Big\}$$

$$\le \sum_{j=1}^{\lceil \log_2 t\rceil} \quad \sum_{k=1}^{2^{j-1}} \mathbb{P}\Big\{\frac{a(t)(N(tk2^{-j+1})-N(t(k-2)2^{-j+1}))}{2^{-j(\alpha-\delta)}}>x\Big\}.$$

In view of (8) we infer

$$\begin{aligned}
&\mathbb{P}\Big\{\frac{a(t)(N(tk2^{-j+1})-N(t(k-2)2^{-j+1}))}{2^{-j(\alpha-\delta)}}>x\Big\}\\
\le\ & \mathbb{P}\Big\{a(t2^{-j+1})(N(tk2^{-j+1})-N(t(k-1)2^{-j+1}))>\frac{a(t2^{-j+1})}{a(t)}2^{-j(\alpha-\delta)-1}x\Big\}\\
+\ & \mathbb{P}\Big\{a(t2^{-j+1})(N(t(k-1)2^{-j+1})-N(t(k-2)2^{-j+1}))>\frac{a(t2^{-j+1})}{a(t)}2^{-j(\alpha-\delta)-1}x\Big\}\\
\le\ & 2f\Big(\frac{a(t2^{-j+1})}{a(t)}2^{-j(\alpha-\delta)-1}x\Big).
\end{aligned}$$

According to Lemma A.5 in [6],

$$a(t2^{-j+1})/a(t)\ge c2^{(j-1)(\alpha-\delta/2)}$$

for some $c>0$, all $t>0$ and all $j=1,\ldots,\lceil \log_2 t\rceil$. Therefore,

$$\mathbb{P}\Big\{\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}>x\Big\}\le \sum_{j=1}^{[\log_2 t]+1} 2^j f(Cx2^{j\delta/2})\le \sum_{j\ge 1}2^j f(Cx2^{j\delta/2})\ \to\ 0$$

as $x\to\infty$, where $C:=c2^{\delta/2-\alpha-1}$. The proof of Lemma 2.1 is complete. □

Proof of Theorem 1.1. We treat two cases separately.
THE CASE OF NONDECREASING h in which necessarily $\beta\ge 0$. The random process X is a.s. nondecreasing, and the random process $Z_{\alpha,\beta}$ is a.s. continuous. While when $\beta=0$ the latter holds by assumption because $Z_{\alpha,0}=V_\lambda$, when $\beta>0$ the continuity follows from Lemma A.8 (b) in [5]. Therefore, according to Remark 2.1 of the already mentioned article [16], the functional convergence on D is a consequence of weak convergence of the finite-dimensional distributions.

In the sequel we assume that $h(t)=0$ for $t<0$. For each $t>0$, put

$$V^{(t)}(u):=a(t)N(tu)\quad\text{and}\quad h_t(u):=h(tu)/h(t),\quad u\ge 0.$$

Using integration by parts we conclude that, for $t>0$ and $u\ge 0$,

$$\begin{aligned}
\frac{a(t)}{h(t)}X(ut) &= \frac{a(t)}{h(t)}\int_{[0,\,u]}h(t(u-y))\mathrm{d}_yN(ty)\\
&= a(t)N(0)\frac{h(tu)-h((tu)-)}{h(t)}+\int_{(0,\,u]}V^{(t)}(y)\mathrm{d}_y(-h_t(u-y)).
\end{aligned}$$

For each $t>0$, the random process $X_t^*:=(X_t^*(u))_{u\ge 0}$ defined by

$$X_t^*(u):=\int_{(0,\,u]}V^{(t)}(y)\mathrm{d}_y(-h_t(u-y)),\quad u\ge 0,$$

has a.s. nondecreasing paths. Note that, for $u>0$,

$$a(t)N(0)\frac{h(tu)-h((tu)-)}{h(t)}\le a(t)N(0)\frac{h(tu)}{h(t)}\ \overset{\mathbb{P}}{\to}\ 0,\quad t\to\infty$$

because h is regularly varying at ∞. For $u = 0$ the latter limit relation is secured by the assumption $\beta > -\alpha$ which ensures $\lim_{t\to\infty} a(t)/h(t) = 0$. Consequently, it is enough to prove the limit theorem for X_t^* (instead of X).

Fix any $n \in \mathbb{N}$. Since $X_t^*(0) = Z_{\alpha,\beta}(0) = 0$ a.s., we only have to show that, for any $0 < u_1 < \ldots < u_n < \infty$ and any $\alpha_1, \ldots, \alpha_n \geq 0$,

$$\alpha_1 X_t^*(u_1) + \ldots + \alpha_n X_t^*(u_n) \overset{\mathrm{d}}{\to} \alpha_1 Z_{\alpha,\beta}(u_1) + \ldots + \alpha_n Z_{\alpha,\beta}(u_n), \quad t \to \infty,$$

where $\overset{\mathrm{d}}{\to}$ denotes convergence in distribution.

For $w > 0$ and $t > 0$, define the measures $\nu_{t,w}$ and ν_w on $[0, w]$ by

$$\nu_{t,w}(c, d] := \frac{h(t(w-c)) - h(t(w-d))}{h(t)}, \quad 0 \leq c < d \leq w,$$

$$\nu_w(c, d] := (w-c)^\beta - (w-d)^\beta, \quad 0 \leq c < d \leq w.$$

Also, we note that

$$\alpha_1 X_t^*(u_1) + \ldots + \alpha_n X_t^*(u_n) = \alpha_1 \int_{(0,\, u_1]} V^{(t)}(y) \nu_{t,u_1}(\mathrm{d}y) + \ldots + \alpha_n \int_{(0,\, u_n]} V^{(t)}(y) \nu_{t,u_n}(\mathrm{d}y).$$

Assume that $\beta > 0$. Invoking (5) in combination with Skorokhod's representation theorem (which ensures the existence of versions which converge a.s. on D) and the fact that $\nu_{t,w} \overset{\mathrm{d}}{\to} \nu_w$ as $t \to \infty$, we obtain with the help of the first part of Lemma 4.4

$$\begin{aligned} \alpha_1 X_t^*(u_1) + \ldots + \alpha_n X_t^*(u_n) &\overset{\mathrm{d}}{\to} \alpha_1 \int_{(0,\, u_1]} V_\lambda(y) \nu_{u_1}(\mathrm{d}y) + \ldots + \alpha_n \int_{(0,\, u_n]} V_\lambda(y) \nu_{u_n}(\mathrm{d}y) \\ &= \alpha_1 Z_{\alpha,\beta}(u_1) + \ldots + \alpha_n Z_{\alpha,\beta}(u_n). \end{aligned}$$

Assume that $\beta = 0$. Then $\nu_{t,w} \overset{\mathrm{d}}{\to} \delta_w$ as $t \to \infty$, where δ_w denotes the degenerate at w distribution. Arguing as before and using the second part of Lemma 4.4 we arrive at

$$\alpha_1 X_t^*(u_1) + \ldots + \alpha_n X_t^*(u_n) \overset{\mathrm{d}}{\to} \alpha_1 V_\lambda(u_1) + \ldots + \alpha_n V_\lambda(u_n) = \alpha_1 Z_{\alpha,0}(u_1) + \ldots + \alpha_n Z_{\alpha,0}(u_n).$$

The case of nonincreasing h in which necessarily $\beta \leq 0$. For $\varepsilon \in (0, 1)$, put

$$I_\varepsilon(u, t) := \frac{a(t)}{h(t)} \sum_{k\geq 0} h(ut - S_k) \,\mathbb{1}_{\{S_k \leq \varepsilon u t\}}, \quad u \geq 0,\ t > 0, \quad I_\varepsilon^*(u) := \int_{[0,\, \varepsilon u]} (u-y)^\beta \mathrm{d}V_\lambda(y), \quad u \geq 0$$

and write

$$\begin{aligned} I_\varepsilon(u,t) &= a(t) \sum_{k\geq 0} \Big(\frac{h(ut - S_k)}{h(t)} - (u - t^{-1}S_k)^\beta\Big) \,\mathbb{1}_{\{S_k \leq \varepsilon u t\}} + a(t) \sum_{k\geq 0} (u - t^{-1}S_k)^\beta \,\mathbb{1}_{\{S_k \leq \varepsilon u t\}} \\ &=: I_{\varepsilon,1}(u,t) + I_{\varepsilon,2}(u,t). \end{aligned}$$

We shall show that

$$I_{\varepsilon,1}(\cdot, t) \overset{J_1}{\Longrightarrow} \Psi(\cdot), \quad t \to \infty, \tag{9}$$

where $\Psi(s) = 0$ for $s \geq 0$, and that

$$I_{\varepsilon,2}(\cdot, t) \overset{J_1}{\Longrightarrow} I_\varepsilon^*(\cdot), \quad t \to \infty, \tag{10}$$

whence

$$I_\varepsilon(\cdot,t)=I_{\varepsilon,1}(\cdot,t)+I_{\varepsilon,2}(\cdot,t)\ \overset{J_1}{\Longrightarrow}\ I^*_\varepsilon(\cdot),\quad t\to\infty.$$

For positive and finite $a<b$ and $t>0$

$$\sup_{a\le u\le b}|I_{\varepsilon,1}(u,t)|\le\sup_{(1-\varepsilon)a\le y\le b}\Big|\frac{h(ty)}{h(t)}-y^\beta\Big|a(t)N(\varepsilon bt)\quad\text{a.s.}$$

Formula (5) ensures that $a(t)N(\varepsilon bt)\overset{\mathrm{d}}{\to}V_\lambda(\varepsilon b)$ as $t\to\infty$. Thus, according to the uniform convergence theorem for regularly varying functions (Theorem 1.5.2 in [2]), the right-hand side of the last centered inequality converges to 0 in probability as $t\to\infty$. This proves (9). Relation (10) follows from Lemma A.2 in [6] which is a deterministic result. We apply it in a standard way: weak convergence (5) together with a.s. continuity of the limit process in (5) ensure with the help of Skorokhod's representation theorem that there exist versions of processes in (5) which converge locally uniformly with probability one.

Now we show that, for any fixed $u\ge 0$,

$$\lim_{\varepsilon\to1-}I^*_\varepsilon(u)=Z_{\alpha,\,\beta}(u)\quad\text{a.s.}\tag{11}$$

Indeed,

$$0\le\int_{[0,\,u]}(u-y)^\beta\mathrm{d}V_\lambda(y)-\int_{[0,\,\varepsilon u]}(u-y)^\beta\mathrm{d}V_\lambda(y)=\int_{(\varepsilon u,\,u]}(u-y)^\beta\mathrm{d}V_\lambda(y).$$

Since $Z_{\alpha,\beta}(u)$ is a.s. finite for all $u\ge0$, the right-hand side converges to 0 a.s. as $\varepsilon\to1-$ by Lebesgue's dominated convergence theorem. Thus, (11) does hold.

To obtain (7), given (11), according to Theorem 3.2 in [1] it suffices to prove that, for all $\theta>0$ and all $0<a<b<\infty$,

$$\lim_{\varepsilon\to1-}\limsup_{t\to\infty}\mathbb{P}\Big\{\sup_{u\in[a,\,b]}\Big(\frac{a(t)}{h(t)}X(ut)-I_\varepsilon(u,t)\Big)>\theta\Big\}=0,$$

or, in an expanded form, that

$$\lim_{\varepsilon\to1-}\limsup_{t\to\infty}\mathbb{P}\Big\{\frac{a(t)}{h(t)}\sup_{u\in[a,\,b]}\sum_{k\ge0}h(ut-S_k)\,\mathbb{1}_{\{\varepsilon ut<S_k\le ut\}}>\theta\Big\}=0.$$

Fix any $\Delta\in(0,(\alpha+\beta)/2)$. By Potter's bound for regularly varying functions (Theorem 1.5.6 in [2]), there exists $c>1$ such that

$$\frac{h(t(u-y))}{h(t)}\le2(u-y)^{\beta-\Delta}$$

for all positive t, u and y satisfying $t(u-y)\ge c$ and $u-y\le1$. Hence, for t large enough, $u\in[a,b]$ and $\varepsilon>0$ such that $(1-\varepsilon)b\le1$,

$$\begin{aligned}&\frac{a(t)}{h(t)}\sum_{k\ge0}h(ut-S_k)\,\mathbb{1}_{\{\varepsilon ut<S_k\le ut\}}=\frac{a(t)}{h(t)}\int_{(\varepsilon u,\,u-c/t]}h(t(u-y))\mathrm{d}_yN(ty)\\ +&\ \frac{a(t)}{h(t)}\int_{(u-c/t,\,u]}h(t(u-y))\mathrm{d}_yN(ty)\le2a(t)\int_{(\varepsilon u,\,u-c/t]}(u-y)^{\beta-\Delta}\mathrm{d}_yN(ty)\\ +&\ \frac{a(t)}{h(t)}h(0)(N(tu)-N(tu-c))=2(-\beta+\Delta)\int_{\varepsilon u}^{u-c/t}a(t)(N(tu)-N(ty))(u-y)^{\beta-\Delta-1}\mathrm{d}y\\ +&\ 2u^{\beta-\Delta}(1-\varepsilon)^{\beta-\Delta}a(t)(N(tu)-N(\varepsilon tu))\\ +&\ \Big(\frac{a(t)}{h(t)}h(0)-2c^{\beta-\Delta}t^{-\beta+\Delta}a(t)\Big)(N(tu)-N(tu-c))=:I_1(u,t)+I_2(u,t)+I_3(u,t).\end{aligned}$$

Further,

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{P}\{2(1-\varepsilon)^{\beta-\Delta}\sup_{u\in[a,\,b]}u^{\beta-\Delta}a(t)(N(tu)-N(\varepsilon tu))>\theta\}\\
=\;&\mathbb{P}\{2(1-\varepsilon)^{\beta-\Delta}\sup_{u\in[a,\,b]}u^{\beta-\Delta}(V_\lambda(u)-V_\lambda(\varepsilon u))>\theta\}\\
\le\;&\mathbb{P}\{2(1-\varepsilon)^{\alpha+\beta-\Delta}b^{\alpha+\beta-\Delta}M_b>\theta\},
\end{aligned}$$

where the equality is a consequence of (5) and the continuous mapping theorem (taking into account that the supremum functional is J_1-continuous), and the inequality follows from (1). It is hard to believe that the distribution of $\sup_{u\in[a,\,b]}u^{\beta-\Delta}(V_\lambda(u)-V_\lambda(\varepsilon u))$ may have a discrete component. Nevertheless, assuming it is the case the latter centered formulae hold true for θ which are the continuity points of the distribution of $\sup_{u\in[a,\,b]}u^{\beta-\Delta}(V_\lambda(u)-V_\lambda(\varepsilon u))$, hence, for all $\theta>0$. Finally, $\alpha+\beta-\Delta>0$ entails

$$\lim_{\varepsilon\to 1-}\limsup_{t\to\infty}\mathbb{P}\{\sup_{u\in[a,\,b]}I_2(u,t)>\theta\}=0.$$

In view of (6) and the fact that functions $t\mapsto a(t)/h(t)$ and $t\mapsto t^{-\beta+\Delta}a(t)$ are regularly varying at ∞ of negative indices $-\alpha-\beta$ and $-\alpha-\beta+\Delta$, respectively, we infer

$$\lim_{t\to\infty}\mathbb{P}\{\sup_{u\in[a,\,b]}I_3(u,t)>\theta\}=0.$$

To complete the proof it remains to check that, for all $\theta>0$ and $T>0$,

$$\lim_{\varepsilon\to 1-}\limsup_{t\to\infty}\mathbb{P}\Big\{\sup_{u\in[0,\,T]}\int_{\varepsilon u}^{u-c/t}a(t)(N(tu)-N(ty))(u-y)^{\beta-\Delta-1}\mathrm{d}y>\theta\Big\}=0. \tag{12}$$

Recall the notation A_t introduced in Lemma 2.1. We have, for $0<\delta<\alpha+\beta-\Delta$ and $x>0$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{u\in[0,T]}\int_{\varepsilon u}^{u-c/t}a(t)(N(tu)-N(ty))(u-y)^{\beta-\Delta-1}\mathrm{d}y>\theta\Big\}\\
=\;&\mathbb{P}\Big\{\cdots,\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}>x\Big\}+\mathbb{P}\Big\{\cdots,\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}\le x\Big\}\\
\le\;&\mathbb{P}\Big\{\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}>x\Big\}+\mathbb{P}\Big\{\sup_{u\in[0,T]}\int_{\varepsilon u}^{u}(u-y)^{\alpha+\beta-\Delta-\delta-1}\mathrm{d}y>\theta/x\Big\}\\
=\;&\mathbb{P}\Big\{\sup_{(u,v)\in A_t}\frac{a(t)(N(ut)-N(vt))}{(u-v)^{\alpha-\delta}}>x\Big\}+\mathbb{P}\Big\{\int_0^{(1-\varepsilon)T}y^{\alpha+\beta-\Delta-\delta-1}\mathrm{d}y>\theta/x\Big\}.
\end{aligned}$$

The second summand converges to 0 as $\varepsilon\to 1-$. Invoking Lemma 2.1 and the fact that the left-hand side does not depend on x, relation (12) holds true. The proof of Theorem 1.1 is complete. □

3 Applications of Theorem 1.1

In this section we give examples of particular input sequences $(S_k)_{k\in\mathbb{N}_0}$ which satisfy the sufficient conditions of Theorem 1.1. We also point out the corresponding limit theorems.

1. Standard random walk. Let $\xi_1, \xi_2, \ldots$ be independent copies of a positive random variable ξ. The random sequence $(S_k)_{k\in\mathbb{N}_0}$ defined by $S_0 := 0$ and $S_k := \xi_1 + \ldots + \xi_k$ for $k \in \mathbb{N}$ is called *(zero-delayed) standard random walk.*

Denote by $N = (N(t))_{t\geq 0}$ the counting process for $(S_k)_{k\in\mathbb{N}_0}$. According to Lemma A.1 in [5], the process N satisfies (6). Assume that $\mathbb{P}\{\xi > t\} \sim t^{-\alpha}\ell(t)$ as $t \to \infty$ for some $\alpha \in (0,1)$ and some ℓ slowly varying at ∞. Invoking Theorem 3.2 in [10] (in which weak convergence of the finite-dimensional distributions is stated) and Remark 2.1 in [16] we infer

$$\mathbb{P}\{\xi > t\}N(t\cdot) \overset{J_1}{\Longrightarrow} W_\alpha(\cdot), \quad t \to \infty,$$

where $W_\alpha := (W_\alpha(u))_{u\geq 0}$ is an inverse α-stable subordinator. This means that

$$W_\alpha(u) := \inf\{s \geq 0 : D_\alpha(s) > u\}, \quad u \geq 0,$$

where $(D_\alpha(t))_{t\geq 0}$ is an α-stable subordinator with $-\log \mathbb{E}e^{-sD_\alpha(1)} = \Gamma(1-\alpha)s^\alpha$, $s \geq 0$, and $\Gamma(\cdot)$ is the gamma function. By Lemma 3.4 in [11], the process W_α is a.s. Hölder continuous with exponent less than α. Hence, $V_\lambda = W_\alpha$ satisfies condition (1) with $\lambda < \alpha$. Using distributional subadditivity of N and Markov's inequality we obtain, for all $x > 0$, $t \geq 1$ and $k \in \mathbb{N}_0$,

$$\begin{aligned}\mathbb{P}\{a(t)(N((k+1)t) - N(kt)) > x\} &\leq \mathbb{P}\{a(t)N(t) > x\} \\ &\leq e^{-x}\mathbb{E}e^{a(t)N(t)} \leq Ce^{-x} =: f(x),\end{aligned}$$

where $C := \sup_{t\geq 1} \mathbb{E}e^{a(t)N(t)}$ is a finite constant by Lemma A.4 in [6]. Further, for $c > 0$,

$$\lim_{x\to\infty} \sum_{j\geq 1} 2^j f(x2^{jc}) = C \lim_{x\to\infty} \sum_{j\geq 1} 2^j e^{-x2^{jc}} = 0,$$

for the latter series converges uniformly in $x \geq 1$. Thus, condition (8) holds.

Using now Theorem 1.1 with $a(t) = \mathbb{P}\{\xi > t\}$, $V_\lambda = W_\alpha$ and h satisfying the conditions of the theorem we conclude that

$$\frac{\mathbb{P}\{\xi > t\}}{h(t)} X(t\cdot) \overset{J_1}{\Longrightarrow} \int_{[0,\,\cdot]} (\cdot - y)^\beta \mathrm{d}W_\alpha(y), \quad t \to \infty$$

on D provided that h is nondecreasing, and on $D(0,\infty)$ provided that h is nonincreasing. The latter limit relation can be found in Theorem 2.1 of [6].

2. Time-changed renewal process.

2.1. Let $\alpha \in (0,1]$ and $(S_k)_{k\in\mathbb{N}_0}$ be defined by

$$S_0 := 0, \quad S_k := ((\xi_1 + \ldots + \xi_k)/\eta)^{1/\alpha}, \quad k \in \mathbb{N},$$

where η is a positive random variable independent of independent and identically distributed random variables $\xi_1, \xi_2, \ldots$ Then $N(t) = N^*(t^\alpha \eta)$ for $t \geq 0$, where $(N^*(t))_{t\geq 0}$ is the counting process for the zero-delayed random walk with jumps ξ_k.

In what follows we assume that $\mu := \mathbb{E}\xi_1 < \infty$ and that $\mathbb{E}e^{s\eta} < \infty$ for some s in a right vicinity of zero. By the strong law of large numbers for renewal processes in combination with Dini's lemma,

$$\lim_{t\to\infty} \sup_{u\in[0,\,T]} \left| t^{-\alpha}N(ut) - \mu^{-1}\eta u^\alpha \right| = 0 \quad \text{a.s.}$$

for all $T > 0$. Thus, condition (5) holds with $a(t) = t^{-\alpha}$ and $V_\alpha(u) = \mu^{-1}\eta u^\alpha$ for $u \geq 0$. Subadditivity of the function $x \mapsto x^\alpha$ on $[0,\infty)$ entails that V_α is a.s. Hölder continuous with exponent α. In particular, inequality (1) holds with $M_T = \mu^{-1}\eta$. For all $q > 0$, $0 < a < b < \infty$ and large t,

$$\begin{aligned} t^{-q} \sup_{u\in[a,\,b]} (N(ut) - N(ut-1)) &= t^{-q} \sup_{u\in[a,\,b]} (N^*((ut)^\alpha \eta) - N^*((ut-1)^\alpha \eta)) \\ &\leq t^{-q} \sup_{u\in[a,\,b]} (N^*((u\eta^{1/\alpha} t)^\alpha) - N^*((u\eta^{1/\alpha}t)^\alpha - \eta)). \end{aligned}$$

According to Lemma A.1 in [5], for any $x > 0$,

$$t^{-q} \sup_{u\in[a,\,b]} (N^*((uxt)^\alpha) - N^*((uxt)^\alpha - x^\alpha)) \overset{\mathbb{P}}{\to} 0, \quad t \to \infty.$$

As a consequence, an application of Lebesgue's dominated convergence theorem enables us to conclude that relation (6) holds for $(N(t))_{t\geq 0}$ defined here.

Using distributional subadditivity of N^* and subadditivity of $x \mapsto x^\alpha$ on $[0, \infty)$ we write, for all $x > 0$, $t \geq 1$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} \mathbb{P}\{a(t)(N((k+1)t) - N(kt)) > x\} &= \mathbb{P}\{a(t)(N^*(((k+1)t)^\alpha \eta) - N^*((kt)^\alpha \eta)) > x\} \\ &\leq \mathbb{P}\{a(t)N^*(\eta t^\alpha ((k+1)^\alpha - k^\alpha)) > x\} \\ &\leq \mathbb{P}\{a(t)N^*(\eta t^\alpha) > x\}. \end{aligned}$$

Let $\sqrt{x}t^\alpha > 1$. Distributional subadditivity of N^* ensures that, for any $n \in \mathbb{N}$ and $y > 0$,

$$\mathbb{P}\{N^*(n) > y\} \leq \mathbb{P}\{N_1^*(1) + \ldots + N_n^*(1) > y\}, \tag{13}$$

where $N_1^*(1), N_2^*(1), \ldots$ are independent copies of $N^*(1)$. As a result, for any $s \in (0, -\log \mathbb{P}\{\xi_1 = 0\})$ ($-\log 0$ is interpreted as $+\infty$)

$$\mathbb{E}e^{sN^*(n)/n} \leq \mathbb{E}e^{s(N_1^*(1)+\ldots+N_n^*(1))/n} = \mathbb{E}e^{sN^*(1)} < \infty,$$

where finiteness is ensured by Theorem 2.1(c) in [7]. For any $r > 1$, there exists $n \in \mathbb{N}$ such that $r \in (n-1, n]$. Since the process N^* is a.s. nondecreasing,

$$N^*(r)/r \leq N^*(n)/(n-1) \leq 2N^*(n)/n \quad \text{a.s.}$$

and thereupon $\mathbb{E}e^{vN^*(r)/r} \leq \mathbb{E}e^{2vN^*(1)} < \infty$ for $v \in (0, -\log \mathbb{P}\{\xi_1 = 0\}/2)$. By Markov's inequality,

$$\begin{aligned} \mathbb{P}\{a(t)N^*(\eta t^\alpha) > x\} &= \mathbb{P}\{a(t)N^*(\eta t^\alpha) > x, \eta > \sqrt{x}\} + \mathbb{P}\{a(t)N^*(\eta t^\alpha) > x, \eta \leq \sqrt{x}\} \\ &\leq \mathbb{P}\{\eta > \sqrt{x}\} + \mathbb{P}\{N^*(\sqrt{x}t^\alpha)/(\sqrt{x}t^\alpha) > \sqrt{x}\} \\ &\leq \mathbb{E}e^{s\eta}e^{-s\sqrt{x}} + \mathbb{E}e^{vN^*(\sqrt{x}t^\alpha)/(\sqrt{x}t^\alpha)}e^{-v\sqrt{x}} \\ &\leq C_1 e^{-s\sqrt{x}} + C_2 e^{-v\sqrt{x}} =: f_1(x), \end{aligned}$$

where $C_1 := \mathbb{E}e^{s\eta} < \infty$ and $C_2 := \mathbb{E}e^{2vN^*(1)} < \infty$. Thus, for $c > 0$,

$$\lim_{x\to\infty} \sum_{j\geq 1} 2^j f_1(x2^{jc}) = C_1 \lim_{x\to\infty} \sum_{j\geq 1} 2^j e^{-s\sqrt{x}2^{jc/2}} + C_2 \lim_{x\to\infty} \sum_{j\geq 1} 2^j e^{-v\sqrt{x}2^{jc/2}} = 0,$$

for the latter series converge uniformly in $x \geq 1$.

Let $\sqrt{x}t^\alpha \le 1$. By Markov's inequality, for $t \ge 1$,

$$\begin{aligned}\mathbb{P}\{a(t)N^*(\eta t^\alpha) > x\} &\le \mathbb{P}\{\eta > \sqrt{x}\} + \mathbb{P}\{N^*(\sqrt{x}t^\alpha) > xt^\alpha\} \\ &\le \mathbb{P}\{\eta > \sqrt{x}\} + \mathbb{P}\{N^*(1) > x\} \le C_1 e^{-s\sqrt{x}} + C_3 e^{-vx} =: f_2(x),\end{aligned}$$

where $C_3 := \mathbb{E}e^{vN^*(1)} < \infty$. Thus, for $c > 0$,

$$\lim_{x\to\infty} \sum_{j\ge 1} 2^j f_2(x2^{jc}) = 0,$$

thereby showing that condition (8) holds with $f := f_1 + f_2$.

According to Theorem 1.1 with $a(t) = t^{-\alpha}$, $V_\alpha(u) = \mu^{-1}\eta u^\alpha$, $u \ge 0$ and h satisfying the conditions of the theorem,

$$\frac{1}{t^\alpha h(t)} X(t\cdot) \overset{J_1}{\Longrightarrow} \mu^{-1}\alpha \mathrm{B}(\alpha, \beta+1)\eta(\cdot)^{\alpha+\beta}, \quad t \to \infty,$$

where $\mathrm{B}(\cdot,\cdot)$ is a Beta function, on D provided that h is nondecreasing, and on $D(0,\infty)$ provided that h is nonincreasing.

2.2. Let η_1, $\eta_2, \ldots$ be independent copies of η independent also of ξ_1, $\xi_2, \ldots$ For $\tau \in (0,1]$, put

$$S_0^* = T_0 := 0, \quad S_n^* := \xi_1 + \ldots + \xi_n, \quad T_n := \eta_1 + \ldots + \eta_n, \quad n \in \mathbb{N},$$

$$S_0 := 0, \quad S_n := T_{\lfloor (S_n^*)^{1/\tau} \rfloor}, \quad n \in \mathbb{N}. \tag{14}$$

Then $N(t) = N^*(N_1^\tau(t))$ for $t \in \mathbb{R}$, where $N_1(t) := \sum_{n\ge 1} \mathbb{1}_{\{T_n \le t\}}$, $t \in \mathbb{R}$ (it is clear that $N_1(t) = 0$ for $t \le 0$). Indeed, using the notation $\lfloor x \rfloor$ for the integer part of $x \in \mathbb{R}$ we infer

$$\{T_{\lfloor (S_n^*)^{1/\tau} \rfloor} \le t\} = \{N_1(t) + 1 > \lfloor (S_n^*)^{1/\tau} \rfloor\} = \{N_1(t) \ge (S_n^*)^{1/\tau}\} = \{S_n^* \le N_1^\tau(t)\},$$

whence $N(t) = \sum_{n\ge 0} \mathbb{1}_{\{S_n \le t\}} = \sum_{n\ge 0} \mathbb{1}_{\{S_n^* \le N_1^\tau(t)\}} = N^*(N_1^\tau(t))$. Note that if ξ_1 has an exponential distribution, the so defined process $(N(t))_{t\ge 0}$ is a particular instance of a Cox process (also known as a doubly stochastic Poisson process).

Assume, as before, that $\mu = \mathbb{E}\xi_1 \in (0,\infty)$ and additionally that $\mathbb{P}\{\eta_1 > t\} \sim t^{-\rho}\ell(t)$ as $t \to \infty$ for $\rho \in (0,1)$ and ℓ slowly varying at ∞. Then

$$(\mathbb{P}\{\eta_1 > t\})^\tau N_1^\tau(t\cdot) \overset{J_1}{\Longrightarrow} W_\rho^\tau(\cdot), \quad t \to \infty,$$

where W_ρ is an inverse ρ-stable subordinator (see the point devoted to standard random walks). Using the strong law of large numbers for N^* and Lemma 4.3 we obtain

$$(\mathbb{P}\{\eta_1 > t\})^\tau N^*(N_1^\tau(t\cdot)) \overset{J_1}{\Longrightarrow} \mu^{-1} W_\rho^\tau(\cdot), \quad t \to \infty.$$

Thus, condition (5) holds with $\alpha = \rho\tau$, $a(t) = (\mathbb{P}\{\eta_1 > t\})^\tau$ and $V_\lambda = \mu^{-1}W_\rho^\tau$.

Let us show that we can take in the role of λ any positive number less than $\rho\tau$. Using subadditivity of $x \to x^\tau$ on $[0,\infty)$, we write, for any $T > 0$ and $0 \le x, y \le T$,

$$\Big| W_\rho^\tau(x) - W_\rho^\tau(y) \Big| \le \Big| W_\rho(x) - W_\rho(y) \Big|^\tau \le M_T^\tau |x-y|^{\lambda_1\tau},$$

where $\lambda_1 \in (0,\rho)$. Conditions (6) and (8) hold for the present $(N(t))_{t\ge 0}$ by Lemmas 4.1 and 4.2, respectively.

By Theorem 1.1 with $\alpha=\rho\tau$, $a(t)=(\mathbb{P}\{\eta_1>t\})^\tau$, $V_\lambda=\mu^{-1}W_\rho^\tau$ and h satisfying the conditions of the theorem,

$$\frac{(\mathbb{P}\{\eta_1>t\})^\tau}{h(t)}X(t\cdot)\ \overset{J_1}{\Longrightarrow}\ \mu^{-1}\int_{[0,\,\cdot]}(\cdot-y)^\beta {\rm d}W_\rho^\tau(y),\quad t\to\infty$$

on D provided that h is nondecreasing, and on $D(0,\infty)$ provided that h is nonincreasing.
2.3. Here, we assume that $(S_n)_{n\in\mathbb{N}}$ is given by (14) with $\tau=1$, allowing however dependence of ξ_1 and η_1. Suppose that $\mathbb{P}\{\xi_1>x\}\sim c_1x^{-\rho_1}$ and $\mathbb{P}\{\eta_1>x\}\sim c_2x^{-\rho_2}$ as $x\to\infty$ for positive c_1,c_2 and $\rho_1,\rho_2\in(0,1)$. This particularly means that, for $x_1,x_2>0$,

$$\lim_{t\to\infty}t\mathbb{P}\{\xi_1>(c_1t)^{1/\rho_1}x_1\}=x_1^{-\rho_1}\quad\text{and}\quad\lim_{t\to\infty}t\mathbb{P}\{\eta_1>(c_2t)^{1/\rho_2}x_2\}=x_2^{-\rho_2}.$$

To control the asymptotics of the joint distribution of ξ_1 and η_1 we assume that, for $x_1,x_2>0$, there exists a limit

$$f(x_1,x_2):=\lim_{t\to\infty}t\mathbb{P}\{\xi_1>(c_1t)^{1/\rho_1}x_1,\eta_1>(c_2t)^{1/\rho_2}x_2\}$$

and consequently a measure ν defined on $\mathbb{K}:=[0,\infty]\backslash\{(0,0)\}$ by

$$\nu\{(u,v)\in\mathbb{K}:u>x_1\text{ or }v>x_2\}=x_1^{-\rho_1}+x_2^{-\rho_2}-f(x_1,x_2),\quad x_1,x_2>0.$$

The measure ν (hence, the function f) cannot be arbitrary. We assume that it satisfies

$$\int_{|\mathbf{x}|\neq0}(|\mathbf{x}|\wedge1)\nu({\rm d}\mathbf{x})<\infty,\tag{15}$$

where $|\mathbf{x}|=\sqrt{x_1^2+x_2^2}$ for $x=(x_1,x_2)\in\mathbb{R}^2$.

Denote by $\mathcal{M}:=\sum_k\delta_{(t_k,\mathbf{j}_k)}$ a Poisson random measure on $[0,\infty)\times\mathbb{K}$ with the intensity measure LEB $\otimes\,\nu$. Here, $\delta_{(t,\mathbf{x})}$ is a probability measure concentrated at $(t,\mathbf{x})\in[0,\infty)\times\mathbb{K}$, and LEB denotes the Lebesgue measure on $[0,\infty)$. Put

$$\mathbf{L}(t):=(L_1(t),L_2(t))=\sum_{k:t_k\leq t}\mathbf{j}_k,\quad t>0.$$

Condition (15) ensures the convergence of the latter series with probability one. The random process $\mathbf{L}$ is a two-dimensional Lévy process with the Lévy measure ν. Its components L_1 and L_2 are (dependent unless $f\equiv0$) drift-free stable subordinators with parameters ρ_1 and ρ_2, respectively. Denote by $L_1^\leftarrow$ and $L_2^\leftarrow$ the corresponding inverse ρ_1 and ρ_2-stable subordinators. By Lemma 6.1 on p. 174 in [13],

$$t\mathbb{P}\Big\{\Big(\frac{\xi_1}{(c_1t)^{1/\rho_1}},\frac{\eta_1}{(c_2t)^{1/\rho_2}}\Big)\in\cdot\Big\}\overset{v}{\to}\ \nu(\cdot),\quad t\to\infty,$$

where $\overset{v}{\to}$ denotes vague convergence on the set of locally finite measure on $\mathbb{K}$. Hence, by Theorem 4 in [14],

$$\Big(\frac{\sum_{k=1}^{[t\cdot]}\xi_k}{(c_1t)^{1/\rho_1}},\frac{\sum_{j=1}^{[t\cdot]}\eta_j}{(c_2t)^{1/\rho_2}}\Big)\ \Rightarrow\ \mathbf{L}(\cdot),\quad t\to\infty\tag{16}$$

in the J_1-topology on $D\times D$. Let us show that this implies

$$\Big(\frac{N^*(t\cdot)}{c_1^{-1}t^{\rho_1}},\frac{N_1(t\cdot)}{c_2^{-1}t^{\rho_2}}\Big)\ \Rightarrow\ (L_1^\leftarrow(\cdot),L_2^\leftarrow(\cdot)),\quad t\to\infty\tag{17}$$

in product J_1-topology on $D \times D$. Indeed, weak convergence of the finite-dimensional distributions in (17) is equivalent to weak convergence of the finite-dimensional distributions in (16). Tightness of the coordinate distributions on the left-hand side of (17) follows from Remark 2.1 in [16] because the coordinates are a.s. nondecreasing, and the limit processes $L_1^{\leftarrow}$ and $L_2^{\leftarrow}$ are a.s. continuous. An application of Lemma 4.3 to (17) yields

$$t^{-\rho_1\rho_2} N^*(N_1(t\cdot)) \overset{J_1}{\Longrightarrow} c_1^{-1} c_2^{-\rho_2} L_1^{\leftarrow}(L_2^{\leftarrow}(\cdot)), \quad t \to \infty.$$

Thus, condition (5) holds with $\alpha = \rho_1\rho_2$, $a(t) = t^{-\rho_1\rho_2}$ and $V_\lambda(u) = c_1^{-1}c_2^{-\rho_2} L_1^{\leftarrow}(L_2^{\leftarrow}(u))$ for $u \geq 0$.

While checking the other conditions of Theorem 1.1 we assume that ξ_1 and η_1 are independent. Let us show that we can take in the role of λ any positive number less than $\rho_1\rho_2$. According to the information provided in point 1, for any $\lambda_i \in (0, \rho_i)$, any $T > 0$, all $0 \leq x, y \leq T$ and some a.s. finite random variables $M_T^{(i)}$ (depending on λ_i) $\left|L_i^{\leftarrow}(x) - L_i^{\leftarrow}(y)\right| \leq M_T^{(i)}|x-y|^{\lambda_i}$, $i = 1, 2$. Therefore, for the same x and y

$$\left|L_1^{\leftarrow}(L_2^{\leftarrow}(x)) - L_1^{\leftarrow}(L_2^{\leftarrow}(y))\right| \leq M^{(1)}_{L_2^{\leftarrow}(T)} \left|L_2^{\leftarrow}(x) - L_2^{\leftarrow}(y)\right|^{\lambda_1} \leq M^{(1)}_{L_2^{\leftarrow}(T)} (M_T^{(2)})^{\lambda_1} |x-y|^{\lambda_1\lambda_2}.$$

In view of independence of $L_1^{\leftarrow}$ and $L_2^{\leftarrow}$ and self-similarity of $L_1^{\leftarrow}$ with index ρ_1, the random variable $M^{(1)}_{L_2^{\leftarrow}(T)}$ has the same distribution as $M_1^{(1)}(L_2^{\leftarrow}(T))^{\rho_1-\lambda_1}$, whence it is a.s. finite. Conditions (6) and (8) hold for the present $(N(t))_{t\geq 0}$ by Lemmas 4.1 and 4.2, respectively.

By Theorem 1.1 with $\alpha = \rho_1\rho_2$, $a(t) = t^{-\rho_1\rho_2}$, $V_\lambda(u) = c_1^{-1}c_2^{-\rho_2} L_1^{\leftarrow}(L_2^{\leftarrow}(u))$ for $u \geq 0$ and h satisfying the conditions of the theorem,

$$\frac{t^{-\rho_1\rho_2}}{h(t)} X(t\cdot) \overset{J_1}{\Longrightarrow} c_1^{-1} c_2^{-\rho_2} \int_{[0,\,\cdot]} (\cdot - y)^\beta \mathrm{d} L_1^{\leftarrow}(L_2^{\leftarrow}(y)), \quad t \to \infty$$

on D provided that h is nondecreasing, and on $D(0,\infty)$ provided that h is nonincreasing.

4 Auxiliary results

Lemma 4.1. *Under the conditions of points 2.2 and 2.3 in Section 3 the random processes* $(N(t))_{t\geq 0}$ *satisfy* (6).

Proof. The argument given below applies to the processes $(N(t))_{t\geq 0}$ appearing in both points 2.2 and 2.3. The principal assumption is independence of ξ_1 and η_1 which entails that of N^* and N_1.

For any $0 < a < b < \infty$ and $t > 0$, the following inequality holds:

$$\sup_{u \in [at,\, bt]} (N^*(N_1^\tau(u)) - N^*(N_1^\tau(u-1))) \leq \max\{R(t), Z(t)\} \leq R(t) + Z(t),$$

where

$$\begin{aligned} R(t) \;&:=\; \sup_{at \leq T_k \leq bt} (N^*(N_1^\tau(T_k)) - N^*(N_1^\tau(T_k - 1))) \\ &\leq\; \sup_{k \leq N_1(bt)} (N^*(N_1^\tau(T_k)) - N^*(N_1^\tau(T_k - 1))) \end{aligned}$$

and

$$\begin{aligned} Z(t) \;&:=\; \sup_{at \leq T_k \leq bt} (N^*(N_1^\tau(T_k + 1)) - N^*(N_1^\tau(T_k))) \\ &\leq\; \sup_{k \leq N_1(bt)} (N^*(N_1^\tau(T_k + 1)) - N^*(N_1^\tau(T_k))). \end{aligned}$$

The assumption $\mathbb{E}\eta = \infty$ together with the weak law of large numbers ensures that, for all $\delta > 0$,

$$\lim_{t\to\infty} \mathbb{P}\{N_1(bt) > \delta t\} = 0.$$

Pick $a \in (0, -\log \mathbb{P}\{\xi = 0\})$ and note that $c := \mathbb{E}e^{aN^*(1)} < \infty$ by Theorem 2.1 (c) in [7]. Further, (13) yields, for each $n \in \mathbb{N}$, $\mathbb{E}e^{aN^*(n)} \le c^n$. Since N^* and N_1 are independent we conclude that

$$\mathbb{E}e^{aN^*(1+N_1(1))} \le \mathbb{E}c^{1+N_1(1)} < \infty,$$

where finiteness is secured by Theorem 2.1 (b) in [7] (recall that, by assumption, $\eta > 0$ a.s.). Analogously,

$$\mathbb{E}e^{aN^*(N_1(1))} = \mathbb{E}e^{aN^*(N_1(1))}\,\mathbb{1}_{\{N_1(1)\ge 1\}} + \mathbb{E}e^{aN^*(N_1(1))}\,\mathbb{1}_{\{N_1(1)=0\}} \le \mathbb{E}c^{N_1(1)} + \mathbb{E}e^{aN^*(0)} < \infty.$$

Using independence of N^* and N_1, distributional subadditivity and monotonicity of N^*, subadditivity of $x \to x^\tau$ on $[0,\infty)$ and Markov's inequality we obtain, for any positive q, ε and δ,

$$\begin{aligned}
\mathbb{P}\{R(t) > \varepsilon t^q\} &= \mathbb{P}\{R(t) > \varepsilon t^q,\, N_1(bt) > \delta t\} + \mathbb{P}\{R(t) > \varepsilon t^q,\, N_1(bt) \le \delta t\}\\
&\le \mathbb{P}\{N_1(bt) > \delta t\} + \mathbb{P}\{\sup_{k\le\lfloor\delta t\rfloor} (N^*(N_1^\tau(T_k)) - N^*(N_1^\tau(T_k - 1))) > \varepsilon t^q\}\\
&\le \mathbb{P}\{N_1(bt) > \delta t\} + \sum_{k=0}^{\lfloor\delta t\rfloor} \mathbb{P}\big\{N^*((N_1(T_k) - N_1(T_k - 1))^\tau) > \varepsilon t^q\big\}\\
&= \mathbb{P}\{N_1(bt) > \delta t\} + \sum_{k=0}^{\lfloor\delta t\rfloor} \mathbb{P}\Big\{N^*\Big(\Big(1 + \sum_{i=1}^{k-1} \mathbb{1}_{\{T_k - T_i < 1\}}\Big)^\tau\Big) > \varepsilon t^q\Big\}\\
&= \mathbb{P}\{N_1(bt) > \delta t\} + \sum_{k=0}^{\lfloor\delta t\rfloor} \mathbb{P}\Big\{N^*\Big(\Big(1 + \sum_{i=1}^{k-1} \mathbb{1}_{\{T_i < 1\}}\Big)^\tau\Big) > \varepsilon t^q\Big\}\\
&\le \mathbb{P}\{N_1(bt) > \delta t\} + \sum_{k=0}^{\lfloor\delta t\rfloor} \mathbb{P}\big\{N^*((1 + N_1(1))^\tau) > \varepsilon t^q\big\}\\
&\le \mathbb{P}\{N_1(bt) > \delta t\} + (\delta t + 1)e^{-a\varepsilon t^q}\mathbb{E}e^{aN^*(1+N_1(1))}.
\end{aligned}$$

Hence,

$$\lim_{t\to\infty} \mathbb{P}\{R(t) > \varepsilon t^q\} = 0.$$

Arguing similarly we obtain, for any positive q, ε and δ,

$$\begin{aligned}
\mathbb{P}\{Z(t)>\varepsilon t^q\} &= \mathbb{P}\{Z(t)>\varepsilon t^q,\, N_1(bt)>\delta t\}+\mathbb{P}\{Z(t)>\varepsilon t^q,\, N_1(bt)\le\delta t\}\\
&\le \mathbb{P}\{N_1(bt)>\delta t\}+\mathbb{P}\{\sup_{k\le\delta t}(N^*(N_1^\tau(T_k+1))-N^*(N_1^\tau(T_k)))>\varepsilon t^q\}\\
&\le \mathbb{P}\{N_1(bt)>\delta t\}+\sum_{k=0}^{\lfloor\delta t\rfloor}\mathbb{P}\{N^*((N_1(T_k+1)+N_1(T_k))^\tau)>\varepsilon t^q\}\\
&= \mathbb{P}\{N_1(bt)>\delta t\}+\sum_{k=0}^{\lfloor\delta t\rfloor}\mathbb{P}\Big\{N^*\Big(\Big(\sum_{i\ge1}\mathbb{1}_{\{T_k<T_{k+i}\le T_k+1\}}\Big)^\tau\Big)>\varepsilon t^q\Big\}\\
&= \mathbb{P}\{N_1(bt)>\delta t\}+\sum_{k=0}^{\lfloor\delta t\rfloor}\mathbb{P}\Big\{N^*\Big(\Big(\sum_{i\ge1}\mathbb{1}_{\{T_{k+i}-T_k\le1\}}\Big)^\tau\Big)>\varepsilon t^q\Big\}\\
&= \mathbb{P}\{N_1(bt)>\delta t\}+\sum_{k=0}^{\lfloor\delta t\rfloor}\mathbb{P}\big\{N^*(N_1^\tau(1))>\varepsilon t^q\big\}\\
&\le \mathbb{P}\{N_1(bt)>\delta t\}+(\delta t+1)e^{-a\varepsilon t^q}\mathbb{E}e^{aN^*(N_1(1))}.
\end{aligned}$$

This entails

$$\lim_{t\to\infty}\mathbb{P}\{Z(t)>\varepsilon t^q\}=0.$$

Thus, the processes $(N(t))_{t\ge0}$ do satisfy condition (6). The proof of Lemma 4.1 is complete. □

Lemma 4.2. *Under the conditions of points 2.2 and 2.3 in Section 3 the random processes* $(N(t))_{t\ge0}$ *satisfy* (8).

Proof. While the first step of the proof will be divided into two parts which correspond to the assumptions of points 2.2 and 2.3, respectively, the second step will be common.
STEP 1 whose purpose is proving finiteness of some exponential moments to be defined below.

Assume that the assumptions of point 2.2 hold. Recall that $a(t)=(\mathbb{P}\{\eta_1>t\})^\tau$ and $\phi(s):=\mathbb{E}e^{sN^*(1)}<\infty$ for $s\in(0,-\log\mathbb{P}\{\xi=0\})$ (see Theorem 2.1 (c) in [7]). In particular, $\mathbb{E}N^*(1)<\infty$. In view of $\log\phi(s)\sim\phi(s)-1\sim\mathbb{E}N^*(1)s$ as $s\to0+$, given $\varepsilon>0$ there exists $s_0>0$ such that $\log\phi(s)\le(\mathbb{E}N^*(1)+\varepsilon)s$ for all $s\in(0,s_0]$. Choose any $b>0$ satisfying $ba(1)\le s_0$. We intend to show that $C^*:=\sup_{t\ge1}\mathbb{E}e^{ba(t)N^*(M_1^\tau(t))}<\infty$, where $M_1(t):=N_1(t)+1$ for $t\ge0$.

Using (13) we obtain, for $t\ge1$,

$$\mathbb{E}[e^{ba(t)N^*(M_1^\tau(t))}\mid M_1(t)]\;\le\;\mathbb{E}[e^{ba(t)(N_1^*(1)+\ldots+N^*_{\lceil M_1^\tau(t)\rceil}(1))}\mid M_1(t)]=e^{\log\phi(ba(t))\lceil M_1^\tau(t)\rceil}.$$

Hence, for $\varepsilon>0$ as above and any $t\ge1$,

$$\mathbb{E}[e^{ba(t)N^*(\lceil M_1^\tau(t)\rceil)}\mid M_1(t)]\le e^{(\mathbb{E}N^*(1)+\varepsilon)ba(t)\lceil M_1^\tau(t)\rceil}.$$

Since

$$\sup_{t\ge1}\mathbb{E}e^{(\mathbb{E}N^*(1)+\varepsilon)ba(t)\lceil M_1^\tau(t)\rceil}\,\mathbb{1}_{\{a(t)\lceil M_1^\tau(t)\rceil>1\}}\le\mathbb{E}e^{(\mathbb{E}N^*(1)+\varepsilon)ba(1)}\sup_{t\ge1}\mathbb{E}e^{(\mathbb{E}N^*(1)+\varepsilon)b\mathbb{P}\{\eta_1>t\}M_1(t)}<\infty,$$

where finiteness is ensured by Lemma A.4 in [6], we infer $C^*<\infty$.

Assume that the assumptions of point 2.3 hold. Recall that $a(t)=t^{-\rho_1\rho_2}$. Put $\psi(s):=\mathbb{E}e^{-s\xi}$ and $\varphi(s):=\mathbb{E}e^{-s\eta}$ for $s\ge0$. We have $\mathbb{E}[e^{-sS_n}|S_n^*]=\varphi^{\lfloor S_n^*\rfloor}(s)\le\varphi^{S_n^*-1}(s)$ and thereupon

$\mathbb{E}e^{-sS_n} \le \psi^n(-\log\varphi(s))/\varphi(s)$. In this case, choose any $b \in (0, -\log \mathbb{P}\{\xi = 0\}/2)$. We are going to check that $C^* := \sup_{t\ge1} \mathbb{E}e^{ba(t)N^*(M_1(t))} < \infty$.

For positive s satisfying $e^{2ba(t)}\psi(-\log\varphi(s)) < 1$, we obtain with the help of Markov's inequality

$$\begin{aligned}\mathbb{E}e^{2ba(t)N(t)} - 1 &= (1-e^{-2ba(t)})\sum_{k\ge1} e^{2ba(t)k}\mathbb{P}\{N(t) \ge k\} = (1-e^{-2ba(t)})\sum_{k\ge1} e^{2ba(t)k}\mathbb{P}\{S_{k-1} \le t\}\\ &\le e^{st}(e^{2ba(t)}-1)\sum_{k\ge0} e^{2ba(t)k}\psi^k(-\log\varphi(s)) = \frac{e^{st}(e^{2ba(t)}-1)}{1-e^{2ba(t)}\psi(-\log\varphi(s))}.\end{aligned}$$

By Corollary 8.1.7 in [2], $1-\psi(s) \sim c_1\Gamma(1-\rho_1)s^{\rho_1}$ and $-\log\varphi(s) \sim 1-\varphi(s) \sim c_2\Gamma(1-\rho_2)s^{\rho_2}$ as $s \to 0+$. This implies that, for any $\kappa > 0$,

$$1-\psi(-\log\varphi(\kappa t^{-1})) \;\sim\; c_1\Gamma(1-\rho_1)(c_2\Gamma(1-\rho_2)\kappa^{\rho_2})^{\rho_1}t^{-\rho_1\rho_2}, \quad t\to\infty.$$

Hence, as $t \to \infty$,

$$\frac{1-e^{-2ba(t)}}{1-\psi(-\log\varphi(ct^{-1}))} \;\sim\; \frac{2b}{c_1\Gamma(1-\rho_1)(c_2\Gamma(1-\rho_2)\kappa^{\rho_2})^{\rho_1}} := A.$$

Choose κ such that $A < 1$. Then the inequality $e^{2ba(t)}\psi(-\log\varphi(s)) < 1$ holds for $s = \kappa t^{-1}$ and large enough t, and

$$\mathbb{E}e^{2ba(t)N(t)} - 1 \le \frac{e^{\kappa}(e^{2ba(t)}-1)}{1-e^{2ba(t)}\psi(-\log\varphi(\kappa t^{-1}))} \;\to\; \frac{e^{\kappa}A}{1-A}, \quad t\to\infty.$$

Thus, we have checked that $\sup_{t\ge1}\mathbb{E}e^{2ba(t)N(t)} < \infty$. Further, using distributional subadditivity of N^* and Hölder's inequality, we write, for $t \ge 1$,

$$\begin{aligned}\mathbb{E}e^{ba(t)N^*(M_1(t))} &\le \mathbb{E}e^{ba(t)(N^*(1)+N^*(N_1(t)))} \le \Big(\mathbb{E}e^{2ba(t)N^*(1)}\mathbb{E}e^{2ba(t)N^*(N_1(t))}\Big)^{1/2}\\ &\le \Big(\mathbb{E}e^{2bN^*(1)}\mathbb{E}e^{2ba(t)N(t)}\Big)^{1/2}.\end{aligned}$$

This gives $C^* < \infty$ because our choice of b ensures finiteness of $\mathbb{E}e^{2bN^*(1)}$.
STEP 2 which completes the proof. Here, under the assumptions of point 2.3 we assume that $\tau = 1$. Using independence of N^* and N_1, distributional subadditivity of N^*, subadditivity of $x \to x^\tau$ on $[0,\infty)$ and the fact that $t \to N^*(t)$ is a.s. nondecreasing we obtain, for $x > 0$, $t \ge 1$ and $k \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{P}\{a(t)(N^*(N_1^\tau((k+1)t)) - N^*(N_1^\tau(kt))) > x \mid (N_1(s))_{s\ge0}\}\\ \le\;& \mathbb{P}\{a(t)N^*(N_1^\tau((k+1)t) - N_1^\tau(kt)) > x \mid (N_1(s))_{s\ge0}\}\\ \le\;& \mathbb{P}\{a(t)N^*((N_1((k+1)t) - N_1(kt))^\tau) > x \mid (N_1(s))_{s\ge0}\}.\end{aligned}$$

Further, using independence of N^* and N_1, distributional subadditivity of $(M_1(t))_{t\ge0}$, the fact that $t \to N^*(t)$ is a.s. nondecreasing and Markov's inequality we infer

$$\begin{aligned}&\mathbb{P}\{a(t)(N^*(N_1^\tau((k+1)t)) - N^*(N_1^\tau(kt))) > x\} \le \mathbb{P}\{a(t)N^*((N_1((k+1)t) - N_1(kt))^\tau) > x\}\\ =\;& \mathbb{E}\mathbb{P}\{a(t)N^*((N_1((k+1)t) - N_1(kt))^\tau) > x \mid (N^*(s))_{s\ge0}\}\\ \le\;& \mathbb{P}\{a(t)N^*(M_1^\tau(t)) > x\} \le C^*e^{-bx} = f(x).\end{aligned}$$

Also, for $c>0$,

$$\lim_{x\to\infty}\sum_{j\geq 1}2^j f(x2^{jc}) = C^* \lim_{x\to\infty}\sum_{j\geq 1}2^j e^{-bx2^{jc}} = 0,$$

for the latter series converges uniformly in $x \geq 1$. Thus, the processes $(N(t))_{t\geq 0}$ do satisfy condition (8). The proof of Lemma 4.2 is complete. □

Finally, we present the last two auxiliary results borrowed from Lemma 2.3 on p. 159 in [4] and Lemma A.5 in [5], respectively.

Lemma 4.3. *The composition mapping* $(x,y)\mapsto x\circ y$ *is* J_1*-continuous at continuous functions* $x:\mathbb{R}^+\to\mathbb{R}^+$ *and nondecreasing continuous functions* $y:\mathbb{R}^+\to\mathbb{R}^+$.

Lemma 4.4. *Assume that* $\lim_{n\to\infty} x_n = x$ *in* D*. Assume also that, as* $n\to\infty$*, finite measures* ν_n *converge weakly to a finite continuous measure* ν *on* $[0,u]$ *for some* $u>0$*. Then*

$$\lim_{n\to\infty}\int_{[0,\,u]} x_n(y)\nu_n(\mathrm{d}y) = \int_{[0,\,u]} x(y)\nu(\mathrm{d}y).$$

If x *is continuous at point* $c\in[0,u]$ *and* $\nu=\delta_c$ *is a Dirac measure at point* c*, then*

$$\lim_{n\to\infty}\int_{[0,\,u]} x_n(y)\nu_n(\mathrm{d}y) = x(c).$$

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