Exponential Rate of Almost Sure Convergence of Intrinsic Martingales in Supercritical Branching Random Walks

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Abstract

We provide sufficient conditions which ensure that the intrinsic martingale in the supercritical branching random walk converges exponentially fast to its limit. The case of Galton-Watson processes is particularly included so that our results can be seen as a generalization of a result given in the classical treatise by Asmussen and Hering. As an auxiliary tool, we prove ultimate versions of two results concerning the exponential renewal measures which may be interesting on its own and which correct, generalize and simplify some earlier works.

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1 Introduction and main result

The Galton-Watson process is the eldest and probably best understood branching process in probability theory. There is a vast literature on different aspects of these processes ranging from simple distributional properties to highly nontrivial results on convergence in function spaces. In particular, in [3, Section

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II.3-4] Asmussen and Hering investigate the rate of the a.s. convergence of the normalized supercritical Galton-Watson process to its limit and among other things point out a criterion for the exponential rate of convergence [3, Theorem 4.1(i)]. The aim of the present paper is to prove a counterpart of this last result for branching random walks (BRW, in short) which form a generalization of the Galton-Watson processes.

We proceed with a formal definition of the BRW. Consider an individual, the ancestor, which we identify with the empty tuple $\varnothing$, located at the origin of the real line at time $n = 0$. At time $n = 1$ the ancestor produces a random number J of offspring which are placed at points of the real line according to a random point process $\mathcal{M} = \sum_{i=1}^{J} \delta_{X_i}$ on $\mathbb{R}$ (particularly, $J = \mathcal{M}(\mathbb{R})$). We enumerate the ancestor's children by $1, 2, \ldots, J$ (note that we do not exclude the case that $J = \infty$ with positive probability). The offspring of the ancestor form the first generation. The population further evolves following the subsequently explained rules. An individual $u = u_1 \ldots u_n$ of the nth generation with position $S(u)$ on the real line produces at time $n+1$ a random number $J(u)$ of offspring which are placed at random locations on $\mathbb{R}$ given by the positions of the random point process $\delta_{S(u)} * \mathcal{M}(u) = \sum_{i=1}^{J(u)} \delta_{S(u)+X_i(u)}$ where $\mathcal{M}(u) = \sum_{i=1}^{J(u)} \delta_{X_i(u)}$ denotes a copy of $\mathcal{M}$ (and $J(u) = \mathcal{M}(u)(\mathbb{R})$). The offspring of individual u are enumerated by $u1 = u_1 \ldots u_n 1, \ldots, uJ(u) = u_1 \ldots u_n J(u)$, the positions of offspring individuals are denoted by $S(ui)$, $i = 1, \ldots, J(u)$. It remains to state that $(\mathcal{M}(u))_{u \in \mathbb{V}}$ is assumed to be a family of i.i.d. point processes. Note that this assumption does not imply anything about the dependence structure of the random variables $X_1(u), \ldots, X_{J(u)}(u)$ for fixed u. The point process of the positions of the nth generation individuals will be denoted by $\mathcal{M}_n$. The sequence of point processes $(\mathcal{M}_n)_{n \in \mathbb{N}_0}$ is then called *branching random walk*. Throughout the article, we assume that $\mathbb{E}\, J > 1$ (supercriticality) which means that the population survives with positive probability. Notice that provided $J < \infty$ a.s., the sequence of generation sizes in the BRW forms a Galton-Watson process.

An important tool in the analysis of the BRW is the Laplace transform of the intensity measure $\xi := \mathbb{E}\, \mathcal{M}$ of $\mathcal{M}$,

$$m : [0, \infty) \to [0, \infty], \qquad \theta \mapsto \int_{\mathbb{R}} e^{-\theta x}\, \xi(dx) \;=\; \mathbb{E} \int_{\mathbb{R}} e^{-\theta x}\, \mathcal{M}(dx).$$

We define $\mathfrak{D}(m) := \{\theta \geq 0 : m(\theta) < \infty\}$ and as a standing assumption, suppose the existence of some $\gamma > 0$ such that $m(\gamma) < \infty$ (equivalently, $\mathfrak{D}(m) \neq \emptyset$). Possibly after the transformation $X_i \mapsto \gamma X_i + \log m(\gamma)$ it is no loss of generality to assume $\gamma = 1$ and

$$m(1) \;=\; \mathbb{E} \int_{\mathbb{R}} e^{-x}\, \mathcal{M}(dx) \;=\; \mathbb{E} \sum_{i=1}^{J} e^{-X_i} \;=\; 1.$$

Put $Y_u := e^{-S(u)}$ and

$$\overline{\Sigma}_n \;:=\; \mathbb{E} \sum_{|u|=n} Y_u\, \delta_{S(u)}, \qquad n \in \mathbb{N},$$

where $\sum_{|u|=n}$ denotes the summation over the individuals of the nth generation, and let $(S_n)_{n\in\mathbb{N}_0}$ denote a zero-delayed random walk with increment distribution $\overline{\Sigma}_1$. We call $(S_n)_{n\in\mathbb{N}_0}$ the *associated random walk* . It is well-known (see *e.g.* [5, Lemma 4.1]) that, for any measurable $f:\mathbb{R}^{n+1}\to[0,\infty)$,

$$\mathbb{E}\, f(S_0,\ldots,S_n) \;=\; \mathbb{E}\sum_{|u|=n} Y_u f(S(u|\,0),S(u|\,1),\ldots,S(u)), \tag{1.1}$$

where for $u=u_1\ldots u_n$ we write $u|\,k$ for the individual $u_1\ldots u_k$, the ancestor of u residing in the kth generation. We note, in passing, that

$$\varphi(t) \;:=\; \mathbb{E}\, e^{-tS_1} \;=\; m(1+t), \quad t\geq 0.$$

Define

$$W_n \;:=\; \int_{\mathbb{R}} e^{-x}\,\mathcal{M}_n(dx) \;=\; \sum_{|u|=n} Y_u, \quad n\in\mathbb{N}_0,$$

and denote the distribution of W_1 by F. Let $\mathcal{F}_n$ be the σ-field generated by the first n generations, *i.e.*, $\mathcal{F}_n=\sigma(\mathcal{M}(u):|u|<n)$ where $|u|<n$ means $u\in\mathbb{N}^k$ for some $k<n$.

It is well-known and easy to check that $(W_n,\mathcal{F}_n)_{n\in\mathbb{N}_0}$ forms a non-negative martingale and thus converges a.s. to a random variable W, say, with $\mathbb{E}\, W\leq 1$. This martingale, which is called the *intrinsic martingale in the BRW*, is of outstanding importance in the asymptotic analysis of the BRW (see *e.g.* [8] and [14]). In this article, we give sufficient conditions for the following statement to hold: for fixed $a>0$

$$\sum_{n\geq 0} e^{an}(W-W_n) \qquad \text{converges a.s.} \tag{1.2}$$

Clearly, (1.2) states that $(W_n)_{n\in\mathbb{N}_0}$ a.s. converges to W exponentially fast.

There are already (at least) two articles which explore the rate of convergence of the intrinsic martingale in the BRW to its limit. In [2] necessary and sufficient conditions were found for the series in (1.2) to converge in L_p, $p>1$. Sufficient conditions for the a.s. convergence of the series

$$\sum_{n\geq 0} f(n)(W-W_n),$$

where f is a function regularly varying at ∞ with an index larger than -1, were obtained in [11]. The results derived in both the paper at hand and [11] form a generalization of the results in [3, Section II.4], where the rate of the a.s. convergence of the normalized supercritical Galton-Watson process to its limit was investigated. We want to remark that the scheme of our proofs borrows heavily from the ideas laid down in [3, Section II.4] but the technical details are much more involved. The source of complication can be easily understood: given $\mathcal{F}_n$, the W_{n+1} in the setting of the Galton-Watson processes is just the sum of finite number of i. i. d. random variables whereas the W_{n+1} in the setting of the BRW is a weighted sum of, possibly infinite, number of i. i. d. random variables.

Before stating our main results, we need some more notation and explanations. If $0 < \inf_{1\le\theta\le 2} m^{1/\theta}(\theta) < 1$, then there exists a $\vartheta_0 \in (1,2]$ such that $m^{1/\vartheta_0}(\vartheta_0) = \inf_{1\le\theta\le 2} m^{1/\theta}(\theta)$. The derivative of the function $\theta \mapsto m^{1/\theta}(\theta)$ is well-defined and negative on $(1,\vartheta_0)$ and the left derivative is well-defined and non-positive on $(1,\vartheta_0]$. From this we conclude that the left derivative of m (to be denoted by m' in what follows) is well-defined and negative on $(1,\vartheta_0]$, *i.e.*,

$$m'(\vartheta_0) < 0.$$

Theorem 1.1. *Let $a > 0$ be given. Assume that*

$$e^a\, m^{1/r}(r) \le 1 \qquad \text{for some } r \in (1,2) \tag{1.3}$$

and define ϑ to be the minimal $r > 1$ such that $e^{ar}m(r) = 1$. Assume further that

$$\mathbb{E}\, W_1^{\vartheta} \ < \ \infty, \tag{1.4}$$

and in case when $a = -\log\inf_{r\ge 1} m^{1/r}(r)$ (which implies $\vartheta = \vartheta_0$) assume that

$$-\frac{\log m(\vartheta_0)}{\vartheta_0} \ < \ -\frac{m'(\vartheta_0)}{m(\vartheta_0)}.$$

Then (1.2) holds true.

Remark 1.2. The point $(\vartheta_0, m^{1/\vartheta_0}(\vartheta_0))$ either belongs to the strictly decreasing branch of the graph $\{(x, m^{1/x}(x)) : x \in \mathfrak{D}(m)\}$, equivalently, $-\log m(\vartheta_0)/\vartheta_0 < -m'(\vartheta_0)/m(\vartheta_0)$, or it is the bottom point of that graph, which is equivalent to $\log m(\vartheta_0)/\vartheta_0 = m'(\vartheta_0)/m(\vartheta_0)$. From this we conclude that the theorem implies (1.2) with $a = -\log m(\vartheta_0)/\vartheta_0$ to hold when the former occurs. Intuitively, while the second situation is somewhat exceptional, the first situation is more or less typical. In conclusion, condition $e^a \inf_{1\le r\le 2} m^{1/r}(r) < 1$ is "typically" sufficient for (1.2) to hold. A similar remark with an obvious modification also applies to Theorem 2.1(a) and Theorem 2.2.

Remark 1.3. Let $p \in (1,2)$. By using a completely different argument, in [2] it was proved that conditions

$$\mathbb{E}\, W_1^r < \infty \ \text{ and } \ e^a m^{1/r}(r) < 1 \ \text{ for some } \ r \in [p,2]$$

are sufficient for the $\mathcal{L}_p$, and hence, the a.s. convergence of $\sum_{n\ge 0} e^{an}(W - W_n)$. Plainly, the conditions of our Theorem 1.1 are weaker.

Remark 1.4. Under the assumptions of Theorem 1.1, the martingale $(W_n)_{n\in\mathbb{N}_0}$ is uniformly integrable, equivalently, $\mathbb{P}\{W > 0\} > 0$. An ultimate criterion of uniform integrability of the intrinsic martingale was recently presented in [1], following earlier investigation in [4, 12, 13].

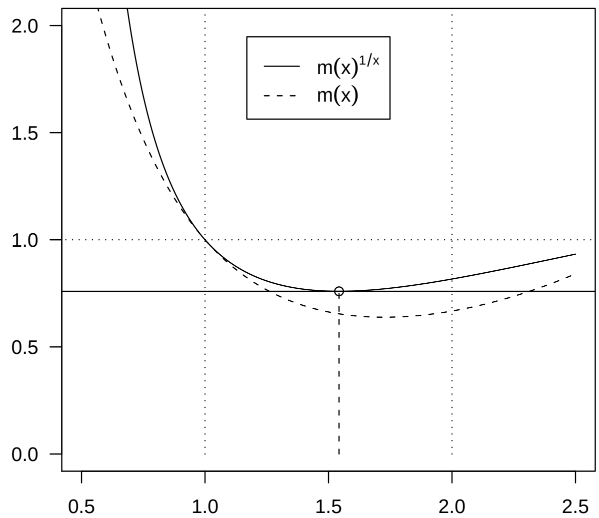

Figure 1: A typical situation in which Theorem 1.1 applies. Here $m(1) = 1$, and m is strictly decreasing in a right neighborhood of 1. The bottom point of the graph of $m^{1/x}(x)$ is marked by a circle. The vertical dashed line connects this point to the x-axis indicating the point ϑ_0. The solid horizontal line and the dotted horizontal line at 1 indicate the interval of possible values of e^{-a} such that $a > 0$ and $e^a m^{1/\vartheta_0} m(\vartheta_0) < 1$. For those a's, the assumptions of Theorem 1.1 are satisfied. The vertical dotted lines at 1 and 2 emphasize the importance of the interval $(1, 2)$ in which ϑ_0 is supposed to be located.

Example 1.5 (Galton-Watson processes). Suppose that $m := \mathbb{E}\, J \in (1, \infty)$ and that $e^{-X_i} = m^{-1}\, \mathbb{1}_{\{i \leq J\}}$, $i \in \mathbb{N}$. Then $(W_n)_{n \in \mathbb{N}_0}$ forms a normalized supercritical Galton-Watson process. Pick p and q such that $p \in (1, 2)$ and $1/p + 1/q = 1$. Theorem 4.1 in [3] proves that

$$W - W_n \;=\; o(m^{-n/q}) \qquad \text{a.s. as } n \to \infty, \tag{1.5}$$

if and only if

$$\mathbb{E}\, W_1^p \;<\; \infty. \tag{1.6}$$

Sufficiency of condition (1.6) for (1.5) to hold follows from our Theorem 1.1. To see this, take $a = q^{-1} \log m$ and notice that equality $m(\theta) = m^{1-\theta}$, $\theta \geq 0$, implies that $e^a m^{1/r}(r) = m^{1/r - 1/p} < 1$ for $r > p$. Therefore, (1.3) holds (with strict inequality). Further, ϑ defined in Theorem 1.1 equals $p \in (1, 2)$ in the present situation, which shows that (1.4) holds. By Theorem 1.1, (1.5) holds.

The rest of the article is organized as follows. In Section 2, we present some auxiliary renewal-theoretic results which correct, generalize and simplify some early results from [9, 10]. Results of this section are an important ingredient of the proof of Theorem 1.1 which is given in Section 3.

2 Ultimate results for the exponential renewal function

For a random variable T with proper distribution which we assume to be non-degenerate at 0 let ψ be its Laplace transform:

$$\psi : [0,\infty) \to (0,\infty], \quad \psi(t) \ := \ \mathbb{E}\, e^{-tT}.$$

In what follows, we denote by ψ' the *left* derivative of ψ.

Set $R := -\log \inf_{t\geq 0} \psi(t)$. Then $R \geq 0$ since $\psi(0) = 1$, and unless $T \geq 0$ a.s., the infimum in the definition of R is attained, *i.e.*, there exists some $\gamma_0 \in [0,\infty)$ such that $\psi(\gamma_0) = e^{-R}$. Note that $\gamma_0 = 0$ is equivalent to $R = 0$ since we assume the distribution of T to be non-degenerate at 0. When $R > 0$ and $a \in (0, R]$, let γ denote the minimal (finite) $t > 0$ satisfying $\psi(t) = e^{-a}$ if such a t exists. Notice that $\gamma = \gamma_0$ if $a = R$ and the infimum is attained. Let $(T_n)_{n\in\mathbb{N}_0}$ be a zero-delayed random walk with a step distributed like T. Whenever γ as above exists, we use it to define a new probability measure $\mathbb{P}_\gamma$ such that

$$\mathbb{P}_\gamma(T_n \in A) \ = \ \psi(\gamma)^{-n}\, \mathbb{E}\, e^{-\gamma T_n}\, \mathbb{1}_{\{T_n\in A\}}, \qquad n \in \mathbb{N}_0, \tag{2.1}$$

for any Borel set $A \subseteq \mathbb{R}$. As a consequence of $\mathbb{E}_\gamma\, e^{\gamma T_1} = \psi(\gamma)^{-1} < \infty$, we have

$$\mathbb{E}_\gamma\, T_1^+ \ < \ \infty. \tag{2.2}$$

Since ψ is non-increasing on $[0, \gamma_0]$ we conclude that if $\gamma_0 > 0$, then

$$\mathbb{E}_{\gamma_0} T_1 \ = \ \mathbb{E}\, e^{-\gamma_0 T} T \ = \ -\,\psi'(\gamma_0)$$

should be non-negative and finite in view of (2.2).

The first theorem in this section investigates finiteness of the *exponential* renewal function

$$V(x) \ := \ \sum_{n\geq 0} e^{an}\, \mathbb{P}(T_n \leq x), \qquad x \in \mathbb{R}\,.$$

Theorem 2.1. *Assume that $\mathbb{P}(T = 0) \neq 1$ and let $a > 0$ be given.*

(a) Assume that $\mathbb{P}(T < 0) > 0$.

(i) If $a \in (0, R)$, then $V(x)$ is finite for every $x \in \mathbb{R}$.

(ii) If $a = R > 0$ and

$$-\psi'(\gamma_0) \ = \ \mathbb{E}\, e^{-\gamma_0 T} T \ > \ 0, \tag{2.3}$$

then $V(x)$ is finite for every $x \in \mathbb{R}$.

(iii) If $a > R$, then $V(x) = +\infty$ for all $x \in \mathbb{R}$.

(iv) If $a = R > 0$ and $\psi'(\gamma_0) = 0$ (equivalently, if (2.3) does not hold), then $V(x) = +\infty$ for all $x \in \mathbb{R}$.

(b) Assume that $\mathbb{P}(T > 0) = 1$. Then $V(x)$ is finite for every $x \in \mathbb{R}$.

(c) Assume that $\mathbb{P}(T \geq 0) = 1$ and $\beta := \mathbb{P}(T = 0) > 0$. Then $R = -\log \beta$ and if $a \in (0, R)$, then $V(x)$ is finite for every $x \in \mathbb{R}$, and if $a \geq R$, then $V(x)$ is infinite for all $x \geq 0$.

Theorem 2.1 constitutes a generalization of Theorem B in [9] but can also be partly deduced (excluding the case $a = R$) from the more general Theorem 2 in [7]. Our contribution here is a streamlined derivation of the exact value of R, a simple proof of dichotomy $a < R$ versus $a > R$ and investigating the most delicate case $a = R$.

The main tool for the analysis in Section 3 is the following result, which provides the asymptotic behavior of the exponential renewal function $V(x)$. Note in advance that Theorem 2.2 will be applied to $(S_n - an)_{n\in\mathbb{N}_0}$ where $(S_n)_{n\in\mathbb{N}_0}$ is the associated random walk of the given BRW.

Theorem 2.2. *Let $a > 0$ be given. Assume that either $a \in (0, R)$ or $a = R$ and (2.3) holds. Then, with γ being the minimal $t > 0$ satisfying $\psi(t) = e^{-a}$,*

$$V(x) \sim \frac{e^{-a}}{\gamma(-\psi'(\gamma))}\, e^{\gamma x}, \qquad \text{as} \quad x \to \infty \tag{2.4}$$

if $(T_n)_{n\in\mathbb{N}_0}$ is a non-arithmetic random walk, and

$$V(\lambda n) \sim \frac{\lambda e^{-a}}{(1 - e^{-\lambda\gamma})(-\psi'(\gamma))}\, e^{\gamma\lambda n}, \qquad \text{as} \qquad n \to \infty \tag{2.5}$$

if $(T_n)_{n\in\mathbb{N}_0}$ is arithmetic with span $\lambda > 0$. Moreover, in the arithmetic case,

$$V(\lambda n) - V(\lambda(n-1)) \sim \frac{\lambda e^{-a}}{(-\psi'(\gamma))}\, e^{\gamma\lambda n}, \qquad \text{as} \qquad n \to \infty. \tag{2.6}$$

Remark 2.3. (a) Theorem 2.2 describes the asymptotics of $V(x)$ whenever it is finite.
(b) Provided $a < R$ or $a = R$ and (2.3) holds, equation $\psi(t) = e^{-a}$ has positive solutions.

Theorem 2.2 is a generalization and correction of Theorem 4 in [10][1] the differences being that

- we do not assume that $\mathbb{E}\,|T|$ is finite;
- the exponential (wrong) rate $a/\mathbb{E}\,T$ claimed in [10] under the assumption $\mathbb{E}\,T \in (0, \infty)$ is replaced by the rate γ in the non-arithmetic case, and a similar substitution is proved to hold true in the arithmetic case;
- unlike [10] we treat, among others, the boundary case $a = R$.

[1] We think that an error in the proof of Theorem 4 of the afore-mentioned paper comes from the end of p. 706 in [10] where the dependence of the real number $\xi(n) \in (\beta, \mu)$ on n cannot be ignored since possibly $\xi(n) \to \mu$ and then $\mathbb{P}(S_n \leq \xi(n))$ does not necessarily decay at an exponential rate (in this footnote we retained the original notation from [10]).

Proof of Theorem 2.1. CASE (a). (i). If $R = 0$, then condition $a \in (0, R)$ cannot hold. So assume that $R > 0$, $a \in (0, R)$ and pick any $x \in \mathbb{R}$. With γ_0 defined at the beginning of the section choose $r \in (0, \gamma_0)$ such that $a < -\log\psi(r)$. Now use Markov's inequality to obtain

$$\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x) \;\leq\; \sum_{n\geq 0} e^{an}e^{rx}\,\mathbb{E}\,e^{-rT_n} \;=\; e^{rx}\sum_{n\geq 0} e^{n(a+\log\psi(r))} \;<\; \infty,$$

which proves the assertion under the assumption (i).

(ii) and (iv). Assume that $a = R > 0$. The function $g(y) := e^{-\gamma_0 y}\,\mathbb{1}_{[0,\infty)}(y)$ is directly Riemann integrable. If (2.3) holds (does not hold), then the random walk $(T_n)_{n\in\mathbb{N}_0}$ is transient (recurrent) under $\mathbb{P}_{\gamma_0}$, the probability measure defined in (2.1). As a consequence, the renewal measure U_{γ_0} of $(T_n)_{n\in\mathbb{N}_0}$ under $\mathbb{P}_{\gamma_0}$ satisfies $U_{\gamma_0}(I) < \infty$ $(= \infty)$ for any open non-empty interval I if (T_n) is non-arithmetic, and for any open non-empty interval I which contains some point $n\lambda$, $n \in \mathbb{Z}$, if (T_n) is arithmetic with span λ, respectively. Therefore, if (2.3) holds, then

$$\begin{aligned}\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x) \;&=\; \sum_{n\geq 0} \mathbb{E}_{\gamma_0}\, e^{\gamma_0 T_n}\,\mathbb{1}_{\{T_n\leq x\}}\\ &=\; e^{\gamma_0 x}\sum_{n\geq 0}\mathbb{E}_{\gamma_0}\, g(x-T_n) \;<\; \infty, \qquad x \in \mathbb{R}\,.\end{aligned}$$

Whereas, if (2.3) does not hold, then

$$\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x) \;=\; e^{\gamma_0 x}\sum_{n\geq 0}\mathbb{E}_{\gamma_0}\, g(x-T_n) \;=\; \infty, \qquad x \in \mathbb{R}\,.$$

(Notice that this argument with γ_0 replaced by γ also applies in the situation of (a)(i).)

(iii). To complete the proof of (a) it remains to check that $V(x) = +\infty$ for all $x \in \mathbb{R}$ provided $a > R$. Notice that the case $R = 0$ is not excluded and is equivalent to $\gamma_0 = 0$. Further notice that ψ assumes its infimum on $[0, \infty)$ since we assume $\mathbb{P}(T < 0) > 0$. Recall that the unique minimizer of ψ is denoted by γ_0 and that $\psi'(\gamma_0)$, the left derivative of ψ at 0, exists and is ≤ 0 if $\gamma_0 > 0$.

SUBCASE (iii-I): $\gamma_0 > 0$. If $\psi'(\gamma_0) < 0$, then, for any $c > 0$, we consider a zero-delayed random walk, $(T_{c,n})_{n\in\mathbb{N}_0}$ say, with steps distributed like $T\,\mathbb{1}_{\{T\geq -c\}}$. Set $\psi_c(t) := \mathbb{E}\,e^{-tT_{c,1}}$ and notice that ψ_c is finite on $[0,\infty)$ and that $R_c := -\log\inf_{t\geq 0}\psi_c(t) \geq R$. If c is large enough, $\psi_c(t) \to \infty$ as $t \to \infty$. Thus ψ_c has a unique minimizer on $[0,\infty)$, γ_c say, and $\psi_c'(\gamma_c) = 0$. It is easily seen that $\gamma_0 \leq \gamma_c$ and that $\gamma_c \downarrow \gamma_0$ as $c \uparrow \infty$. Some elementary analysis now shows that R_c converges to R as $c \to \infty$. Moreover,

$$\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x) \;\geq\; \sum_{n\geq 0} e^{an}\,\mathbb{P}(T_{c,n} \leq x), \qquad x \in \mathbb{R}\,. \tag{2.7}$$

Therefore, if we can prove that provided $a > R_c$ the series $\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_{c,n} \leq x)$ diverges, this will imply (after choosing c sufficiently large) that provided $a > R$

the series $\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x)$ diverges. Thus we have shown that, without loss of generality, we can work under the additional assumption $\psi'(\gamma_0) = 0$. Condition $a > R$ now reads as $\psi(\gamma_0) > e^{-a}$. By using the probability measure $\mathbb{P}_{\gamma_0}$ defined in (2.1) we conclude that $\mathbb{E}_{\gamma_0} T_1 = 0$. Hence, the random walk $(T_n)_{n\in\mathbb{N}_0}$ is recurrent under $\mathbb{P}_{\gamma_0}$. As a consequence, the renewal measure U_{γ_0} of $(T_n)_{n\in\mathbb{N}_0}$ under $\mathbb{P}_{\gamma_0}$ satisfies $U_{\gamma_0}(I) = \infty$ for any open non-empty interval I if (T_n) is non-arithmetic, and for any open non-empty interval I which contains some point $n\lambda$, $n \in \mathbb{Z}$, if (T_n) is arithmetic with span λ, respectively. As a consequence,

$$\begin{aligned}\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x) \;&=\; \sum_{n\geq 0}(\psi(\gamma_0)e^a)^n\,\mathbb{E}_{\gamma_0} e^{\gamma_0 T_n}\,\mathbb{1}_{\{T_n\leq x\}}\\ &\geq\; \int_{(-\infty,x]} e^{\gamma_0 y}\,U_{\gamma_0}(dy) \;=\; \infty\end{aligned}$$

for every $x \in \mathbb{R}$.
Subcase (iii-II): $\gamma_0 = 0$. We have $\psi(t) \in (1,\infty]$ for all $t > 0$. If $T_n \to -\infty$ a.s., then $\mathbb{P}(T_n \leq x) \to 1$, as $n \to \infty$, and the infinite series $\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x)$ diverges. Thus we are left with the situation that either $T_n \to \infty$ a.s. or $(T_n)_{n\in\mathbb{N}_0}$ oscillates. In both cases, $\mathbb{E}\,T_{1,c} \in (0,\infty]$, where $T_{1,c}$ is defined as above, and the Laplace transform ψ_c of $T_{c,1}$ is finite on $[0,\infty)$ and assumes its minimum at some $\gamma_c > 0$ satisfying $\psi_c'(\gamma_c) = 0$. Now we can argue as in the subcase (iii-I) to show that $\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x) = \infty$ for all $x \in \mathbb{R}$.

Case (b). In this case $R = +\infty$. Pick any $x > 0$ ($V(x) = 0$ for $x \leq 0$) and choose $r > 0$ such that $a < -\log\psi(r)$ and proceed in the same way as under (a)(i).

Case (c). Note that we have $\beta \in (0,1)$. Choose $a \in (0, -\log\beta)$. The subsequent proof literally repeats that given under the assumption (a)(i).
Conversely, $\mathbb{P}(T_n = 0) = \beta^n$, $n \in \mathbb{N}_0$. Therefore, if $a \geq -\log\beta$, then $V(0) = +\infty$ which implies that $V(x) = +\infty$ for all $x \geq 0$. □

Proof of Theorem 2.2. By Theorem 2.1, $V(x) < \infty$ for every $x \in \mathbb{R}$. Under $\mathbb{P}_\gamma$, the probability measure defined in (2.1), $(T_n)_{n\in\mathbb{N}_0}$ forms a random walk with Laplace transform

$$\psi_\gamma(t) \;=\; \mathbb{E}_\gamma\, e^{-tT_1} \;=\; e^a\,\mathbb{E}\,e^{-(\gamma+t)T} \;=\; e^a\psi(\gamma+t) \tag{2.8}$$

and drift

$$\nu_\gamma \;=\; -\,\psi_\gamma'(0) \;=\; -\,e^a\psi'(\gamma) \;\in\; (0,\infty). \tag{2.9}$$

For $x \in \mathbb{R}$, we write $V(x)$ in the following form

$$V(x) \;=\; \sum_{n\geq 0}\mathbb{E}_\gamma\, e^{\gamma T_n}\,\mathbb{1}_{\{T_n\leq x\}} \;=\; \int_{(-\infty,x]} e^{\gamma y}\,U_\gamma(dy) \;=:\; e^{\gamma x}Z(x), \tag{2.10}$$

where U_γ denotes the renewal measure of the process $(T_n)_{n\geq 0}$ under $\mathbb{P}_\gamma$.
Assume that $(T_n)_{n\in\mathbb{N}_0}$ is non-arithmetic. Since

$$\begin{aligned} Z(x) \;&=\; e^{-\gamma x}\int_{(-\infty,x]} e^{\gamma y}\,U_\gamma(dy)\\ &=\; \int e^{-\gamma(x-y)}\,\mathbb{1}_{[0,\infty)}(x-y)\,U_\gamma(dy)\end{aligned}$$

and the function $x \mapsto e^{-\gamma x}\,\mathbb{1}_{[0,\infty)}(x)$ is directly Riemann integrable we can invoke the key renewal theorem on the whole line to conclude that

$$\begin{aligned} e^{-\gamma x}\sum_{n\geq 0} e^{an}\,\mathbb{P}(T_n \leq x) \quad &= \quad Z(x) \\ &\underset{x\to\infty}{\longrightarrow} \quad \frac{1}{\nu_\gamma}\int_0^\infty e^{-\gamma y}\,dy \;=\; \frac{1}{\gamma\nu_\gamma}, \end{aligned}$$

where we have used $\nu_\gamma > 0$. This in combination with (2.9) immediately implies (2.4).

Asymptotics (2.5) in the arithmetic case can be treated similarly. Finally, (2.6) follows by an application of (2.5) to

$$e^{-\gamma\lambda n}\left(V(\lambda n) - V(\lambda(n-1))\right) = e^{-\gamma\lambda n}V(\lambda n) - e^{-\gamma\lambda}\left(e^{-\gamma\lambda(n-1)}V(\lambda(n-1))\right).$$

□

3 Proof of Theorem 1.1

For any $u \in \mathbb{V}$, let $W_1(u)$ denote a copy of W_1 but based on the point process $\mathcal{M}(u)$ instead of $\mathcal{M} = \mathcal{M}(\varnothing)$, that is, if $W_1 = \psi(\mathcal{M})$ for an appropriate measurable function ψ, then $W_1(u) := \psi(\mathcal{M}(u))$. In this situation, let

$$\begin{aligned} \widetilde{W}_{n+1} \quad &:= \quad \sum_{|u|=n} Y_u W_1(u)\,\mathbb{1}_{\{e^{an}Y_uW_1(u)\leq 1\}} \\ \text{and} \qquad R_n \quad &:= \quad \mathbb{E}(W_n - \widetilde{W}_{n+1}\,|\,\mathcal{F}_n), \qquad n \in \mathbb{N}_0. \end{aligned}$$

For $a > 0$ define a measure V_a on by

$$V_a(x) \;:=\; V_a((0,x]) \;:=\; \sum_{n\geq 0} e^{an}\,\mathbb{P}(S_n - an \leq \log x) \qquad (x > 0).$$

Given next is a result on the asymptotic behavior of V_a and two integrals involving V_a which play an important role in the proof of Theorem 1.1.

Lemma 3.1. *Let $a > 0$ be given. Assume that*

$$e^a m^{1/r}(r) \leq 1 \qquad \textit{for some } r > 1$$

and define ϑ to be the minimal $r > 1$ such that $e^a m^{1/r}(r) = 1$. In case when $a = -\log\inf_{r\geq 1} m^{1/r}(r)$ (which implies $\vartheta = \vartheta_0$) assume further that

$$-\frac{\log m(\vartheta_0)}{\vartheta_0} \;<\; -\frac{m'(\vartheta_0)}{m(\vartheta_0)}. \tag{3.1}$$

Then, as $x \to \infty$,

$$V_a(x) \;\sim\; \frac{x^{\vartheta-1}}{(\vartheta-1)(e^{a\vartheta}(-m'(\vartheta)) - a)}, \tag{3.2}$$

$$\int_{(0,x]} y\,V_a(dy) \;\sim\; \frac{x^{\vartheta}}{\vartheta(e^{a\vartheta}(-m'(\vartheta)) - a)} \tag{3.3}$$

if the random walk $(S_n)_{n\in\mathbb{N}_0}$ is non-arithmetic. If $(S_n - an)_{n\in\mathbb{N}_0}$ has span $\lambda_a > 0$, then, analogously, as $n \to \infty$

$$V_a(e^{\lambda_a n}) \;\sim\; \frac{\lambda_a e^{(\vartheta-1)\lambda_a n}}{(1-e^{-\lambda_a(\vartheta-1)})(e^{a\vartheta}(-m'(\vartheta))-a)}, \tag{3.4}$$

$$\int_{(0,e^{\lambda_a n}]} y\, V_a(dy) \;\sim\; \frac{\lambda_a e^{\vartheta\lambda_a n}}{(1-e^{-\lambda_a\vartheta})(e^{a\vartheta}(-m'(\vartheta))-a)}. \tag{3.5}$$

Further, if $\vartheta < 2$, then, in the non-arithmetic case, as $x \to \infty$,

$$\int_{(x,\infty)} y^{-1}\, V_a(dy) \;\sim\; \frac{x^{\vartheta-2}}{(2-\vartheta)(e^{a\vartheta}(-m'(\vartheta))-a)}, \tag{3.6}$$

whereas, in the arithmetic case, as $n \to \infty$,

$$\int_{[e^{\lambda_a n},\infty)} y^{-1}\, V_a(dy) \;\sim\; \frac{\lambda_a e^{(\vartheta-2)\lambda_a n}}{(1-e^{(\vartheta-2)\lambda_a})(a^{a\vartheta}(-m'(\vartheta))-a)}. \tag{3.7}$$

Proof. Let φ and φ_a be the Laplace transforms of the increment distributions of the associated random walk $(S_n)_{n\in\mathbb{N}_0}$ defined by (1.1) and of the shifted random walk $(S_n - an)_{n\in\mathbb{N}_0}$, respectively. Our purpose is to check that under the assumptions of the lemma Theorem 2.2 applies to the random walk $(S_n - an)_{n\in\mathbb{N}_0}$ with $\psi = \varphi_a$ and $R = -\log\inf_{t\geq 0}\varphi_a(t)$.

By definition, $\varphi_a(t) = e^{at}\varphi(t) = e^{at}m(1+t)$ which implies that

$$e^a\varphi_a(r-1) \;=\; e^{ar}m(r) \;\leq\; 1. \tag{3.8}$$

Therefore, in the notation of Theorem 2.2 condition $a \leq R$ holds. In the case $a = -\log\inf_{t\geq 1} m^{1/t}(t)$, we have $a = -\log\inf_{t\geq 0}\varphi_a(t) = R$ in view of (3.8). With $\psi = \varphi_a$, γ_0 defined on p. 6 equals $\vartheta_0 - 1$. Therefore, condition (2.3) reads $\varphi_a'(\vartheta_0 - 1) < 0$ and is a consequence of (3.1). In any case, Theorem 2.2 applies with γ being the minimal $t > 0$ satisfying $\varphi_a(t) = e^{-a}$ that is, $\gamma = \vartheta - 1$, and yields

$$V_a(x) \;\sim\; \frac{e^{-a}}{(\vartheta-1)(-\varphi_a'(\vartheta-1))}\, x^{\vartheta-1} \qquad (x\to\infty), \tag{3.9}$$

in case when $(S_n - an)_{n\in\mathbb{N}_0}$ is non-arithmetic, and

$$V_a(e^{\lambda_a n}) \;\sim\; \frac{\lambda_a e^{-a}}{(1-e^{-\lambda_a(\vartheta-1)})(-\varphi_a'(\vartheta-1))}\, e^{(\vartheta-1)\lambda_a n} \qquad (n\to\infty), \tag{3.10}$$

in case when $(S_n - an)_{n\in\mathbb{N}_0}$ is arithmetic with span λ_a.

Now first notice that (3.9) proves (3.2) and that (3.10) implies (3.4). Secondly, in the non-arithmetic case, asymptotics (3.3) and (3.6) follow from (3.2) by integration by parts and subsequent application of Propositions 1.5.8 and 1.5.10 in [6], respectively. Finally, in the lattice case, asymptotics (3.5) and (3.7) follow by an elementary analysis from (3.4) and the corresponding asymptotic for $V_a(e^{\lambda_a n}) - V_a(e^{\lambda_a(n-1)})$, which can be derived from (2.6). We omit the details. □

Lemma 3.2. *Let $a > 0$ be given. Assume that*

$$e^a m^{1/r}(r) \;\leq\; 1 \qquad \text{for some } r > 1$$

and define ϑ to be the minimal $r > 1$ such that $e^a m^{1/r}(r) = 1$. In case when $a = -\log \inf_{r\geq 1} m^{1/r}(r)$ (which implies $\vartheta = \vartheta_0$) assume further that

$$-\frac{\log m(\vartheta_0)}{\vartheta_0} \;<\; -\frac{m'(\vartheta_0)}{m(\vartheta_0)}.$$

Then $\mathbb{E}\, W_1^{\vartheta} < \infty$ implies

$$\sum_{n\geq 0} \mathbb{P}(\widetilde{W}_{n+1} \neq W_{n+1}) \;<\; \infty \quad \text{and} \quad \mathbb{E} \sum_{n\geq 0} e^{an} R_n \;<\; \infty. \tag{3.11}$$

If, moreover, $\vartheta < 2$, then $(M_n)_{n\in\mathbb{N}_0}$ is an $\mathcal{L}^2$-bounded martingale, where

$$M_n \;:=\; \sum_{k=0}^{n} e^{ak}(\widetilde{W}_{k+1} - W_k + R_k), \qquad n \in \mathbb{N}_0\,.$$

Remark 3.3. Note that the second infinite series in (3.11) is well-defined since all summands are non-negative. Indeed, for any $n \in \mathbb{N}_0$, by the independence of $W_1(v)$ and $\mathcal{F}_n$ for $|v| = n$,

$$\begin{aligned}
R_n \;&=\; \sum_{|v|=n} Y_v\, \mathbb{E}(1 - W_1(v)\, \mathbb{1}_{\{e^{an}Y_v W_1(v)\leq 1\}} \mid \mathcal{F}_n) \\
&=\; \sum_{|v|=n} Y_v\, \mathbb{E}(W_1(v) - W_1(v)\, \mathbb{1}_{\{e^{an}Y_v W_1(v)\leq 1\}} \mid \mathcal{F}_n) \\
&=\; \sum_{|v|=n} Y_v \int_{(e^{-an}Y_v^{-1},\infty)} x\, F(dx) \;\geq\; 0 \qquad \text{a.s.}
\end{aligned}$$

Proof of Lemma 3.2.

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{P}(\widetilde{W}_{n+1} \neq W_{n+1}) \;&\leq\; \sum_{n\geq 0} \mathbb{E} \sum_{|v|=n} \mathbb{P}(e^{an} Y_v W_1(v) > 1 \mid \mathcal{F}_n) \\
&=\; \sum_{n\geq 0} \mathbb{E} \sum_{|v|=n} Y_v e^{S(v)} \int \mathbb{1}_{(e^{-an}e^{S(v)},\,\infty)}(x)\, F(dx) \\
&=\; \sum_{n\geq 0} \mathbb{E}\, e^{S_n} \int \mathbb{1}_{(e^{S_n - an},\,\infty)}(x)\, F(dx) \\
&=\; \int \sum_{n\geq 0} \mathbb{E}\, e^{S_n}\, \mathbb{1}_{\{e^{S_n - an} < x\}}\; F(dx) \\
&=\; \int \sum_{n\geq 0} e^{an}\, \mathbb{E}\, e^{S_n - an}\, \mathbb{1}_{(0,x)}(e^{S_n - an})\, F(dx) \\
&=\; \int \int_{(0,x)} y\, V_a(dy)\, F(dx).
\end{aligned}$$

Using (3.3) or (3.5), respectively, yields

$$\sum_{n\geq 0}\mathbb{P}(\widetilde{W}_{n+1}\neq W_{n+1}) \;\leq\; \int O\left(x^{\vartheta}\right)\,F(dx) \;<\; \infty.$$

Concerning the second series in (3.11), we obtain by using the calculations from Remark 3.3 that

$$\begin{aligned}
\mathbb{E}\sum_{n\geq 0} e^{an}R_n \;&=\; \sum_{n\geq 0} e^{an}\,\mathbb{E}\sum_{|v|=n} Y_v \int_{(e^{-an}Y_v^{-1},\infty)} x\,F(dx)\\
&=\; \sum_{n\geq 0} e^{an}\,\mathbb{E}\int_{(e^{S_n-an},\infty)} x\,F(dx)\\
&=\; \int x\sum_{n\geq 0} e^{an}\,\mathbb{P}(S_n-an<\log x)\,F(dx)\\
&\leq\; \int xV_a(x)\,F(dx).
\end{aligned}$$

In view of (3.2) and (3.4), we conclude that

$$\begin{aligned}
\mathbb{E}\sum_{n\geq 0} e^{an}R_n \;&\leq\; \int x\,O\left(x^{\vartheta-1}\right)\,F(dx)\\
&=\; \int O\left(x^{\vartheta}\right)\,F(dx) \;<\; \infty.
\end{aligned}$$

Now we turn to the final assertion of the lemma. Since $R_n = \mathbb{E}(W_n - \widetilde{W}_{n+1}\,|\,\mathcal{F}_n)$, we have $\widetilde{W}_{n+1} - W_n + R_n = \widetilde{W}_{n+1} - \mathbb{E}(\widetilde{W}_{n+1}\,|\,\mathcal{F}_n)$ a.s. In particular, $(M_n)_{n\geq 0}$ constitutes a martingale. It remains to prove that $(M_n)_{n\in\mathbb{N}_0}$ is $\mathcal{L}^2$-bounded. For this purpose, note that

$$\begin{aligned}
\mathbb{E}(e^{an}(\widetilde{W}_{n+1}-W_n+R_n))^2 \;&=\; e^{2an}\,\mathbb{E}\left(\widetilde{W}_{n+1}-W_n+R_n\right)^2\\
&=\; e^{2an}\,\mathbb{E}\,\mathrm{Var}(\widetilde{W}_{n+1}\,|\,\mathcal{F}_n)\\
&\leq\; e^{2an}\,\mathbb{E}\sum_{|v|=n} Y_v^2\int_{(0,\,e^{-an}Y_v^{-1}]} x^2\,F(dx).
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\sum_{n\geq 0} \mathbb{E}(e^{an}(\widetilde{W}_{n+1} - W_n + R_n))^2 \\
&\quad\leq \sum_{n\geq 0} e^{2an}\, \mathbb{E} \sum_{|v|=n} Y_v^2 \int_{(0,\, e^{-an}Y_v^{-1}]} x^2\, F(dx) \\
&\quad= \sum_{n\geq 0} e^{an}\, \mathbb{E} \sum_{|v|=n} Y_v e^{an-S(v)} \int_{(0,\, e^{S(v)-an}]} x^2\, F(dx) \\
&\quad= \sum_{n\geq 0} e^{an}\, \mathbb{E}\, e^{-(S_n-an)} \int_{(0,\, e^{S_n-an}]} x^2\, F(dx) \\
&\quad= \int_{(0,\infty)} x^2 \sum_{n\geq 0} e^{an}\, \mathbb{E}\, e^{-(S_n-an)}\, \mathbb{1}_{\{e^{S_n-an}\geq x\}}\, F(dx) \\
&\quad= \int_{(0,\infty)} x^2 \int_{[x,\infty)} y^{-1}\, V_a(dy)\, F(dx).
\end{aligned}
$$

Now use (3.6) or (3.7) to obtain the finiteness of the last expression. □

The proof of the next result can be found in Remark 3.2 in [2].

Lemma 3.4. (1.2) *holds if and only if*

$$\sum_{n\geq 0} e^{an}(W_{n+1} - W_n) \qquad \textit{converges a.s.}$$

Proof of Theorem 1.1. Under the assumptions of the theorem, Lemma 3.2 implies that $(M_n)_{n\in\mathbb{N}_0}$ is an $\mathcal{L}^2$-bounded martingale, in particular,

$$\sum_{n\geq 0} e^{an}(\widetilde{W}_{n+1} - W_n + R_n) \qquad \text{converges a.s.}$$

This is equivalent to the a.s. convergence of the series $\sum_{n\geq 0} e^{an}(\widetilde{W}_{n+1} - W_n)$ since $\sum_{n\geq 0} e^{an}R_n < \infty$ a.s. in view of (3.11). Another appeal to (3.11) yields $\sum_{n\geq 0} \mathbb{P}(\widetilde{W}_{n+1} \neq W_{n+1}) < \infty$ which directly implies $\mathbb{P}(\widetilde{W}_{n+1} \neq W_{n+1} \text{ i.o.}) = 0$ by an application of the Borel-Cantelli Lemma. Hence,

$$\sum_{n\geq 0} e^{an}(W_{n+1} - W_n) \qquad \text{converges a.s.,}$$

which is equivalent to (1.2) by Lemma 3.4. □

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