Functional limit theorems for the maxima of perturbed random walk and divergent perpetuities in the M_1-topology

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Abstract

Let (ξ_1, η_1), $(\xi_2, \eta_2), \ldots$ be a sequence of i.i.d. two-dimensional random vectors. In the earlier article Iksanov and Pilipenko (2014) weak convergence in the J_1-topology on the Skorokhod space of $n^{-1/2} \max_{0\leq k\leq [n\cdot]} (\xi_1 + \ldots + \xi_k + \eta_{k+1})$ was proved under the assumption that contributions of $\max_{0\leq k\leq n} (\xi_1 + \ldots + \xi_k)$ and $\max_{1\leq k\leq n} \eta_k$ to the limit are comparable and that $n^{-1/2}(\xi_1 + \ldots + \xi_{[n\cdot]})$ is attracted to a Brownian motion. In the present paper, we continue this line of research and investigate a more complicated situation when $\xi_1 + \ldots + \xi_{[n\cdot]}$, properly normalized without centering, is attracted to a centered stable Lévy process, a process with jumps. As a consequence, weak convergence normally holds in the M_1-topology. We also provide sufficient conditions for the J_1-convergence. For completeness, less interesting situations are discussed when one of the sequences $\max_{0\leq k\leq n} (\xi_1 + \ldots + \xi_k)$ and $\max_{1\leq k\leq n} \eta_k$ dominates the other. An application of our main results to divergent perpetuities with positive entries is given.

Key words: functional limit theorem; J_1-topology; M_1-topology; perpetuity; perturbed random walk

1 Introduction and results

1.1 Introduction

Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η). No condition is imposed on the dependence structure between ξ and η. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Further, let $(S_n)_{n\in\mathbb{N}_0}$ be the zero-delayed ordinary random walk with increments ξ_n for $n \in \mathbb{N}$, i.e., $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n \in \mathbb{N}$. Then define its perturbed variant $(T_n)_{n\in\mathbb{N}}$, that we call *perturbed random walk*, by

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}. \tag{1}$$

A large sample of articles on the so defined perturbed random walks can be traced via the references given in the recent book [4].

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Denote by $D := D[0,\infty)$ the Skorokhod space of real-valued right-continuous functions which are defined on $[0,\infty)$ and have finite limits from the left at each positive point. Throughout the paper we assume that D is equipped with either the J_1-topology or the M_1-topology. We refer to [2, 7] and [14] for comprehensive accounts of the J_1- and the M_1-topologies, respectively. We write $\mathcal{M}_p(A)$ for the set of Radon point measures on a locally compact metric space A. The $\mathcal{M}_p(A)$ is endowed with vague convergence. More information on these can be found in [9]. Throughout the paper $\overset{\mathrm{J_1}}{\Rightarrow}$ and $\overset{\mathrm{M_1}}{\Rightarrow}$ will mean weak convergence on the Skorokhod space D when endowed with the J_1-topology and the M_1-topology, respectively. The notation $\Rightarrow$ without superscript is normally followed by a specification of the topology and the space involved. Finally, we write $\overset{\mathrm{v}}{\to}$ and $\overset{\mathrm{P}}{\to}$ to denote vague convergence and convergence in probability, respectively.

In the present paper we are interested in weak convergence on D of $\max_{0\le k\le [n\cdot]}(S_k+\eta_{k+1})$, properly normalized without centering, as $n\to\infty$. It should not come as a surprise that the maxima exhibit three types of different behaviors depending on the asymptotic interplay of $A_n := \max\limits_{0\le k\le n} S_k$ and $B_n := \max\limits_{1\le k\le n+1}\eta_k$, namely, on whether

(I) A_n dominates B_n;

(II) A_n is dominated by B_n;

(III) A_n and B_n are comparable.

Relying essentially upon findings in [6] (see also [13]) three functional limit theorems for the maxima of perturbed random walks, properly rescaled without centering, were proved in [4] under the assumption that $\mathbb{E}\xi^2<\infty$. Throughout the remainder of the paragraph we assume that the most interesting alternative (III) prevails. The situation treated in [6] was relatively simple because the limit process for $S_{[n\cdot]}/n^{1/2}$ was a Brownian motion, a process with continuous paths. As a consequence, the convergence took place in the J_1-topology on D, and, more surprisingly, the contributions of (S_k) and (η_j) turned out asymptotically independent, despite the possible strong dependence of ξ and η. In the present paper we treat a more delicate case where the distribution of ξ belongs to the domain of attraction of an α-stable distribution, $\alpha\in(0,2)$, so that the limit process for $S_{[n\cdot]}$, properly normalized, is an α-stable Lévy process. We shall show that the presence of jumps in the latter process destroys dramatically an idyllic picture pertaining to the Brownian motion scenario: the convergence typically holds in the weaker M_1-topology on D, and the aforementioned asymptotic independence only occurs in some exceptional cases where ξ and η are themselves asymptotically independent in an appropriate sense.

Throughout the paper we assume that, as $x\to\infty$,

$$\mathbb{P}\{|\xi|>x\}\sim x^{-\alpha}\ell(x) \tag{2}$$

and that

$$\mathbb{P}\{\xi>x\}\sim c_1\mathbb{P}\{|\xi|>x\},\quad \mathbb{P}\{-\xi>x\}\sim c_2\mathbb{P}\{|\xi|>x\} \tag{3}$$

for some $\alpha\in(0,2)$, some ℓ slowly varying at ∞, some nonnegative c_1 and c_2 summing up to one. The assumptions mean that the distribution of ξ belongs to the domain of attraction of an α-stable distribution. To ensure weak convergence of $S_{[n\cdot]}$ without centering we assume that $\mathbb{E}\xi=0$ when $\alpha\in(1,2)$ and that the distribution of ξ is symmetric when $\alpha=1$. Then the classical Skorokhod theorem (Theorem 2.7 in [12]) tells us that

$$\frac{S_{[n\cdot]}}{a(n)}\overset{\mathrm{J_1}}{\Rightarrow}\mathcal{S}_\alpha(\cdot),\quad n\to\infty, \tag{4}$$

where $a(x)$ is a positive function satisfying $\lim_{x\to\infty} x\mathbb{P}\{|\xi| > a(x)\} = 1$ and $\mathcal{S}_\alpha := (\mathcal{S}_\alpha(t))_{t\geq 0}$ is an α-stable Lévy process with the characteristic function

$$\mathbb{E}\exp(\mathrm{i}z\mathcal{S}_\alpha(1)) = \exp\big(|z|^\alpha(\Gamma(2-\alpha)/(\alpha-1))(\cos(\pi\alpha/2) - \mathrm{i}(c_1-c_2)\sin(\pi\alpha/2)\mathrm{sign}z)\big), \quad z\in\mathbb{R}$$

when $\alpha\in(0,2)$, $\alpha\neq 1$, here, $\Gamma(\cdot)$ denotes the Euler gamma function, and

$$\mathbb{E}\exp(\mathrm{i}z\mathcal{S}_\alpha(1)) = \exp(-2^{-1}\pi|z|), \quad z\in\mathbb{R}$$

when $\alpha = 1$. Note that the Lévy measure ν^* of $\mathcal{S}_\alpha$ is given by

$$\nu^*((x,\infty]) = c_1 x^{-\alpha}, \quad \nu^*((-\infty,-x]) = c_2 x^{-\alpha}, \quad x>0, \tag{5}$$

where $c_1 = c_2 = 1/2$ when $\alpha = 1$.

Put $E := [-\infty,+\infty]\times[0,\infty]\backslash\{(0,0)\}$. For a Radon measure ρ on E, let $N^{(\rho)} := \sum_k \varepsilon_{(\theta_k,\, i_k,\, j_k)}$ be a Poisson random measure on $[0,\infty)\times E$ with mean measure $\mathbb{LEB}\times\rho$, where $\varepsilon_{(t,\,x,\,y)}$ is the probability measure concentrated at $(t,x,y)\subset[0,\infty)\times E$, $\mathbb{LEB}$ is the Lebesgue measure on $[0,\infty)$.

1.2 Case (III): the main result

Theorem 1.1 which is the main result of the present paper treats the most complicated situation (III) when the contributions of $\max_{0\leq k\leq n} S_k$ and $\max_{1\leq k\leq n+1}\eta_k$ to the asymptotic behavior of $\max_{0\leq k\leq n}(S_k + \eta_{k+1})$ are comparable. At this point it is worth stressing that ξ and η are assumed arbitrarily dependent which makes the analysis nontrivial. We stipulate hereafter that the supremum over the empty set equals zero.

Theorem 1.1. *Suppose that conditions* (2) *and* (3) *hold, and that the distribution tails of* $|\xi|$ *and* η *are comparable in the sense that* $\mathbb{P}\{\eta > x\}\sim c\mathbb{P}\{|\xi|>x\}$ *as* $x\to\infty$, *for some* $c>0$, *and that*

$$x\mathbb{P}\Big\{\frac{(\xi,\eta^+)}{a(x)}\in\cdot\Big\} \overset{\mathrm{v}}{\to} \nu, \quad x\to\infty \tag{6}$$

on $\mathcal{M}_p(E)$. *Then*

$$\frac{\max_{0\leq k\leq[n\cdot]}(S_k+\eta_{k+1})}{a(n)} \overset{\mathrm{M}_1}{\Rightarrow} \sup_{0\leq s\leq\cdot}\mathcal{S}^*_\alpha(s)\vee\sup_{\theta_k\leq\cdot}(\mathcal{S}^*_\alpha(\theta_k-)+j_k), \quad n\to\infty, \tag{7}$$

where (θ_k, i_k, j_k) *are the atoms of a Poisson random measure* $N^{(\nu)}$ *and* $\mathcal{S}^*_\alpha$ *is a copy of* $\mathcal{S}_\alpha$ *whose Lévy- Itô representation is built upon the Poisson random measure* $\sum_k\varepsilon_{(\theta_k,i_k)}$, *that is,*

$$\mathcal{S}^*_\alpha(t) := \lim_{\delta\downarrow 0}\Bigg(\sum_{\theta_k\leq t} i_k\,\mathbb{1}_{\{|i_k|>\delta\}} - ta(\delta)\Bigg), \quad t\geq 0,$$

where $a(\delta) := 0$ *if* $\alpha\in(0,1]$ *and* $a(\delta) := \int_{|s|>\delta} s\nu^*(\mathrm{d}s)$ *if* $\alpha\in(1,2)$; *here,* ν^* *is the Lévy measure defined in* (5).

Under the additional assumption

$$\nu\{(x,y): 0<y<x\} = 0, \tag{8}$$

the convergence in (7) *holds in the* J_1-*topology on* D.

We refer to Theorems 19.2 and 19.3 in [11] or Theorem 3.12.2 in [10] for more details concerning the Lévy-Itô decomposition appearing in Theorem 1.1.

We proceed with a number of remarks.

Remark 1.2. It is perhaps worth stating explicitly that $N^{(\nu)}(\cdot \times \cdot \times [0,\infty]) = \sum_k \varepsilon_{(\theta_k,\, i_k)}$ is a Poisson random measure on $[0,\infty) \times ([-\infty,+\infty]\backslash\{0\})$ with mean measure $\mathbb{LEB} \times \nu^*$, where ν^* is the Lévy measure of $\mathcal{S}_\alpha$ given in (5). Analogously, $N^{(\nu)}(\cdot \times [-\infty,+\infty] \times \cdot) = \sum_k \varepsilon_{(\theta_k,\, j_k)}$ is a Poisson random measure on $[0,\infty) \times (0,\infty]$ with mean measure $\mathbb{LEB} \times \mu_c$, where μ_c is a measure on $(0,\infty]$ defined by

$$\mu_c\big((x,\infty]\big) = cx^{-\alpha}, \quad x > 0.$$

Remark 1.3. The limit process in (7) is not a Markov process. Other distributional properties of this process are unknown.

Remark 1.4. Condition (8) obviously holds if the measure ν is concentrated on the axes. This is the case whenever ξ and η are independent and also in many cases when these are dependent. For instance, take $\xi = |\log W|$ and $\eta = |\log(1-W)|$ satisfying (6) for a random variable W taking values in $(0,1)$ (details can be found in the proof of Theorem 1.1 in [5]).

Suppose now that $\eta = r\xi$ for some $r > 0$. Then the restriction of ν to the first quadrant concentrates on the line $y = rx$. Hence, condition (8) holds if, and only if, $r \geq 1$.

Let R be a positive random variable such that $\mathbb{P}\{R > x\} \sim x^{-\alpha}$ as $x \to \infty$ and ζ a random variable which is independent of R and takes values in $[-\pi,\pi)$. Setting $\xi = R\cos\zeta$ and $\eta = R\sin\zeta$ we obtain

$$\mathbb{P}\{\xi > x\} \ \sim\ (\mathbb{E}(\cos\zeta)^\alpha\, \mathbb{1}_{\{|\zeta|<\pi/2\}})x^{-\alpha},\ \mathbb{P}\{-\xi > x\} \ \sim\ (\mathbb{E}|\cos\zeta|^\alpha\, \mathbb{1}_{\{|\zeta|>\pi/2\}})x^{-\alpha}$$

as $x \to \infty$ by the Lebesgue dominated convergence theorem. Furthermore,

$$x\mathbb{P}\bigg\{\frac{(\xi,\eta^+)}{a(x)} \in \cdot\bigg\} \ \overset{\text{v}}{\to}\ \nu, \quad x \to \infty,$$

where $a(x) = (\mathbb{E}|\cos\zeta|^\alpha)^{1/\alpha} x^{1/\alpha}$ and ν is the image of the measure

$$\frac{\alpha\mathbb{E}(\cos\zeta)^\alpha\, \mathbb{1}_{\{\zeta\in(-\pi/2,\pi/2)\}}}{\mathbb{E}|\cos\zeta|^\alpha}\, \mathbb{1}_{(r,\varphi)\in(0,\infty)\times[0,\pi)}\, r^{-\alpha-1}\mathrm{d}r\mathbb{P}\{\zeta\in\mathrm{d}\varphi\}$$

under the mapping $(r,\varphi) \to (r\cos\varphi, r\sin\varphi)$. Condition (8) is equivalent to $\mathbb{P}\{\zeta \in (0,\pi/4)\} = 0$.

Remark 1.5. Weak convergence of nondecreasing processes in the M_1-topology is not as strong as it might appear. Actually, it is equivalent to weak convergence of finite-dimensional distributions just because a sequence of nondecreasing processes is always tight on D equipped with the M_1-topology. This follows from the fact that the M_1-oscillation

$$\omega_\delta(f) := \sup_{t_1\leq t\leq t_2,\, 0\leq t_2-t_1\leq\delta} M(f(t_1), f(t), f(t_2))$$

of a nondecreasing function f equals zero, where $M(x_1,x_2,x_3) := 0$, if $x_2 \in [x_1,x_3]$, and $M(x_1,x_2,x_3) := \min(|x_2-x_1|, |x_3-x_2|)$, otherwise.

Remark 1.6. From a look at Theorem 2.1 underlying the proof of Theorem 1.1 it should be clear that a counterpart of Theorem 1.1 also holds when replacing the input vectors $(\xi_k,\eta_k)_{k\in\mathbb{N}}$ with arrays $\big(\xi_k^{(n)},\eta_k^{(n)}\big)_{k\in\mathbb{N}}$ for each $n \in \mathbb{N}$. We however refrain from formulating such a generalization, for we are not aware of any potential applications of such a result.

1.3 Cases (I) and (II): complementary results

Propositions 1.7 and 1.9 given next are concerned with the simpler situations (I) and (II), respectively.

Proposition 1.7. *Suppose that conditions* (2) *and* (3) *hold, and that the distribution tail of* $|\xi|$ *is heavier than that of* η*, i.e.,*

$$\lim_{x\to\infty}\frac{\mathbb{P}\{\eta>x\}}{\mathbb{P}\{|\xi|>x\}}=0. \tag{9}$$

Then

$$\frac{\max\limits_{0\le k\le[n\cdot]}(S_k+\eta_{k+1})}{a(n)}\quad\overset{\mathrm{J}_1}{\Rightarrow}\quad\sup_{0\le s\le\cdot}\mathcal{S}_\alpha(s),\quad n\to\infty. \tag{10}$$

Remark 1.8. When $\alpha\in(0,1)$ and $c_1=0$, the right-hand side in (10) is the zero function because $\mathcal{S}_\alpha$ is then the negative of an α-stable subordinator (recall that a subordinator is a nondecreasing Lévy process). In this setting there are two possibilities: either $\sup_{k\ge0}(S_k+\eta_{k+1})<\infty$ a.s. or $\sup_{k\ge0}(S_k+\eta_{k+1})=\infty$ a.s. Plainly, if the first alternative prevails, much more than (10) can be said, namely, $\max\limits_{0\le k\le[n\cdot]}(S_k+\eta_{k+1})/r_n$ converges to the zero function in the J_1-topology on D for any positive sequence (r_n) diverging to ∞.

Now we intend to give examples showing that either of possibilities can hold. By Theorem 2.1 in [1], the supremum of $S_k+\eta_{k+1}$ is finite a.s. if, and only if,

$$\int_{(0,\infty)}\frac{x}{\int_0^x\mathbb{P}\{-\xi>y\}\mathrm{d}y}\mathrm{d}\mathbb{P}\{\xi\le y\}<\infty\quad\text{and}\quad\int_{(0,\infty)}\frac{x}{\int_0^x\mathbb{P}\{-\xi>y\}\mathrm{d}y}\mathrm{d}\mathbb{P}\{\eta\le y\}<\infty \tag{11}$$

If $\mathbb{P}\{-\xi>x\}\sim x^{-\alpha}$, $\mathbb{P}\{\xi>x\}\sim x^{-\alpha}(\log x)^{-2}$ as $x\to\infty$ and $\mathbb{E}(\eta^+)^\alpha<\infty$, then both inequalities in (11) hold, whereas if $\mathbb{P}\{-\xi>x\}\sim x^{-\alpha}$, $\mathbb{P}\{\xi>x\}\sim x^{-\alpha}(\log x)^{-1}$ as $x\to\infty$, then the first integral in (11) diverges.

Proposition 1.9. *Suppose that conditions* (2) *and* (3) *hold, and that the distribution tail of* $|\xi|$ *is lighter than that of* η*, i.e.,*

$$\lim_{x\to\infty}\frac{\mathbb{P}\{\eta>x\}}{\mathbb{P}\{|\xi|>x\}}=\infty \tag{12}$$

and that $\mathbb{P}\{\eta>x\}$ *is regularly varying at* ∞ *of index* $-\beta$ *(necessarily* $\beta\in(0,\alpha]$*). Let* $b(x)$ *be a positive function which satisfies* $\lim\limits_{x\to\infty}x\mathbb{P}\{\eta>b(x)\}=1$*. Then*

$$\frac{\max\limits_{0\le k\le[n\cdot]}(S_k+\eta_{k+1})}{b(n)}\quad\overset{\mathrm{J}_1}{\Rightarrow}\quad\sup_{\theta_k\le\cdot}j_k,\quad n\to\infty, \tag{13}$$

where (θ_k,j_k) *are the atoms of a Poisson random measure on* $[0,\infty)\times(0,\infty]$ *with mean measure* $\mathbb{LEB}\times\mu$*, where* μ *is a measure on* $(0,\infty]$ *defined by*

$$\mu\big((x,\infty]\big)=x^{-\beta},\quad x>0.$$

1.4 An application to perpetuities

Whenever the random series $\sum_{k\ge0}e^{T_{k+1}}$ converges a.s., where $(T_k)_{k\in\mathbb{N}}$ is defined in (1), its sum is called *perpetuity* due to its occurrence in the realm of insurance and finance as a sum of discounted payment streams. When the random series diverges, it is natural to investigate weak convergence on D of its partial sums, properly rescaled, as the number of summands becomes large. Some results of this flavor can be found in [3] and [4] (in these works many references to earlier one-dimensional results can be found). Here, we prove functional limit theorems in the situations that, for the most part, remained untouched.

Theorem 1.10. *In the settings of Theorem 1.1 and Propositions 1.7 and 1.9 functional limit theorems hold with* $\log\left(\sum_{k=0}^{[n\cdot]} e^{T_{k+1}}\right)$ *replacing* $\max_{0\le k\le [n\cdot]} T_{k+1}$*. For instance, under the assumptions of Proposition 1.7*

$$\frac{\log\left(\sum_{k=0}^{[n\cdot]} e^{T_{k+1}}\right)}{a(n)} \overset{\mathrm{J}_1}{\Rightarrow} \sup_{0\le s\le \cdot} \mathcal{S}_\alpha(s), \quad n\to\infty. \tag{14}$$

Remark 1.11. Under the additional assumption that ξ and η are independent weak convergence of one-dimensional distributions in (14) was proved in Theorem 2.1(c) of [8]. To the best of our knowledge the other parts of our Theorem 1.10 are new.

2 Proof of Theorem 1.1

For each $n\in\mathbb{N}$, let $\left(x_i^{(n)}, y_i^{(n)}\right)_{i\in\mathbb{N}}$ be a sequence of $\mathbb{R}^2$-valued vectors. Put $S_0^{(n)} := 0$,

$$S_k^{(n)} := \sum_{i=1}^k x_i^{(n)}, \quad k\in\mathbb{N}, \quad T_k^{(n)} := S_k^{(n)} + y_{k+1}^{(n)}, \quad k\in\mathbb{N}_0$$

and then define the piecewise constant functions

$$f_n(t) := \sum_{k\ge 0} S_k^{(n)}\, \mathbb{1}_{[\frac{k}{n}, \frac{k+1}{n})}(t), \quad g_n(t) := \max_{0\le k\le [nt]} T_k^{(n)}, \quad t\ge 0$$

where $\mathbb{1}_A(x) = 1$ if $x\in A$ and $=0$, otherwise.

To proceed, we have to recall the notation $E = [-\infty, +\infty]\times[0,\infty]\backslash\{(0,0)\}$. The proof of Theorem 1.1 is essentially based on the following deterministic result along with the continuous mapping theorem.

Theorem 2.1. *Let* $f_0\in D$ *and* $\nu_0 = \sum_k \varepsilon_{(t_k, x_k, y_k)}$ *be a Radon measure on* $[0,\infty)\times E$ *satisfying* $\nu_0(\{0\}\times E) = 0$ *and* $t_k\ne t_j$ *for* $k\ne j$*. Suppose that*

$$\lim_{n\to\infty} f_n = f_0 \tag{15}$$

in the J_1*-topology on* D *and that*

$$\nu_n := \sum_{i\ge 1} \varepsilon_{(i/n, x_i^{(n)}, y_i^{(n)})}\, \mathbb{1}_{\{y_i^{(n)}>0\}} \overset{\mathrm{v}}{\to} \nu_0, \quad n\to\infty \tag{16}$$

on $\mathcal{M}_p([0,\infty)\times E)$*. Then*

$$\lim_{n\to\infty} g_n = g_0 := \sup_{0\le s\le \cdot} f_0(s) \vee \sup_{t_k\le \cdot} (f_0(t_k-) + y_k)$$

in the M_1*-topology on* D*. This convergence holds in the* J_1*-topology on* D *under the additional assumption*

$$\nu_0([0,\infty)\times\{(x,y): 0<y<x\}) = 0. \tag{17}$$

Remark 2.2. Suppose that f_0 is continuous and that the set of points (t_k, y_k) with $y_k > 0$ is dense in $[0, \infty)$. Then

$$g_0(\cdot) = \sup_{t_k \le \cdot} (f_0(t_k) + y_k).$$

Furthermore, condition (17) holds automatically, and condition (16) is equivalent to

$$\sum_{i \ge 1} \varepsilon_{(i/n,\, y_i^{(n)})} \mathbb{1}_{\{y_i^{(n)} > 0\}} \stackrel{\mathrm{v}}{\to} \sum_k \varepsilon_{(t_k, y_k)} \quad n \to \infty.$$

on $\mathcal{M}_p([0,\infty) \times (0,\infty])$. This is the setting of Theorem 1.3 in [6].

Remark 2.3. Here, we discuss the necessity of condition (17) for the J_1-convergence. Suppose that in the setting of Theorem 2.1 there exists $k \in \mathbb{N}$ such that $0 < y_k < x_k$, so that condition (17) does not hold. Now we give an example in which the J_1-convergence in Theorem 2.1 fails to hold. With $x_i^{(n)} = y_i^{(n)} = 0$ for $i \neq [n/2]$, $x_{[n/2]}^{(n)} = 2$ and $y_{[n/2]}^{(n)} = 1$ we have $f_n(t) = 2\, \mathbb{1}_{[[n/2]/n, \infty)}(t)$ and

$$g_n(t) = \begin{cases} 0, & t < \frac{[n/2]-1}{n}, \\ 1, & \frac{[n/2]-1}{n} \le t < \frac{[n/2]}{n}, \\ 2, & \frac{[n/2]}{n} \le t. \end{cases}$$

Plainly, condition (16) holds with $\nu_0 = \varepsilon_{(1/2, 2, 1)}$. Setting $f_0(t) = g(t) := 2\, \mathbb{1}_{[1/2,\infty)}(t)$ we conclude that $\lim_{n\to\infty} f_n = f_0$ in the J_1-topology and $\lim_{n\to\infty} g_n = g$ in the M_1-topology. On the other hand, g_n has a jump of magnitude 1 at point $[n/2]/n$. Furthermore, this magnitude does not converge to 2, the size of the limit jump at point 1/2. This precludes the J_1-convergence.

Lemma 2.4 given next collects known criteria for the convergence of nondecreasing functions in the J_1- and M_1-topologies. For part (a), see Corollary 12.5.1 and Lemma 12.5.1 in [14]. While one implication of part (b) is standard, the other follows from Theorem 2.15 on p. 342 and Lemma 2.22 on p. 343 in [7].

Lemma 2.4. *Let $(h_n)_{n \in \mathbb{N}_0}$ be a sequence of nondecreasing functions in D.*
(a) $\lim_{n\to\infty} h_n = h_0$ in the M_1-topology on D if, and only if, $h_n(t)$ converges to $h_0(t)$ for each t in a dense subset of continuity points of h_0 including zero.
(b) $\lim_{n\to\infty} h_n = h_0$ in the J_1-topology on D if, and only if, $\lim_{n\to\infty} h_n = h_0$ in the M_1-topology on D and for any discontinuity point s of h_0 there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} s_n = s$,

$$\lim_{n\to\infty} h_n(s_n-) = h(s-) \quad \textit{and} \quad \lim_{n\to\infty} h_n(s_n) = h(s).$$

Proof of Theorem 2.1. We start by showing that $g_0 \in D$. Since g_0 is nondecreasing, it has finite limits from the left on $(0,\infty)$. Using right-continuity of f_0 we obtain

$$\begin{aligned} g_0(t) &\le \lim_{\delta\to 0+} g_0(t+\delta) = \lim_{\delta\to 0+} \Big(\sup_{0\le s<t+\delta} f_0(s) \vee \sup_{t_k \le t+\delta} (f_0(t_k-) + y_k) \Big) \\ &= \sup_{0\le s\le t} f_0(s) \vee \lim_{\delta\to 0+} \sup_{t_k\le t+\delta} (f_0(t_k-) + y_k) \\ &\le \sup_{0\le s\le t} f_0(s) \vee \sup_{t_k \le t} (f_0(t_k-) + y_k) \vee \lim_{\delta\to 0+} \sup_{t<t_k\le t+\delta} (f_0(t_k-) + y_k) \\ &\le \sup_{0\le s\le t} f_0(s) \vee \sup_{t_k \le t} (f_0(t_k-) + y_k) \vee \big(\lim_{\delta\to 0+} \sup_{t<t_k\le t+\delta} f_0(t_k-) + \lim_{\delta\to 0+} \sup_{t<t_k\le t+\delta} y_k \big) \\ &= \sup_{0\le s\le t} f_0(s) \vee \sup_{t_k \le t} (f_0(t_k-) + y_k) \vee (f_0(t) + 0) = g_0(t) \end{aligned}$$

for any $t > 0$ which proves right-continuity of g_0.

Proof of the M_1-convergence. Since $f_0, g_0 \in D$, they have at most countably many discontinuities. Hence, the set

$$K := \{T \ge 0 : \nu_0(\{T\} \times E) = 0; \quad T \text{ is a continuity point of } f_0 \text{ and continuity point of } g_0\}$$

is dense in $[0, \infty)$. Since g_n is nondecreasing for each $n \in \mathbb{N}$, according to Lemma 2.4 (a), it suffices to prove that

$$\lim_{n\to\infty} g_n(T) = g_0(T) \tag{18}$$

for all $T \in K$. Observe that $g_0(0) = 0$ as a consequence of $f_n(0) = f_0(0) = 0$ and $\nu_0(\{0\}\times E) = 0$. The last condition ensures that $g_n(0) = y_1^{(n)}$ converges to zero as $n \to \infty$. This proves that relation (18) holds for $T = 0$. Thus, in what follows we assume that $T \in K$ and $T > 0$.

Fix any such a T. There exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ that vanishes as $k \to \infty$ and such that its generic element denoted by ε is a continuity point of the nonincreasing function

$$x \mapsto \nu_0([0, T] \times [-\infty, +\infty] \times (x, \infty]),$$

so that $\nu_0([0, T] \times [-\infty, +\infty] \times \{\varepsilon\}) = 0$. Put $E_\varepsilon := [-\infty, +\infty] \times (\varepsilon, \infty]$. Condition (16) implies that $\nu_0([0, T] \times E_\varepsilon) = \nu_n([0, T] \times E_\varepsilon) = m$ for large enough n and some $m \in \mathbb{N}_0$, where the finiteness of m is secured by the fact that ν_0 is a Radon measure. The case $m = 0$ is trivial. Hence, in what follows we assume that $m \in \mathbb{N}$. Denote by $(\bar{t}_i, \bar{x}_i, \bar{y}_i)_{1\le i\le m}$ an enumeration of the points of ν_0 in $[0, T] \times E_\varepsilon$ with

$$\bar{t}_1 < \bar{t}_2 < \ldots < \bar{t}_m \tag{19}$$

and by $\big(\bar{t}_i^{(n)}, \bar{x}_i^{(n)}, \bar{y}_i^{(n)}\big)_{1\le i\le m}$ the analogous enumeration of the points ν_n in $[0, T] \times E_\varepsilon$. Note that $\bar{t}_1 > 0$ in view of the assumption $\nu_0(\{0\} \times E) = 0$, whereas the assumption $t_k \ne t_j$ for $k \ne j$ ensures that the inequalities in (19) are strict. Then

$$\lim_{n\to\infty} \sum_{i=1}^{m} (|\bar{t}_i^{(n)} - \bar{t}_i| + |\bar{x}_i^{(n)} - \bar{x}_i| + |\bar{y}_i^{(n)} - \bar{y}_i|) = 0. \tag{20}$$

Later on we shall need the following relation

$$f_n(\bar{t}_i^{(n)} - 1/n) = f_n(\bar{t}_i^{(n)}-) \to f_0(\bar{t}_i-), \quad n \to \infty \tag{21}$$

for $i = 1, \ldots, m$. To prove it, fix any $i = 1, \ldots, m$ and assume that $\bar{t}_i$ is a discontinuity point of f_0. Then condition (15) in combination with $f_n(\bar{t}_i^{(n)}) - f_n(\bar{t}_i^{(n)}-) = \bar{x}_i^{(n)} \to \bar{x}_i \ne 0$ as $n \to \infty$ entails $\bar{x}_i = f_0(\bar{t}_i) - f_0(\bar{t}_i-)$ and (21) (see the proof of Proposition 2.1 on p. 337 in [7]). If $\bar{t}_i$ is a continuity point of f_0, (21) holds trivially. Arguing similarly, we also obtain

$$\lim_{n\to\infty} \max_{t\in[0, t_i^{(n)} - 2/n]} f_n(t) = \sup_{t\in[0,\bar{t}_i)} f_0(t) \tag{22}$$

for $i = 1, \ldots, m$.

We first work with functions $g_{n,\varepsilon}$ and $g_{0,\varepsilon}$ which are counterparts of g_n and g_0 based on the restrictions of ν_n and ν_0 to $[0, T] \times E_\varepsilon$. Put

$$A_{n,T} := \{j \in \mathbb{N}_0 : 0 \le j \le [nT],\ (j + 1)/n \ne \bar{t}_i^{(n)} \text{ for } i = 1, \ldots, m\}.$$

Now we are ready to write a basic decomposition

$$\begin{aligned} g_{n,\varepsilon}(T) &:= \max_{0\le i\le [nT]} \big(x_1^{(n)}+\ldots+x_i^{(n)}+y_{i+1}^{(n)}\,\mathbb{1}_{\{y_{i+1}^{(n)}>\varepsilon\}}\big)\\ &= \max_{i\in A_{n,T}} f_n(i/n)\vee \max_{1\le k\le m}\Big(f_n(\bar t_k^{(n)}-1/n)+\bar y_k^{(n)}\Big)\\ &= \max_{0\le i\le [nT]} f_n(i/n)\vee \max_{1\le k\le m}\Big(f_n(\bar t_k^{(n)}-1/n)+\bar y_k^{(n)}\Big)\\ &= \max_{t\in[0,T]} f_n(t)\vee \max_{1\le k\le m}\Big(f_n(\bar t_k^{(n)}-1/n)+\bar y_k^{(n)}\Big), \end{aligned} \tag{23}$$

the third equality following from the fact that, for integer $i\in[0,[nT]]$ such that $i/n=\bar t_k^{(n)}-1/n$ for some $k=1,\ldots,m$, we have $f_n(i/n)<\max_{1\le k\le m}\Big(f_n(\bar t_k^{(n)}-1/n)+\bar y_k^{(n)}\Big)$ because all the $\bar y_k^{(n)}$ are positive.

It is convenient to state the following known result as a lemma, for it will be used twice in the subsequent proof.

Lemma 2.5. *Let s_0 be a continuity point of f_0 and $(s_n)_{n\in\mathbb{N}}$ a sequence of positive numbers converging to s_0 as $n\to\infty$. Then $\lim_{n\to\infty}\sup_{t\in[0,s_n]} f_n(t)=\max_{t\in[0,s_0]} f_0(t)$.*

Proof. We first observe that $\sup_{t\in[0,s_0]} f_0(t)=\max_{t\in[0,s_0]} f_0(t)$ because s_0 is a continuity point of f_0 (hence, of the supremum). It is well-known (and easily checked) that (15) entails

$$\lim_{n\to\infty}\sup_{0\le t\le\cdot} f_n(t)=\sup_{0\le t\le\cdot} f_0(t) \tag{24}$$

in the J_1-topology on D. In particular, $\lim_{n\to\infty}\sup_{t\in[0,s_n]} f_n(t)=\max_{t\in[0,s_0]} f_0(t)$. □

Recalling that T is a continuity point of f_0 and using Lemma 2.5 with $s_n=T$ for all $n\in\mathbb{N}_0$ we infer

$$\lim_{n\to\infty}\max_{t\in[0,T]} f_n(t)=\sup_{t\in[0,T]} f_0(t)$$

and thereupon

$$\lim_{n\to\infty} g_{n,\varepsilon}(T)=\sup_{s\in[0,T]} f_0(s)\vee\sup_{\bar t_k\le T}\,(f_0(\bar t_k-)+\bar y_k):=g_{0,\varepsilon}(T) \tag{25}$$

having utilized (20) and (21) for the second supremum.

Further, we claim that

$$\sup_{t\ge0}|g_0(t)-g_{0,\varepsilon}(t)|\le\varepsilon\quad\text{and}\quad\sup_{t\ge0}|g_n(t)-g_{n,\varepsilon}(t)|\le\varepsilon. \tag{26}$$

We only prove the first inequality, the proof of the second being analogous and simpler. Write

$$\begin{aligned}|g_0(t)-g_{0,\varepsilon}(t)| &= g_0(t)-g_{0,\varepsilon}(t)=\sup_{s\in[0,t]} f_0(s)\vee\sup_{t_k\le t}\,(f_0(t_k-)+y_k)\\ &- \sup_{s\in[0,t]} f_0(s)\vee\sup_{\bar t_k\le t}\,(f_0(\bar t_k-)+\bar y_k)\end{aligned}$$

for all $t\ge0$. There are two possibilities: either

$$\sup_{t_k\le t}\,(f_0(t_k-)+y_k)=\sup_{t_k\le t,\,t_k\ne\bar t_k}(f_0(t_k-)+y_k)\vee\sup_{\bar t_k\le t}\,(f_0(\bar t_k-)+\bar y_k)=\sup_{\bar t_k\le t}\,(f_0(\bar t_k-)+\bar y_k)$$

in which case $|g_0(t) - g_{0,\varepsilon}(t)| = 0$ for all $t \geq 0$, i.e., the first inequality in (26) holds, or

$$\sup_{t_k \leq t} (f_0(t_k-) + y_k) = \sup_{t_k \leq t,\, t_k \neq \bar{t}_k} (f_0(t_k-) + y_k).$$

Observe that

$$\sup_{t_k \leq t,\, t_k \neq \bar{t}_k} (f_0(t_k-) + y_k) \leq \sup_{s \in [0,t]} f_0(s) + \varepsilon$$

as a consequence of $y_k \leq \varepsilon$ for all $k \in \mathbb{N}$ such that $t_k \neq \bar{t}_k$, and that

$$\sup_{s \in [0,t]} f_0(s) \vee \sup_{\bar{t}_k \leq t} (f_0(\bar{t}_k-) + \bar{y}_k) \geq \sup_{s \in [0,t]} f_0(s).$$

Hence, the first inequality in (26) holds in this case, too.

It remains to note that

$$\begin{aligned} |g_n(T) - g_0(T)| &\leq |g_n(T) - g_{n,\varepsilon}(T)| + |g_{n,\varepsilon}(T) - g_{0,\varepsilon}(T)| + |g_{0,\varepsilon}(T) - g_0(T)| \\ &\leq 2\varepsilon + |g_{n,\varepsilon}(T) - g_{0,\varepsilon}(T)| \end{aligned}$$

and then first let n tend to ∞ and use (25), and then let ε go to zero through the sequence (ε_k). This shows that $\lim_{n\to\infty} g_n(T) = g_0(T)$. The proof of the M_1-convergence is complete.

Proof of the J_1-convergence. We intend to prove that whenever $\bar{s}$ is a discontinuity point of $g_{0,\varepsilon}$ there is a sequence $(s_n)_{n\in\mathbb{N}}$ converging to $\bar{s}$ for which

$$\lim_{n\to\infty} g_{n,\varepsilon}(s_n) = g_{0,\varepsilon}(\bar{s}) \quad \text{and} \quad \lim_{n\to\infty} g_{n,\varepsilon}(s_n-) = g_{0,\varepsilon}(\bar{s}-). \tag{27}$$

Now we explain that (27) entails

$$\lim_{n\to\infty} g_n = g_0 \tag{28}$$

in the J_1-topology on D, which is the desired result. From the first part of the proof we know that $\lim_{n\to\infty} g_{n,\varepsilon} = g_{0,\varepsilon}$ in the M_1-topology on D. Thus, if (27) holds, we conclude that

$$\lim_{n\to\infty} g_{n,\varepsilon} = g_{0,\varepsilon} \tag{29}$$

in the J_1-topology on D by Lemma 2.4(b). Let $r \in [0,T]$ be a continuity point of g_0, where $T \in K$ (see the first part of the proof for the definition of K). In order to prove (28) it suffices to show that $\lim_{n\to\infty} g_n = g_0$ in the J_1-topology on $D[0,r]$ or, equivalently, that $\lim_{n\to\infty} \rho(g_n, g_0) = 0$ where ρ is the standard Skorokhod metric on $[0,r]$. Since r is also a continuity point of $g_{0,\varepsilon}$, relation (29) ensures that $\lim_{n\to\infty} \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) = 0$. We proceed by writing

$$\begin{aligned} \rho(g_n, g_0) &\leq \rho(g_n, g_{n,\varepsilon}) + \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) + \rho(g_{0,\varepsilon}, g_0) \leq \sup_{0\leq t\leq r} |g_n(t) - g_{n,\varepsilon}(t)| + \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) \\ &+ \sup_{0\leq t\leq r} |g_{0,\varepsilon}(t) - g_0(t)| \leq 2\varepsilon + \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) \end{aligned}$$

having utilized the fact that ρ is dominated by the uniform metric on $[0,r]$ for the penultimate inequality and (26) for the last. Now sending $n \to \infty$ and then letting ε approach zero through the sequence (ε_k) proves $\lim_{n\to\infty} \rho(g_n, g_0) = 0$ and thereupon (28).

Passing to the proof of (27) and recalling that $\bar{s}$ is a discontinuity point of $g_{0,\varepsilon}$ we consider two cases.

Case 1: $\bar{s}$ is a discontinuity point of $g_{0,\varepsilon}$ and a continuity point of f_0.

We claim that $\bar{s} = \bar{t}_k$ for some $k = 1, \ldots, m$. Indeed, if $g_{0,\varepsilon}(\bar{s}) = \sup_{t\in[0,\bar{s}]} f_0(t)$, then $\bar{s}$ is a continuity point of $g_{0,\varepsilon}$, a contradiction. Thus, we must have $g_{0,\varepsilon}(\bar{s}) = \max_{\bar{t}_j\leq\bar{s}} (f_0(\bar{t}_j-) + \bar{y}_j)$. The points $\bar{t}_1, \ldots \bar{t}_m$ are the only discontinuities of $x \mapsto \max_{\bar{t}_j\leq x} (f_0(\bar{t}_j-) + \bar{y}_j)$ on $[0,\infty)$. Therefore, $\bar{s} = \bar{t}_k$ for some $k = 1, \ldots, m$, as claimed.

With this k, set $s_n = t_k^{(n)} - 1/n$. Analogously to (23) we obtain

$$g_{n,\varepsilon}\big(\bar{t}_k^{(n)} - 1/n\big) = \max_{t\in[0,\bar{t}_k^{(n)}-2/n]} f_n(t) \vee \max_{1\leq j\leq k} \big(f_n(\bar{t}_j^{(n)} - 1/n) + \bar{y}_j^{(n)}\big) \tag{30}$$

and

$$g_{n,\varepsilon}\big((\bar{t}_k^{(n)}-1/n)-\big) = g_{n,\varepsilon}\big(\bar{t}_k^{(n)}-2/n\big) = \max_{t\in[0,\bar{t}_k^{(n)}-2/n]} f_n(t)\vee \max_{1\leq j\leq k-1} \big(f_n(\bar{t}_j^{(n)}-1/n)+\bar{y}_j^{(n)}\big). \tag{31}$$

We shall now show

$$\lim_{n\to\infty} g_{n,\varepsilon}(\bar{t}_k^{(n)} - 1/n) = \sup_{t\in[0,\bar{t}_k]} f_0(t) \vee \max_{\bar{t}_j\leq\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big) = g_{0,\varepsilon}(\bar{t}_k)$$

and

$$\lim_{n\to\infty} g_{n,\varepsilon}((\bar{t}_k^{(n)} - 1/n)-) = \sup_{t\in[0,\bar{t}_k]} f_0(t) \vee \max_{\bar{t}_j<\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big) = g_{0,\varepsilon}(\bar{t}_k-).$$

Indeed, while the limit relations

$$\lim_{n\to\infty} \max_{1\leq j\leq k} \big(f_n(\bar{t}_j^{(n)} - 1/n) + \bar{y}_j^{(n)}\big) = \max_{\bar{t}_j\leq\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big) \tag{32}$$

and

$$\lim_{n\to\infty} \max_{1\leq j\leq k-1} \big(f_n(\bar{t}_j^{(n)} - 1/n) + \bar{y}_j^{(n)}\big) = \max_{\bar{t}_j<\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big) \tag{33}$$

are secured by (20) and (21), the limit relation

$$\lim_{n\to\infty} \max_{t\in[0,\bar{t}_k^{(n)}-2/n]} f_n(t) = \sup_{t\in[0,\bar{t}_k]} f_0(t)$$

holds in view of Lemma 2.5 with $s_n = \bar{t}_k^{(n)} - 1/n$ for $n \in \mathbb{N}$ and $s_0 = \bar{t}_k$. Thus, formula (27) has been proved in Case 1.

CASE 2: $\bar{s}$ is a discontinuity point of both $g_{0,\varepsilon}$ and f_0.

SUBCASE 2.1: $\bar{s} = \bar{t}_k$ for some $k = 1, \ldots, m$. We intend to check that (27) holds with $s_n = t_k^{(n)} - 1/n$. Using formulae (30) and (31) and recalling (22), (32) and (33) we infer

$$\lim_{n\to\infty} g_{n,\varepsilon}((\bar{t}_k^{(n)} - 1/n)-) = \sup_{t\in[0,\bar{t}_k)} f_0(t) \vee \max_{\bar{t}_j<\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big) = g_{0,\varepsilon}(\bar{t}_k-)$$

and

$$\lim_{n\to\infty} g_{n,\varepsilon}(\bar{t}_k^{(n)} - 1/n) = \sup_{t\in[0,\bar{t}_k)} f_0(t) \vee \max_{\bar{t}_j\leq\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big).$$

Since

$$f_0(\bar{t}_k) = f_0(\bar{t}_k-) + \bar{x}_k \leq f_0(\bar{t}_k-) + \bar{y}_k.$$

in view of (17), we conclude that

$$\sup_{t\in[0,\bar{t}_k)} f_0(t) \vee \max_{\bar{t}_j\leq\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big) = \sup_{t\in[0,\bar{t}_k]} f_0(t) \vee \max_{\bar{t}_j\leq\bar{t}_k} \big(f_0(\bar{t}_j-) + \bar{y}_j\big) = g_{0,\varepsilon}(\bar{t}_k),$$

thereby finishing the proof of (27) in this subcase.

SUBCASE 2.2: $\bar{s} \notin \{\bar{t}_1, \dots, \bar{t}_m\}$. Let r be a continuity point of f_0 satisfying $r > \bar{s}$. Recall that (15) entails (24). Hence, there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of continuous strictly increasing functions of $[0, r]$ onto $[0, r]$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,r]} |\lambda_n(t) - t| = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{s\in[0,r]} \Big| \sup_{t\in[0,\lambda_n(s)]} f_0(t) - \sup_{t\in[0,s]} f_n(t)\Big| = 0.$$

In particular, $\lim_{n\to\infty} \sup_{t\in[0,s_n)} f_n(t) = \sup_{t\in[0,\bar{s})} f_0(t)$ and $\lim_{n\to\infty} \sup_{t\in[0,s_n]} f_n(t) = \sup_{t\in[0,\bar{s}]} f_0(t)$, where $s_n := \lambda_n(\bar{s})$. We shall show that (27) holds with this choice of s_n. To this end, it only remains to note that

$$\lim_{n\to\infty} \max_{\bar{t}_j^{(n)}<s_n} \big(f_n(\bar{t}_j^{(n)})+\bar{y}_j^{(n)}\big) = \lim_{n\to\infty} \max_{\bar{t}_j^{(n)}\le s_n} \big(f_n(\bar{t}_j^{(n)})+\bar{y}_j^{(n)}\big) = \max_{\bar{t}_j\le\bar{s}} \big(f_0(\bar{t}_j-)+\bar{y}_j\big) = \max_{\bar{t}_j<\bar{s}} \big(f_0(\bar{t}_j-)+\bar{y}_j\big)$$

as a consequence of $\bar{s} \notin \{\bar{t}_1, \dots, \bar{t}_m\}$ and (20). Therefore,

$$\begin{aligned}\lim_{n\to\infty} g_{n,\varepsilon}(s_n-) &= \lim_{n\to\infty} \sup_{t\in[0,s_n)} f_n(t) \vee \max_{\bar{t}_j^{(n)}<s_n} \big(f_n(\bar{t}_j^{(n)}) + \bar{y}_j^{(n)}\big)\\ &= \sup_{t\in[0,\bar{s})} f_0(t) \vee \max_{\bar{t}_j<\bar{s}} \big(f_0(\bar{t}_j) + \bar{y}_j^{(n)}\big) = g_{0,\varepsilon}(\bar{s}-)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} g_{n,\varepsilon}(s_n) &= \lim_{n\to\infty} \sup_{t\in[0,s_n]} f_n(t) \vee \max_{\bar{t}_j^{(n)}\le s_n} \big(f_n(\bar{t}_j^{(n)}) + \bar{y}_j^{(n)}\big)\\ &= \sup_{t\in[0,\bar{s}]} f_0(t) \vee \max_{\bar{t}_j\le\bar{s}} \big(f_0(\bar{t}_j) + \bar{y}_j^{(n)}\big) = g_{0,\varepsilon}(\bar{s})\end{aligned}$$

which proves (27).

The proof of Theorem 2.1 is complete. □

Proof of Theorem 1.1. By Corollary 6.1 on p. 183 in [9], condition (6) entails

$$\sum_{l\ge1} \varepsilon_{(l/n,\, \xi_l/a(n),\, \eta_l^+/a(n))} \Rightarrow \sum_k \varepsilon_{(\theta_k, i_k, j_k)}, \quad n\to\infty$$

on $\mathcal{M}_p([0,\infty)\times E)$ and thereupon

$$\Big(\sum_{l\ge1} \mathbb{1}_{\{\xi_l\neq0\}}\, \varepsilon_{(l/n,\, \xi_l/a(n))}, \sum_{l\ge1} \varepsilon_{(l/n,\, \xi_l/a(n),\, \eta_l^+/a(n))}\Big) \Rightarrow \Big(\sum_k \mathbb{1}_{\{i_k\neq0\}}\, \varepsilon_{(\theta_k, i_k)}, \sum_k \varepsilon_{(\theta_k, i_k, j_k)}\Big) \tag{34}$$

as $n\to\infty$ on $\mathcal{M}_p([0,\infty)\times([-\infty,+\infty]\backslash\{0\}))\times\mathcal{M}_p([0,\infty)\times E)$ because the first coordinates are just the restrictions of the second from $[0,\infty)\times E$ on $[0,\infty)\times([-\infty,+\infty]\backslash\{0\})$.

In the proof of Corollary 7.1 on p. 218 in [9] it is shown that the convergence of the first coordinates in (34) implies $S_{[n\cdot]}/a(n) \overset{\mathrm{J}_1}{\Rightarrow} \mathcal{S}_\alpha(\cdot)$ as $n\to\infty$. Starting with *full* relation (34), i.e., that involving the two coordinates, exactly the same reasoning as in [9] leads to the conclusion

$$\Big(\frac{S_{[n\cdot]}}{a(n)}, \sum_{l\ge1} \varepsilon_{(l/n,\, \xi_l/a(n),\, \eta_l^+/a(n))}\Big) \Rightarrow \Big(\mathcal{S}_\alpha^*(\cdot), \sum_k \varepsilon_{(\theta_k, i_k, j_k)}\Big), \quad n\to\infty$$

or, equivalently,

$$\Big(\frac{S_{[n\cdot]}}{a(n)}, \sum_{l\geq 1}\varepsilon_{(l/n,\, \xi_l/a(n),\, \eta_l/a(n))}\, \mathbb{1}_{\{\eta_l>0\}}\Big) \ \Rightarrow\ \Big(\mathcal{S}^*_\alpha(\cdot),\, \sum_k \varepsilon_{(\theta_k, i_k, j_k)}\Big), \quad n\to\infty$$

in the product topology on $D\times\mathcal{M}_p([0,\infty)\times E)$. By the Skorokhod representation theorem there are versions which converge a.s. Retaining the original notation for these versions we want to apply Theorem 2.1 with $f_n(\cdot)=S_{[n\cdot]}/a(n)$, $f_0=\mathcal{S}^*_\alpha$, $\nu_n=\sum_{l\geq 1}\varepsilon_{\{l/n,\, \xi_l/a(n),\, \eta_l/a(n)\}}\,\mathbb{1}_{\{\eta_l>0\}}$ and $\nu_0=N^{(\nu)}=\sum_k \varepsilon_{(\theta_k,i_k,j_k)}$. We already know that conditions (15) and (16) are fulfilled a.s. It is obvious that $N^{(\nu)}(\{0\}\times E)=0$ a.s. In order to show that $N^{(\nu)}$ does not have clustered jumps a.s. i.e., $\theta_k\neq\theta_j$ for $k\neq j$ a.s., it suffices to check this property for $N^{(\nu)}(([0,T]\times[-\infty,+\infty]\times(\delta,\infty])\cap\cdot)$ with $T>0$ and $\varepsilon>0$ fixed. This is done on p. 223 in [9]. Hence Theorem 2.1 is indeed applicable with our choice of f_n and ν_n, and (7) follows. □

3 Proofs of Propositions 1.7 and 1.9 and Theorem 1.10

Proof of Proposition 1.7. Fix any $T>0$. Note that (9) entails

$$\lim_{x\to\infty} x\mathbb{P}\{\eta>\varepsilon a(x)\}=0 \qquad (35)$$

for all $\varepsilon>0$ because $a(x)$ is regularly varying at ∞ (of index $1/\alpha$). Since, for all $\varepsilon>0$,

$$\begin{aligned}\mathbb{P}\{\sup_{0\leq s\leq T}\eta_{[ns]+1}>\varepsilon a(n)\} &= 1-\big(\mathbb{P}\{\eta\leq\varepsilon a(n)\}\big)^{[nT]+1}\\ &\leq ([nT]+1)\mathbb{P}\{\eta>\varepsilon a(n)\}\to 0\end{aligned}$$

as $n\to\infty$ in view of (35), we infer

$$\frac{\sup_{0\leq s\leq T}\eta_{[ns]+1}}{a(n)} \overset{\mathrm{P}}{\to} 0, \quad n\to\infty.$$

This in combination with (4) enables us to conclude that

$$\frac{S_{[n\cdot]}+\eta_{[n\cdot]+1}}{a(n)} \overset{\mathrm{J_1}}{\Rightarrow} \mathcal{S}_\alpha(\cdot), \quad n\to\infty$$

by Slutsky's lemma. Relation (10) now follows by the continuous mapping theorem because the supremum functional is continuous in the J_1-topology. □

Proof of Proposition 1.9. To begin with, we note that $\lim_{n\to\infty}(b(n)/a(n))=\infty$ as a consequence of (12). Consequently,

$$S_{[n\cdot]}/b(n) \overset{\mathrm{J_1}}{\Rightarrow} \Xi(\cdot)$$

in view of (4), where $\Xi(t):=0$ for $t\geq 0$. Further, according to Theorem 3.6 on p. 62 in combination with Corollary 6.1 on p. 183 in [9], regular variation of $\mathbb{P}\{\eta>x\}$ ensures that

$$\sum_{k\geq 0}\varepsilon_{(k/n,\eta_{k+1}/b(n))}\,\mathbb{1}_{\{\eta_{k+1}>0\}} \ \Rightarrow\ N:=\sum_k \varepsilon_{(\theta_k,j_k)}, \quad n\to\infty$$

on $\mathcal{M}_p([0,\infty)\times(0,\infty])$ and thereupon

$$\Big(S_{[n\cdot]}/b(n), \sum_{k\geq 0}\mathbb{1}_{\{\eta_{k+1}>0\}}\,\varepsilon_{(k/n,\eta_{k+1}/b(n))}\Big) \ \Rightarrow\ \big(\Xi(\cdot),N\big), \quad n\to\infty$$

on $D \times \mathcal{M}_p([0,\infty) \times (0,\infty])$ equipped with the product topology. Arguing as in the proof of Theorem 1.1 we obtain (13) by an application of Remark 2.2 with $f_n(\cdot) = S_{[n\cdot]}/b(n)$, $f_0 = \Xi$, $\nu_n = \sum_{k\geq 0} \varepsilon_{(k/n, \eta_{k+1}/b(n))} \mathbb{1}_{\{\eta_{k+1}>0\}}$ and $\nu_0 = N$. The condition $N((a,b) \times (0,\infty]) \geq 1$ a.s. whenever $0 < a < b$ required in Remark 2.2 holds because $\mu((0,\infty]) = \infty$. □

Proof of Theorem 1.10. The limit relations of Theorem 1.1 and Propositions 1.7 and 1.9 can be written in a unified form as

$$\frac{\max_{0\leq k\leq [n\cdot]} T_{k+1}}{c(n)} \Rightarrow X(\cdot), \quad n \to \infty$$

in the J_1- or the M_1-topology on D. Using this limit relation together with the inequality

$$\max_{0\leq k\leq n} T_{k+1} \leq \log\left(\sum_{k=0}^{n} e^{T_{k+1}}\right) \leq \log(n+1) + \max_{0\leq k\leq n} T_{k+1}$$

and the fact that $\lim_{n\to\infty} \frac{\log n}{c(n)} = 0$ we arrive at the desired conclusion

$$\frac{\log \sum_{k=0}^{[n\cdot]} e^{T_{k+1}}}{c(n)} \Rightarrow X(\cdot), \quad n \to \infty$$

in the J_1- or M_1-topology on D. □

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References

[1] G. Alsmeyer, A. Iksanov and M. Meiners, *Power and exponential moments of the number of visits and related quantities for perturbed random walks.* J. Theoret. Probab. **28** (2015), 1–40.

[2] P. Billingsley, *Convergence of probability measures.* Wiley, 1968.

[3] D. Buraczewski and A. Iksanov, *Functional limit theorems for divergent perpetuities in the contractive case.* Electron. Commun. Probab. **20**, article 10 (2015), 1–14.

[4] A. Iksanov, *Renewal theory for perturbed random walks and similar processes.* Birkhäuser, 2016.

[5] A. M. Iksanov, A. V. Marynych and V. A. Vatutin, *Weak convergence of finite-dimensional distributions of the number of empty boxes in the Bernoulli sieve.* Theory Probab. Appl. **59** (2015), 87–113.

[6] A. Iksanov and A. Pilipenko, *On the maximum of a perturbed random walk.* Stat. Probab. Letters. **92** (2014), 168–172.

[7] J. Jacod and A. N. Shiryaev, *Limit theorems for stochastic processes.* 2nd Edition, Springer, 2003.

[8] S. T. Rachev and G. Samorodnitsky, *Limit laws for a stochastic process and random recursion arising in probabilistic modelling.* Adv. Appl. Probab. **27** (1995), 185–202.

[9] S. I. Resnick, *Heavy-tail phenomena. Probabilistic and statistical modeling.* Springer, 2007.

[10] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes: stochastic models with infinite variance.* Chapman & Hall, 1994.

[11] K. Sato, *Lévy processes and infinitely divisible distributions.* Cambridge University Press, 1999.

[12] A. V. Skorohod, *Limit theorems for stochastic processes with independent increments*. Theor. Probab. Appl. **2** (1957), 138–171.

[13] Y. Wang, *Convergence to the maximum process of a fractional Brownian motion with shot noise*. Stat. Probab. Letters. **90** (2014), 33–41.

[14] W. Whitt, *Stochastic-process limits: an introduction to stochastic-process limits and their application to queues*. Springer, 2002.