A FUNCTIONAL LIMIT THEOREM FOR THE PROFILE OF RANDOM RECURSIVE TREES

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ABSTRACT. Let $X_n(k)$ be the number of vertices at level k in a random recursive tree with $n+1$ vertices. We prove a functional limit theorem for the vector-valued process $(X_{[n^t]}(1),\dots,X_{[n^t]}(k))_{t\geq 0}$, for each $k\in\mathbb{N}$. We show that after proper centering and normalization, this process converges weakly to a vector-valued Gaussian process whose components are integrated Brownian motions. This result is deduced from a functional limit theorem for Crump-Mode-Jagers branching processes generated by increasing random walks with increments that have finite second moment.

1. Introduction and main results

A (deterministic) *recursive tree* with n vertices is a rooted tree with vertices labeled with $1,2\dots,n$ that satisfies the following property: the root is labeled with 1, and the labels of the vertices on the unique path from the root to any other vertex (labeled with $m\in\{2,\dots,n\}$) form an increasing sequence. There are $(n-1)!$ different recursive trees with n vertices, and we denote them $T_{1,n},T_{2,n},\dots,T_{(n-1)!,n}$. A random object $\mathcal{T}_n$ is called *random recursive tree* with n vertices if

$$\mathbb{P}\{\mathcal{T}_n=T_{i,n}\}=\frac{1}{(n-1)!},\quad i=1,2,\dots,(n-1)!.$$

A simple way to generate a random recursive tree is as follows. At time 0 start with a tree consisting of a single vertex (the root) labeled with 1. At each time n, given a recursive tree with $n+1$ vertices, choose one vertex uniformly at random and add to this vertex an offspring labeled by n. The random tree obtained at time n has the same distribution as $\mathcal{T}_{n+1}$. We refer the reader to Chapter 6 of [6] for more information.

For $k\in\mathbb{N}$, let $X_n(k)$ denote the number of vertices at level k in a random recursive trees on $n+1$ vertices. The level of a vertex is, by definition, its distance to the root. The root is at level 0. The function $k\mapsto X_n(k)$ is usually referred to as the *profile* of the tree. In Theorem 3 of [8] it was shown by using analytic tools that for any fixed $k\in\mathbb{N}$,

$$\frac{(k-1)!\sqrt{2k-1}\big(X_n(k)-(\log n)^k/k!\big)}{(\log n)^{k-1/2}}\ \underset{n\to\infty}{\overset{\mathrm{d}}{\longrightarrow}}\ \text{normal}(0,1). \tag{1.1}$$

Furthermore, the uniform in $k=1,2,\dots,o(\log n)$ rate of convergence in the uniform metric was obtained. The profiles of random recursive trees (along with closely related binary search trees) have been much studied at the central limit regime levels $k(n)=\log n+c\sqrt{\log n}+o(\sqrt{\log n})$, $c\in\mathbb{R}$, and at the large deviation regime levels of the form $k(n)\sim\alpha n$, $\alpha>0$; see [4, 5, 7, 16, 17]. Apart from [8], we are aware of only one work studying vertices of random recursive trees at a fixed level. It is shown in [1] that the proportion of vertices at level $k\in\mathbb{N}$ having more than $t\log n$ descendants converges to $(1-t)^k$ a.s. Also, a Poisson limit theorem is proved in [1] for the number of vertices at fixed level k that have a fixed number of descendants.

In this paper we are interested in weak convergence of the random process $\big(X_{[n^t]}(1),\dots,X_{[n^t]}(k)\big)_{t\geq 0}$ for each $k\in\mathbb{N}$, properly normalized and centered, as $n\to\infty$. The latter vector might be called the *low levels profile*.

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Theorem 1.1. *The following functional limit theorem holds for the low levels profile of a random recursive tree:*

$$\left(\frac{(k-1)!\big(X_{[n^{(\cdot)}]}(k)-((\log n)\cdot)^k/k!\big)}{(\log n)^{k-1/2}}\right)_{k\in\mathbb{N}} \underset{n\to\infty}{\Rightarrow} \left(\int_{[0,\,\cdot]}(\cdot-y)^{k-1}\mathrm{d}B(y)\right)_{k\in\mathbb{N}} \tag{1.2}$$

in the product J_1-topology on $D^{\mathbb{N}}$, where $(B(u))_{u\geq 0}$ is a standard Brownian motion and $D=D[0,\infty)$ is the Skorokhod space.

Remark 1.2. While the stochastic integral $R_1(s):=\int_{[0,\,s]}\mathrm{d}B(y)$ on the right-hand side of (1.2) is interpreted as $B(s)$, the other stochastic integrals can be defined via integration by parts which yields

$$R_k(s):=\int_{[0,\,s]}(s-y)^{k-1}\mathrm{d}B(y)=(k-1)!\int_0^{s_1}\int_0^{s_2}\ldots\int_0^{s_{k-1}}B(y)\mathrm{d}y\mathrm{d}s_{k-1}\ldots\mathrm{d}s_2$$

for integer $k\geq 2$ and $s\geq 0$, where $s_1=s$. Depending on whether the left- or right-hand representation is used the latter process is known in the literature as a Riemann-Liouville process or an integrated Brownian motion. It can be checked (details can be found in Section 2 of [12]) that $R_k(s)$ has the same distribution as $\sqrt{s^{2k-1}/(2k-1)}B(1)$ for each $s\geq 0$ and $k\in\mathbb{N}$. In particular, $\mathbb{E}\,R_k^2(s)=s^{2k-1}/(2k-1)$. Along similar lines one can also show that

$$\mathbb{E}\,R_k(s)R_l(u)=\int_0^{u\wedge s}(s-y)^{k-1}(u-y)^{l-1}\mathrm{d}y=\begin{cases}\sum_{j=0}^{l-1}\binom{l-1}{j}\frac{1}{k+j}s^{k+j}(u-s)^{l-1-j}, & \text{if } u\geq s\geq 0,\\ \sum_{j=0}^{k-1}\binom{k-1}{j}\frac{1}{l+j}u^{l+j}(s-u)^{k-1-j}, & \text{if } 0\leq u<s\end{cases}$$

for $k,l\in\mathbb{N}$. Observe that the aforementioned distributional equality shows that taking in (1.2) $(\cdot)=1$ and any fixed k we obtain (1.1). Moreover, taking $(\cdot)=1$ and $k=1,2,\ldots$, we obtain the following multivariate central limit theorem for the low levels profile:

$$\left(\frac{(k-1)!\big(X_n(k)-(\log n)^k/k!\big)}{(\log n)^{k-1/2}}\right)_{k\in\mathbb{N}} \overset{\mathrm{d}}{\underset{n\to\infty}{\longrightarrow}} (R_k(1))_{k\in\mathbb{N}}$$

weakly on $\mathbb{R}^{\mathbb{N}}$ endowed with the product topology, where the limit is a centered Gaussian process with covariance function

$$\mathbb{E}\,R_k(1)R_l(1)=\frac{1}{k+l-1},\quad k,l\in\mathbb{N}.$$

2. Our approach and an auxiliary tool

In order to explain our approach that we use to prove Theorem 1.1 we need more notation.

Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. positive random variables with generic copy ξ. Denote by $S:=(S_n)_{n\in\mathbb{N}}$ the ordinary random walk with jumps ξ_n for $n\in\mathbb{N}$, that is, $S_n=\xi_1+\ldots+\xi_n$, $n\in\mathbb{N}$. Further, we define the renewal process $(N(t))_{t\in\mathbb{R}}$ by

$$N(t):=\sum_{k\geq 1}\mathbb{1}_{\{S_k\leq t\}},\quad t\in\mathbb{R}.$$

Set $U(t):=\mathbb{E}\,N(t)$ for $t\in\mathbb{R}$, so that, with a slight abuse of terminology, U is the renewal function. Clearly, $N(t)=0$ a.s. and $U(t)=0$ for $t<0$.

Next, we recall the construction of a Crump-Mode-Jagers branching process in the special case when it is generated by the random walk S. At time $\tau_0=0$ there is one individual, the ancestor. The ancestor produces offspring (the first generation) with birth times given by a point process $\mathcal{Z}=\sum_{n\geq 1}\delta_{S_n}$ on $\mathbb{R}_+:=[0,\infty)$. The first generation produces the second generation. The shifts of birth times of the second generation individuals with respect to their mothers' birth times are distributed according to independent copies of the same point process $\mathcal{Z}$. The second generation produces the third one, and so on. All individuals act independently of each other. Equivalently, one may consider a branching random walk. In this case, the points of $\mathcal{Z}$ are interpreted as the positions of the first generation individuals. Each individual in the first generation produces individuals from the second generation whose displacements with respect to the position of their respective mother are given by an independent copy of $\mathcal{Z}$, and so on.

For $k\in\mathbb{N}$, denote by $Y_k(t)$ the number of the kth generation individuals with birth times $\leq t$. Plainly, $Y_1(t)=N(t)$ for $t\geq 0$. We recall that $0!=1$. For $n\in\mathbb{N}$, denote by τ_n the birth time of the nth individual (in the chronological order of birth times, excluding the ancestor).

Now we are ready to point out the basic observation for the proof of Theorem 1.1: if ξ has an exponential distribution of unit mean, then the following distributional equality of stochastic processes holds true:

$$(X_{[n^s]}(k))_{s\geq 0, k\in\mathbb{N}} \overset{\mathrm{d}}{=} (Y_k(\tau_{[n^s]}))_{s\geq 0, k\in\mathbb{N}}. \tag{2.1}$$

In the following, we shall simply identify these processes. Formula (2.1) follows from the fact observed by B. Pittel, see p. 339 in [18], that the tree formed by the individuals in combination with their family relations at time τ_n is a version of a random recursive tree with $n+1$ vertices. To give a short explanation, imagine that a random recursive tree is generated in continuous time as follows. Start at time 0 with one vertex, the root. At any time, any vertex in the tree generates with intensity 1 a single offspring, and all vertices act independently. Then, the birth times of the vertices at the first level form a Poisson point process with intensity 1. More generally, if some vertex was born at time t, then the birth times of its offspring minus t form an independent copy of the Poisson point process. This system can be identified with the Crump-Mode-Jagers process generated by an ordinary random walk with jumps having the exponential distribution of unit mean. If τ_n is the birth time of the nth vertex, then the genealogical tree of the vertices with birth times in the interval $[0, \tau_n]$ is a random recursive tree. The embedding into a continuous time process just described was used in [5, 17, 18].

Theorem 2.1 given next is our main technical tool for proving Theorem 1.1. We stress that here, the distribution of ξ is not assumed exponential, so that Theorem 2.1 is far more general than what is needed to treat random recursive trees.

Theorem 2.1. *Suppose that $\sigma^2 := \operatorname{Var}\xi \in (0,\infty)$. Then*

$$\left(\frac{(k-1)!\big(Y_k(t\cdot) - (t\cdot)^k/(k!\mu^k)\big)}{\sqrt{\sigma^2\mu^{-2k-1}t^{2k-1}}}\right)_{k\in\mathbb{N}} \underset{t\to\infty}{\Rightarrow} (R_k(\cdot))_{k\in\mathbb{N}} \tag{2.2}$$

in the product J_1-topology on $D^{\mathbb{N}}$, where $\mu := \mathbb{E}\,\xi < \infty$.

For $i\in\mathbb{N}$, consider the 1st generation individual born at time S_i and denote by $Y_j^{(i)}(t)$ for $j\in\mathbb{N}$ the number of her successors in the $(j+1)$st generation with birth times $\leq t+S_i$. By the branching property $(Y_j^{(1)}(t))_{t\geq 0}$, $(Y_j^{(2)}(t))_{t\geq 0}, \ldots$ are independent copies of $(Y_j(t))_{t\geq 0}$ which are independent of S. With this at hand we are ready to write the basic representation

$$Y_k(t) = \sum_{i\geq 1} Y_{k-1}^{(i)}(t-S_i), \quad t\geq 0, k\geq 2.$$

Note that, for $k\geq 2$, $(Y_k(t))_{t\geq 0}$ is a particular instance of a random process with immigration at the epochs of a renewal process which is a renewal shot noise process with random and independent response functions (the term was introduced in [15]; see also [13] for a review).

For $t\geq 0$ and $k\in\mathbb{N}$, set $U_k(t) := \mathbb{E}\,Y_k(t)$ and observe that, $U_1(t) = U(t)$ and

$$U_k(t) = \int_{[0,\,t]} U_{k-1}(t-y)\mathrm{d}U(y) = \int_{[0,\,t]} U(t-y)\mathrm{d}U_{k-1}(y).$$

Our strategy of the proof of Theorem 2.1 is the following. Using a decomposition

$$\begin{aligned} Y_k(t) - \frac{t^k}{k!\mu^k} &= \sum_{j\geq 1}\big(Y_{k-1}^{(j)}(t-S_j) - U_{k-1}(t-S_j)\,\mathbb{1}_{\{S_j\leq t\}}\big) \\ &+ \left(\sum_{j\geq 1} U_{k-1}(t-S_j)\,\mathbb{1}_{\{S_j\leq t\}} - \mu^{-1}\int_0^t U_{k-1}(y)\mathrm{d}y\right) \\ &+ \left(\mu^{-1}\int_0^t U_{k-1}(y)\mathrm{d}y - \frac{t^k}{k!\mu^k}\right) =: Y_{k,1}(t) + Y_{k,2}(t) + Y_{k,3}(t) \end{aligned}$$

for $k\geq 2$, we shall prove three statements: for all $T>0$,

$$\frac{\sup_{0\leq s\leq T}|Y_{k,1}(st)|}{t^{k-1/2}} \overset{\mathbb{P}}{\to} 0, \quad t\to\infty; \tag{2.3}$$

$$\lim_{t\to\infty} t^{-(k-1/2)}\sup_{0\leq s\leq T}|Y_{k,3}(st)| = 0, \tag{2.4}$$

and

$$\left(\frac{Y_1(t\cdot)-\mu^{-1}(t\cdot)}{\sqrt{\sigma^2\mu^{-3}t}}, \frac{(k-1)!Y_{k,2}(t\cdot)}{\sqrt{\sigma^2\mu^{-2k-1}t^{2k-1}}}\right)_{k\geq 2} \underset{t\to\infty}{\Rightarrow} (R_k(\cdot))_{k\in\mathbb{N}} \tag{2.5}$$

in the product J_1-topology on $D^{\mathbb{N}}$. Plainly, (2.3), (2.4) and (2.5) entail (2.2). Weak convergence of the coordinates in (2.5) is known: see Theorem 3.1 on p. 162 in [11] for the first coordinate and Theorem 1.1 in [12] for the others.

3. Proof of Theorem 1.1

Applying Theorem 2.1 to exponentially distributed ξ of unit mean (so that $\mu=\sigma^2=1$) we obtain

$$\left(\frac{(k-1)!\big(Y_k((\log n)\cdot)-((\log n)\cdot)^k/k!\big)}{(\log n)^{k-1/2}}\right)_{k\in\mathbb{N}} \underset{n\to\infty}{\Rightarrow} \big(R_k(\cdot)\big)_{k\in\mathbb{N}} \tag{3.1}$$

in the product J_1-topology on $D^{\mathbb{N}}$.

It is a classical fact that τ_n is the sum of n independent exponentially distributed random variables of means $1,2,\ldots,n$, whence $\lim_{n\to\infty}(\tau_n/\log n)=1$ a.s. Arguing as in the proof of Theorem 3 in [9] we conclude that, for each $T>0$, $\lim_{n\to\infty}\sup_{0\leq s\leq T}|\tau_{[n^s]}/\log n-\psi(s)|=0$ a.s., where $\psi(s)=s$ for $s\geq 0$. This in combination with (3.1) gives

$$\left(\left(\frac{(k-1)!\big(Y_k(\log n\cdot)-((\log n)\cdot)^k/k!\big)}{(\log n)^{k-1/2}}\right)_{k\in\mathbb{N}}, \frac{\tau_{[n^{(\cdot)}]}}{\log n}\right) \underset{n\to\infty}{\Rightarrow} \big(\big(R_k(\cdot)\big)_{k\in\mathbb{N}}, \psi(\cdot)\big)$$

in the product J_1-topology on $D^{\mathbb{N}}\times D$.

It is well-known (see, for instance, Lemma 2.3 on p. 159 in [11]) that, for fixed $j\in\mathbb{N}$, the composition mapping $((x_1,\ldots,x_j),\varphi)\mapsto(x_1\circ\varphi,\ldots,x_j\circ\varphi)$ is continuous at vectors $(x_1,\ldots,x_j):\mathbb{R}_+^j\to\mathbb{R}^j$ with continuous coordinates and nondecreasing continuous $\varphi:\mathbb{R}_+\to\mathbb{R}_+$, where $\mathbb{R}_+:=[0,\infty)$. Since R_k is a.s. continuous and ψ is nonnegative, nondecreasing and continuous, we can invoke the continuous mapping theorem to infer (1.2) with $Y_k(\tau_{[n^{(\cdot)}]})$ replacing $X_{[n^{(\cdot)}]}(k)$. In view of (2.1) this completes the proof of Theorem 1.1.

4. Proof of Theorem 2.1

It is well known that

$$-1\leq U(t)-t/\mu\leq c_0,\quad t\geq 0 \tag{4.1}$$

for appropriate constant $c_0>0$ whenever $\mathbb{E}\,\xi^2<\infty$. While the left-hand inequality follows from Wald's identity $t\leq\mathbb{E}\,S_{N(t)+1}=\mu(U(t)+1)$, the right-hand inequality is Lorden's inequality (see [3] for a short probabilistic proof in the situation where ξ has a nonlattice distribution). If the distribution of ξ is nonlattice, one can take $c_0=\operatorname{Var}\xi/\mathbb{E}\,\xi^2$, whereas if the distribution of ξ is δ-lattice, (4.1) holds with $c_0=2\delta/\mu+\operatorname{Var}\xi/\mathbb{E}\,\xi^2$. We shall need the following consequence of (4.1):

$$|U(t)-t/\mu|\leq c,\quad t\geq 0 \tag{4.2}$$

where $c=\max(c_0,1)$.

Lemma 4.1. *Under the assumption* $\mathbb{E}\,\xi^2<\infty$

$$\left|U_k(t)-\frac{t^k}{k!\mu^k}\right|\leq\sum_{i=0}^{k-1}\binom{k}{i}\frac{t^ic^{k-i}}{i!\mu^i},\quad k\in\mathbb{N},\ t\geq 0. \tag{4.3}$$

Proof. By using the mathematical induction we first show that

$$\left|\int_{[0,\,t]}(t-z)^m\mathrm{d}U(z)-\frac{t^{m+1}}{(m+1)\mu}\right|\leq ct^m,\quad m\in\mathbb{N}_0\,. \tag{4.4}$$

When $m=0$, (4.4) is a consequence of (4.2). Assuming that (4.4) holds for $m=j-1$ we obtain

$$\left|\int_{[0,\,t]}(t-z)^j\mathrm{d}U(z)-\frac{t^{j+1}}{(j+1)\mu}\right|=\left|j\int_0^t\left(\int_{[0,\,s]}(s-z)^{j-1}\mathrm{d}U(z)-\frac{s^j}{j\mu}\right)\mathrm{d}s\right|\leq j\int_0^t cs^{j-1}\mathrm{d}s=ct^j$$

which completes the proof of (4.4).

To prove (4.3) we once again use the mathematical induction. When $k = 1$, (4.3) coincides with (4.2). Assuming that (4.3) holds for $k \le j$ and appealing to (4.4) we infer

$$\begin{aligned}
&\left|U_{j+1}(t) - \frac{t^{j+1}}{(j+1)!\mu^{j+1}}\right| \\
\le\ & \int_{[0,\,t]} \left|U_j(t-z) - \frac{(t-z)^j}{j!\mu^j}\right| \mathrm{d}U(z) + \frac{1}{j!\mu^j}\left|\int_{[0,\,t]} (t-z)^j \mathrm{d}U(z) - \frac{t^{j+1}}{(j+1)\mu}\right| \\
\le\ & \int_{[0,\,t]} \sum_{i=0}^{j-1} \binom{j}{i} \frac{c^{j-i}}{i!\mu^i}(t-z)^i \mathrm{d}U(z) + \frac{ct^j}{j!\mu^j} \\
\le\ & \sum_{i=0}^{j-1} \binom{j}{i} \frac{c^{j+1-i}t^i}{i!\mu^i} + \sum_{i=0}^{j-1} \binom{j}{i} \frac{c^{j-i}t^{i+1}}{(i+1)!\mu^{i+1}} + \frac{ct^j}{j!\mu^j} \\
\le\ & c^{j+1} + \sum_{i=1}^{j-1} \left(\binom{j}{i} + \binom{j}{i-1}\right) \frac{c^{j+1-i}t^i}{i!\mu^i} + \frac{(j+1)ct^j}{j!\mu^j} = \sum_{i=0}^{j} \binom{j+1}{i} \frac{c^{j+1-i}t^i}{i!\mu^i}
\end{aligned}$$

□

Lemma 4.2. *Under the assumption* $\mathbb{E}\,\xi^2 < \infty$, *for* $k \in \mathbb{N}$,

$$D_k(t) := \operatorname{Var} Y_k(t) = O(t^{2k-1}), \quad t \to \infty \tag{4.5}$$

and, for $k \ge 2$,

$$\mathbb{E}[(Y_{k,1}(t))^2] = O(t^{2k-2}), \quad t \to \infty. \tag{4.6}$$

Proof. Using a decomposition

$$\begin{aligned}
Y_k(t) - U_k(t) &= \sum_{j\ge 1} \big(Y_{k-1}^{(j)}(t-S_j) - U_{k-1}(t-S_j)\big) \mathbb{1}_{\{S_j \le t\}} \\
&+ \left(\sum_{j\ge 1} U_{k-1}(t-S_j)\, \mathbb{1}_{\{S_j\le t\}} - U_k(t)\right) =: Y_{k,1}(t) + Y_{k,2}^*(t)
\end{aligned}$$

we infer

$$D_k(t) = \mathbb{E}[(Y_{k,1}(t))^2] + \mathbb{E}[(Y^*_{k,2}(t))^2]. \tag{4.7}$$

We start by proving the asymptotic relation

$$\begin{aligned}
\mathbb{E}[(Y^*_{k,2}(t))^2] &= \operatorname{Var}\left(\sum_{i\ge1} U_{k-1}(t-S_i)\, \mathbb{1}_{\{S_i\le t\}}\right) \\
&= \mathbb{E}\left(\sum_{i\ge1} U_{k-1}(t-S_i)\, \mathbb{1}_{\{S_i\le t\}}\right)^2 - U_k^2(t) = O(t^{2k-1}), \quad t\to\infty
\end{aligned} \tag{4.8}$$

for $k \ge 2$. To this end, we need the following formula

$$\mathbb{E}\left(\sum_{i\ge1} U_{k-1}(t-S_i)\, \mathbb{1}_{\{S_i\le t\}}\right)^2 = 2\int_{[0,\,t]} U_{k-1}(t-y)U_k(t-y)\mathrm{d}U(y) + \int_{[0,\,t]} U_{k-1}^2(t-y)\mathrm{d}U(y). \tag{4.9}$$

PROOF OF (4.9). Write

$$\mathbb{E}\left(\sum_{i\ge1} U_{k-1}(t-S_i)\, \mathbb{1}_{\{S_i\le t\}}\right)^2 = 2\,\mathbb{E}\sum_{1\le i<j} U_{k-1}(t-S_i)U_{k-1}(t-S_j)\, \mathbb{1}_{\{S_j\le t\}} + \mathbb{E}\sum_{i\ge1} U_{k-1}^2(t-S_i)\, \mathbb{1}_{\{S_i\le t\}}.$$

It is clear that the second expectation is equal to the second summand on the right-hand side of (4.9). Thus, it remains to show that the first expectation is equal to the first summand on the right-hand side of (4.9):

$$\begin{aligned}
&\mathbb{E}\sum_{1\leq i<j} U_{k-1}(t-S_i)U_{k-1}(t-S_j)\,\mathbb{1}_{\{S_j\leq t\}}\\
=\;& \mathbb{E}\sum_{i\geq 1} U_{k-1}(t-S_i)\big(U_{k-1}(t-S_{i+1})\,\mathbb{1}_{\{S_{i+1}\leq t\}} + U_{k-1}(t-S_{i+2})\,\mathbb{1}_{\{S_{i+2}\leq t\}}+\ldots\big)\\
=\;& \mathbb{E}\sum_{i\geq 1} U_{k-1}(t-S_i)\,\mathbb{1}_{\{S_i\leq t\}}\,\mathbb{E}\big(U_{k-1}(t-S_i-\xi_{i+1})\,\mathbb{1}_{\{\xi_{i+1}\leq t-S_i\}}\\
+\;& U_{k-1}(t-S_i-\xi_{i+1}-\xi_{i+2})\,\mathbb{1}_{\{\xi_{i+1}+\xi_{i+2}\leq t-S_i\}}+\ldots|S_i\big)\\
=\;& \mathbb{E}\sum_{i\geq 1} U_{k-1}(t-S_i)\int_{[0,\,t-S_i]} U_{k-1}(t-S_i-y)\mathrm{d}U(y)\,\mathbb{1}_{\{S_i\leq t\}} = \mathbb{E}\sum_{i\geq 1} U_{k-1}(t-S_i)U_k(t-S_i)\,\mathbb{1}_{\{S_i\leq t\}}\\
=\;& \int_{[0,\,t]} U_{k-1}(t-y)U_k(t-y)\mathrm{d}U(y).
\end{aligned}$$

Before we proceed let us note that (4.4) implies that, for integer $m\leq 2k-3$,

$$\int_{[0,\,t]}(t-y)^m\mathrm{d}U(y)=o(t^{2k-1}),\quad t\to\infty,$$

that

$$\int_{[0,\,t]}(t-y)^{2k-2}\mathrm{d}U(y)=O(t^{2k-1}),\quad t\to\infty$$

and that

$$\int_{[0,\,t]}(t-y)^{2k-1}\mathrm{d}U(y)\leq \frac{t^{2k}}{2k\mu}+ct^{2k-1},\quad t\geq 0.$$

Using these relations in combination with (4.3) yields

$$\mathbb{E}\Big(\sum_{i\geq 1}U_{k-1}(t-S_i)\,\mathbb{1}_{\{S_i\leq t\}}\Big)^2\leq \frac{2}{(k-1)!k!\mu^{2k-1}}\int_{[0,\,t]}(t-y)^{2k-1}\mathrm{d}U(y)+O(t^{2k-1})\leq \frac{t^{2k}}{(k!)^2\mu^{2k}}+O(t^{2k-1})$$

as $t\to\infty$. Further,

$$U_k^2(t)=\frac{t^{2k}}{(k!)^2\mu^{2k}}+\frac{2t^k}{k!\mu^k}\Big(U_k(t)-\frac{t^k}{k!\mu^k}\Big)+\Big(U_k(t)-\frac{t^k}{k!\mu^k}\Big)^2=\frac{t^{2k}}{(k!)^2\mu^{2k}}+O(t^{2k-1}),\quad t\to\infty$$

having utilized (4.3). The last two asymptotic relations entail

$$\mathbb{E}[(Y^*_{k,2}(t))^2]=\mathbb{E}\Big(\sum_{i\geq 1}U_{k-1}(t-S_i)\,\mathbb{1}_{\{S_i\leq t\}}\Big)^2-U_k^2(t)=O(t^{2k-1}),\quad t\to\infty.$$

The proof of (4.8) is complete.

To prove (4.5) we shall use the mathematical induction. If $k=1$, (4.5) holds true by Lemma 5.1. Assume that (4.5) holds for $k=m-1\geq 2$. Then given $\delta>0$ there exist $t_0>0$ and $c_m>0$ such that $D_{m-1}(t)\leq c_m t^{2m-3}$ whenever $t\geq t_0$. Consequently,

$$\begin{aligned}
\mathbb{E}[(Y_{m,1}(t))^2] &= \mathbb{E}\sum_{i\geq 1}D_{m-1}(t-S_i)\,\mathbb{1}_{\{S_i\leq t\}}=\int_{[0,\,t-t_0]}D_{m-1}(t-x)\mathrm{d}U(x)\\
&+ \int_{(t-t_0,\,t]}D_{m-1}(t-x)\mathrm{d}U(x)\leq c_m\int_{[0,\,t-t_0]}(t-x)^{2m-3}\mathrm{d}U(x)\\
&+ \sup_{0\leq y\leq t_0}D_{m-1}(y)(U(t)-U(t-t_0))\\
&\leq c_m t^{2m-3}U(t)+\sup_{0\leq y\leq t_0}D_{m-1}(y)(U(t_0)+1)=O(t^{2m-2}),\quad t\to\infty \qquad (4.10)
\end{aligned}$$

having utilized subadditivity of $U(t)+1$ and the elementary renewal theorem which states that $U(t)\sim t/\mu$ as $t\to\infty$. Using (4.7) and (4.8) we conclude that (4.5) holds for $k=m$. Relation (4.6) is now an immediate consequence of (4.10). □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. PROOF OF (2.4). In view of (4.3) we infer

$$\begin{aligned}\mu \sup_{0\le s\le T}|Y_{k,3}(st)| &\le \sup_{0\le s\le T}\int_0^{st}\Big|U_{k-1}(y)-\frac{y^{k-1}}{(k-1)!\mu^{k-1}}\Big|\mathrm{d}y\\ &\le \sup_{0\le s\le T}\int_0^{st}\sum_{i=0}^{k-2}\binom{k-1}{i}\frac{y^i c^{k-1-i}}{i!\mu^i}\mathrm{d}y\\ &\le \sum_{i=0}^{k-2}\binom{k-1}{i}\frac{(Tt)^{i+1}c^{k-1-i}}{(i+1)!\mu^i}=O(t^{k-1})\end{aligned}$$

for all $T>0$. This proves (2.4).
PROOF OF (2.3). It suffices to check that, for integer $k\ge 2$,

$$\lim_{t\to\infty} t^{-(k-1/2)}Y_{k,1}(t)=0 \quad \text{a.s.} \tag{4.11}$$

To this end, we pick $\delta\in(1,2)$ and note that for each $t\ge 0$, there exists $m\in\mathbb{N}_0$ such that $t\in[m^\delta,(m+1)^\delta)$ and

$$\begin{aligned}t^{-(k-1/2)}Y_{k,1}(t) &\le m^{-\delta(k-1/2)}\sum_{i\ge1}\big(Y_{k-1}^{(i)}((m+1)^\delta-S_i)-U_{k-1}((m+1)^\delta-S_i)\,\mathbb{1}_{\{S_i\le(m+1)^\delta\}}\big)\\ &+ m^{-\delta(k-1/2)}\sum_{i\ge1}\big(U_{k-1}((m+1)^\delta-S_i)-U_{k-1}(m^\delta-S_i)\big)\,\mathbb{1}_{\{S_i\le m^\delta\}}\\ &+ m^{-\delta(k-1/2)}\sum_{i\ge1}U_{k-1}((m+1)^\delta-S_i)\,\mathbb{1}_{\{m^\delta<S_i\le(m+1)^\delta\}}\\ &\le m^{-\delta(k-1/2)}Y_{k,1}((m+1)^\delta)\\ &+ m^{-\delta(k-1/2)}((U((m+1)^\delta-m^\delta)+1)U_{k-2}((m+1)^\delta)N(m^\delta)\\ &+ U_{k-1}((m+1)^\delta-m^\delta)N((m+1)^\delta)))\end{aligned}$$

where $U_0(t):=1$ for $t\ge0$. For the last inequality we have used monotonicity of the functions U_i, $i\in\mathbb{N}$ and the following estimate which is essentially based on subadditivity and monotonicity of $U+1$:

$$\begin{aligned}U_i(t+s)-U_i(t) &= \int_{[0,\,t]}(U(t+s-z)-U(t-z))\mathrm{d}U_{i-1}(z)+\int_{(t,\,t+s]}U(t+s-z)\mathrm{d}U_{i-1}(z)\\ &\le (U(s)+1)U_{i-1}(t)+U(s)(U_{i-1}(t+s)-U_{i-1}(t))\\ &\le (U(s)+1)U_{i-1}(t+s)\end{aligned}$$

for $t,s\ge0$ and $i\ge2$.

Similarly,

$$\begin{aligned}t^{-(k-1/2)}Y_{k,1}(t) &\ge (m+1)^{-\delta(k-1/2)}Y_{k,1}(m)\\ &- (m+1)^{-\delta(k-1/2)}((U((m+1)^\delta-m^\delta)+1)U_{k-2}((m+1)^\delta)N(m^\delta)\\ &+ U_{k-1}((m+1)^\delta-m^\delta)N((m+1)^\delta)).\end{aligned}$$

By the strong law of large numbers for the renewal processes and Lemma 4.1 $N(m)\sim\mu^{-1}m$ a.s. and, for $j\in\mathbb{N}$, $U_j(m)\sim\mu^{-j}(j!)^{-1}m^j$ as $m\to\infty$, respectively, whence, as $m\to\infty$,

$$m^{-\delta(k-1/2)}((U((m+1)^\delta-m^\delta)+1)U_{k-2}((m+1)^\delta)N(m^\delta) \sim \frac{\delta}{(k-2)!\mu^k}\frac{1}{m^{1-\delta/2}} \quad \text{a.s.}$$

and

$$m^{-\delta(k-1/2)}U_{k-1}((m+1)^\delta-m^\delta)N((m+1)^\delta) \sim \frac{\delta^{k-1}}{(k-1)!\mu^k}\frac{1}{m^{k-(1+\delta/2)}} \quad \text{a.s.}$$

Since $\delta<2$ and $k\ge2$, the right-hand sides of the last two relations converge to zero a.s. Hence, (4.11) is a consequence of

$$\lim_{\mathbb{N}\ni m\to\infty} m^{-\delta(k-1/2)}Y_{k,1}(m^\delta) = 0 \quad \text{a.s.} \tag{4.12}$$

By Markov's inequality in combination with (4.6) $\mathbb{P}\{|Y_{k,1}(m^\delta)| > m^{\delta(k-1/2)}\gamma\} = O(m^{-\delta})$ as $m \to \infty$ for all $\gamma > 0$ which entails (4.12) by the Borel-Cantelli lemma.

Proof of (2.5). We already know that the distributions of the coordinates in (2.5) are tight. Thus, it remains to check weak convergence of finite-dimensional distributions, that is, for all $n \in \mathbb{N}$, all $0 \le s_1 < s_2 < \ldots < s_n < \infty$ and all integer $j \ge 2$

$$\left(\frac{Y_1^\ast(s_i t)}{a_1(t)}, \frac{Y_{k,2}(s_i t)}{a_k(t)}\right)_{2\le k\le j,\,1\le i\le n} \ \overset{\mathrm{d}}{\underset{t\to\infty}{\longrightarrow}} \ (R_k(s_i))_{1\le k\le j,\,1\le i\le n}, \tag{4.13}$$

where $Y_1^\ast(t) := Y_1(t) - \mu^{-1}t$ and $a_k(t) := \sqrt{\sigma^2\mu^{-2k-1}t^{2k-1}}/(k-1)!$ for $k \in \mathbb{N}$ (recall that $0! = 1$). If $s_1 = 0$ we have $Y_1^\ast(s_1 t) = Y_{k,2}(s_1 t) = R_i(s_1) = 0$ a.s. for $k \ge 2$ and $i \in \mathbb{N}$. Hence, in what follows we assume that $s_1 > 0$.

By Theorem 3.1 on p. 162 in [11]

$$\frac{N(t\cdot) - \mu^{-1}(\cdot)}{\sqrt{\sigma^2\mu^{-3}t}} \ \underset{t\to\infty}{\Rightarrow} \ B$$

in the J_1-topology on D. By Skorokhod's representation theorem there exist versions $\widehat{N}$ and $\widehat{B}$ such that

$$\lim_{t\to\infty} \sup_{0\le y\le T} \left|\frac{\widehat{N}(ty) - \mu^{-1}ty}{\sqrt{\sigma^2\mu^{-3}t}} - \widehat{B}(y)\right| = 0 \quad \text{a.s.} \tag{4.14}$$

for all $T > 0$. This implies that (4.13) is equivalent to

$$\left(\frac{(k-1)!\mu^{k-1}\widehat{V}_k(t, s_i)}{t^{k-1}}\right)_{1\le k\le j,\,1\le i\le n} \ \overset{\mathrm{d}}{\underset{t\to\infty}{\longrightarrow}} \ (R_k(s_i))_{1\le k\le j,\,1\le i\le n}, \tag{4.15}$$

where, for $t, y \ge 0$, $\widehat{V}_1(t, y) := \widehat{B}(y)$ and $\widehat{V}_k(t, y) := \int_{(0,\,y]} \widehat{B}(x)\mathrm{d}_x(-U_{k-1}(t(y-x)),\ k \ge 2$. As far as the coordinates involving $\widehat{V}_1$ are concerned the equivalence is an immediate consequence of (4.14). As for the other coordinates, integration by parts yields, for $s > 0$ fixed and $k \ge 2$,

$$\begin{aligned}\int_{[0,\,st]} \frac{U_{k-1}(st-x)}{t^{k-1}}\mathrm{d}_x\frac{\widehat{N}(x) - \mu^{-1}x}{\sqrt{\sigma^2\mu^{-3}t}} \ &= \ \int_{(0,\,s]} \left(\frac{\widehat{N}(tx) - \mu^{-1}tx}{\sqrt{\sigma^2\mu^{-3}t}} - \widehat{B}(x)\right)\mathrm{d}_x\frac{-U_{k-1}(t(s-x))}{t^{k-1}} \\ &+ \ \int_{(0,\,s]} \widehat{B}(x)\mathrm{d}_x\frac{-U_{k-1}(t(s-x))}{t^{k-1}}.\end{aligned}$$

Denoting by $J(t)$ the first term on the right-hand side, we infer

$$|J(t)| \le \sup_{0\le y\le s} \left|\frac{\widehat{N}(ty) - \mu^{-1}ty}{\sqrt{\sigma^2\mu^{-3}t}} - \widehat{B}(y)\right|(t^{-(k-1)}U_{k-1}(st))$$

which tends to zero a.s. as $t \to \infty$ in view of (4.14) and Lemma 4.1 which implies that $\lim_{t\to\infty} t^{-(k-1)}U_{k-1}(st) = s^{k-1}/((k-1)!\mu^{k-1})$.

For $t, y \ge 0$, set $V_1(t, y) := B(y)$ and $V_k(t, y) := \int_{(0,\,y]} B(x)\mathrm{d}_x(-U_{k-1}(t(y-x)),\ k \ge 2$. We note that (4.15) is equivalent to

$$\left(\frac{(k-1)!\mu^{k-1}V_k(t, s_i)}{t^{k-1}}\right)_{1\le k\le j,\,1\le i\le n} \ \overset{\mathrm{d}}{\underset{t\to\infty}{\longrightarrow}} \ (R_k(s_i))_{1\le k\le j,\,1\le i\le n} \tag{4.16}$$

because the left-hand sides of (4.15) and (4.16) have the same distribution. Both the limit and the converging vectors in (4.16) are Gaussian. Hence, it suffices to prove that

$$\begin{aligned}\lim_{t\to\infty} t^{-(k+l-2)}\,\mathbb{E}\,V_k(t, s)V_l(t, u) \ &= \ \frac{1}{(k-1)!(l-1)!\mu^{k+l-2}}\,\mathbb{E}\,R_k(s)R_l(u) \\ &= \ \frac{1}{(k-1)!(l-1)!\mu^{k+l-2}}\int_0^{s\wedge u} (s-y)^{k-1}(u-y)^{l-1}\mathrm{d}y\end{aligned} \tag{4.17}$$

for $k, l \in \mathbb{N}$ and $s, u > 0$. We only consider the cases where $0 < s \le u$ and $k, l \ge 2$, the case $s > u$ being similar and the cases where k or/and l is/are equal to 1 being simpler.

We start by writing

$$\begin{aligned}\mathbb{E}\, V_k(t,s)V_l(t,u) &= \int_0^s U_{k-1}(t(s-y))U_{l-1}(t(u-y))\mathrm{d}y \\ &= \int_0^s \Big(U_{k-1}(t(s-y)) - \frac{t^{k-1}(s-y)^{k-1}}{(k-1)!\mu^{k-1}}\Big)U_{l-1}(t(u-y))\mathrm{d}y \\ &+ \frac{t^{k-1}}{(k-1)!\mu^{k-1}}\int_0^s (s-y)^{k-1}\Big(U_{l-1}(t(u-y)) - \frac{t^{l-1}(u-y)^{l-1}}{(l-1)!\mu^{l-1}}\Big)\mathrm{d}y \\ &+ \frac{t^{k+l-2}}{(k-1)!(l-1)!\mu^{k+l-2}}\int_0^s (s-y)^{k-1}(u-y)^{l-1}\mathrm{d}y.\end{aligned}$$

Denoting by $J_1(t)$ and $J_2(t)$ the first and the second summand on the right-hand side, respectively, we infer with the help of Lemma 4.1:

$$\begin{aligned}J_1(t) &\le \int_0^s \sum_{i=0}^{k-2}\binom{k-1}{i}\frac{t^i(s-y)^i}{i!\mu^i}U_{l-1}(t(u-y))\mathrm{d}y \\ &\le U_{l-1}(tu)\sum_{i=0}^{k-2}\binom{k-1}{i}\frac{t^i s^{i+1}}{(i+1)!\mu^i} = O(t^{k+l-3})\end{aligned}$$

as $t\to\infty$ because the sum is $O(t^{k-2})$ and $U_{l-1}(tu) = O(t^{l-1})$. Arguing similarly we obtain $J_2(t) = O(t^{k+l-3})$ as $t\to\infty$, and (4.17) follows. The proof of Theorem 2.1 is complete. □

5. Appendix

Lemma 5.1 is stated in a greater generality than we need in the present paper because we believe that this result is of some importance for the renewal theory.

Lemma 5.1. *Assume that the distribution of ξ is nondegenerate and $\mathbb{E}\,\xi^p<\infty$ for some $p\ge 2$. Then $\mathbb{E}\,|N(t)-U(t)|^p \sim \mathbb{E}\,|Z|^p t^{p/2}$ as $t\to\infty$, where Z is a normally distributed random variable with mean zero and variance $\sigma^2\mu^{-3}$, $\mu=\mathbb{E}\,\xi$ and $\sigma^2=\mathrm{Var}\,\xi$.*

Proof. Theorem 8.4 on p. 98 in [11] states the result holds with $\mu^{-1}t$ replacing $U(t)$. Using the inequality (see p. 282 in [10]) $(a+b)^p \le a^p + p2^{p-1}(ab^{p-1}+ab^{p-1}) + b^p$ for $a,b\ge 0$ together with $\mathbb{E}\,|X| \le (\mathbb{E}\,|X|^p)^{1/p}$ yields

$$\begin{aligned}\mathbb{E}\,|N(t)-U(t)|^p &\le \mathbb{E}\,|N(t)-\mu^{-1}t|^p + p2^{p-1}(\mathbb{E}\,|N(t)-\mu^{-1}t|^p)^{1/p}(U(t)-\mu^{-1}t)^{p-1} \\ &+ p2^{p-1}\,\mathbb{E}\,|N(t)-\mu^{-1}t|^{p-1}(U(t)-\mu^{-1}t) + (U(t)-\mu^{-1}t)^p.\end{aligned}$$

Recalling (4.1) we arrive at $\limsup_{t\to\infty} t^{-p/2}\,\mathbb{E}\,|N(t)-\mu^{-1}t|^p \le \mathbb{E}\,|Z|^p$. The converse inequality for the lower limit follows from the central limit theorem for $N(t)$, formula (4.1) and Fatou's lemma. □

Remark 5.2. It is worth stating explicitly that when $p>2$ the assumption $\mathbb{E}\,\xi^p<\infty$ in Lemma 5.1 cannot be dispensed with. According to Remark 1.2 in [14], there exist distributions of ξ such that $\mathbb{E}\,\xi^2<\infty$ and $\lim_{t\to\infty} t^{-p/2}\,\mathbb{E}\,|N(t)-U(t)|^p=\infty$ for every $p>2$.

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