

Moment convergence of first-passage times in renewal theory

Alexander Iksanov^a, Alexander Marynych^{a,b}, Matthias Meiners^{c,*}

^a*Faculty of Cybernetics, National T. Shevchenko University of Kiev, 01033 Kiev, Ukraine*

^b*Department of Mathematics and Computer Science, University of Münster, 48149 Münster
Germany*

^c*Department of Mathematics, Technical University of Darmstadt, 64289 Darmstadt,
Germany*

Abstract

Let ξ_1, ξ_2, \dots be independent copies of a positive random variable ξ , $S_0 = 0$, and $S_k = \xi_1 + \dots + \xi_k$, $k \in \mathbb{N}_0$. Define $N(t) = \inf\{k \in \mathbb{N} : S_k > t\}$ for $t \geq 0$. The process $(N(t))_{t \geq 0}$ is the first-passage time process associated with $(S_k)_{k \geq 0}$. It is known that if the law of ξ belongs to the domain of attraction of a stable law or $\mathbb{P}(\xi > t)$ varies slowly at ∞ , then $N(t)$, suitably shifted and scaled, converges in distribution as $t \rightarrow \infty$ to a random variable W with a stable law or a Mittag-Leffler law. We investigate whether there is convergence of the power and exponential moments to the corresponding moments of W . Further, the analogous problem for first-passage times of subordinators is considered.

Keywords: Exponential moment, Lévy process, power moment, renewal process, subordinator

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1. Introduction and results

Setup. Let ξ_1, ξ_2, \dots be independent copies of a positive random variable ξ . We set $\mu := \mathbb{E}[\xi] \in (0, \infty]$, and then $\sigma^2 := \mathbb{V}\text{ar}[\xi]$ whenever μ is finite. Throughout the paper, we assume that the law of ξ is non-degenerate, that is, $\mathbb{P}(\xi = c) < 1$ for all $c > 0$. Define

$$S_0 := 0, \quad S_k := \xi_1 + \dots + \xi_k, \quad k \in \mathbb{N},$$

and

$$N(t) := \#\{k \in \mathbb{N}_0 : S_k \leq t\} = \inf\{k \in \mathbb{N} : S_k > t\}, \quad t \geq 0.$$

The stochastic process $(N(t))_{t \geq 0}$ is called *first-passage time process* associated with $(S_k)_{k \geq 0}$. The term ‘renewal counting process’ is also used.

*Corresponding author

Objective. It is known (see, for instance, [1, Proposition A.1]) that if the law of ξ is in the domain of attraction of a stable law or $\mathbb{P}(\xi > t)$ varies slowly at ∞ , then

$$\frac{N(t) - b(t)}{a(t)} \xrightarrow{d} W \quad \text{as } t \rightarrow \infty \quad (1.1)$$

where “ \xrightarrow{d} ” denotes convergence in distribution, W is a non-degenerate random variable, and $b(t) \in \mathbb{R}$, $a(t) > 0$ are suitable shifting and scaling functions, respectively.

The purpose of this note is to answer the question: when does (1.1) imply convergence of the corresponding power and exponential moments, finite or infinite? The motivation for writing a short note on this problem comes from the fact that the moment convergence of first-passage time processes repeatedly turned out to be an important technical step in other works on processes bearing some regenerative or renewal structure. For instance, Theorems 1.1 and 1.4 below are essential ingredients in our work on the finite-dimensional convergence of shot noise processes [2]. Theorem 1.5 is used to prove convergence of shot noise processes to fractionally integrated inverse stable subordinators [3]. Corollary 1.6 is used in the proof of Theorem 3.3 in [4]. Consequently, we found it useful to have one paper which contains the complete results on convergence of power and exponential moments for renewal counting processes.

Before we state our results we briefly recall the different regimes in which (1.1) holds.

Domains of attraction. The law of a random variable ξ is in the domain of attraction of an α -stable law, $\alpha \in (0, 2]$ or $\mathbb{P}\{\xi > t\}$ varies slowly at ∞ if one of the following alternatives prevails¹:

- (A1) $\mu < \infty$ and $\sigma^2 := \text{Var}[\xi] < \infty$;
- (A2) $\mu < \infty$ but $\sigma^2 = \infty$ and $\ell_2(t) := \mathbb{E}[\xi^2 \mathbb{1}_{\{\xi \leq t\}}]$ is slowly varying at ∞ ;
- (A3) $\mathbb{P}(\xi > t) = t^{-\alpha} \ell(t)$ for some $\alpha \in (1, 2)$ and a function ℓ slowly varying at ∞ ;
- (A4) $\mathbb{P}(\xi > t) = t^{-\alpha} \ell(t)$ for some $\alpha \in [0, 1)$ and a function ℓ slowly varying at ∞ .

We refer to [5, Section 2.6] for details. The convergence of the first-passage time process in (1.1) can now be described more precisely:

- (N1) if (A1) holds, then $b(t) = t/\mu$, $a(t) = \sigma\mu^{-3/2}c(t)$, $c(t) = \sqrt{t}$, and W is a standard normal random variable;
- (N2) if (A2) holds, then $b(t) = t/\mu$, $a(t) = \mu^{-3/2}c(t)$ where $c(t)$ is a positive function satisfying $\lim_{t \rightarrow \infty} t\ell_2(c(t))c(t)^{-2} = 1$, and W is a standard normal random variable;

¹ Here, we do not treat the case where $\mathbb{P}(\xi > t)$ is regularly varying of index -1 at ∞ as it appears less frequently in applications and requires cumbersome calculations that would impair the character of this paper as a brief note.

(N3) if (A3) holds, then $b(t) = t/\mu$, $a(t) = \mu^{-(1+\alpha)/\alpha}c(t)$ where $c(t)$ is a positive function such that $\lim_{t \rightarrow \infty} t\ell(c(t))c(t)^{-\alpha} = 1$, and W is a random variable with characteristic function given by²

$$\psi(\lambda) = \exp\left\{-|\lambda|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i \sin(\pi\alpha/2) \operatorname{sgn}(\lambda))\right\}, \quad \lambda \in \mathbb{R} \quad (1.2)$$

where $\Gamma(\cdot)$ denotes Euler's gamma function;

(N4) if (A4) holds, then $b(t) = 0$, $a(t) = 1/\mathbb{P}(\xi > t)$, and W has a Mittag-Leffler distribution with parameter α (exponential with mean 1 if $\alpha = 0$), that is, W has moment generating function

$$\mathbb{E}[e^{\theta W}] = E_\alpha\left(\frac{\theta}{\Gamma(1-\alpha)}\right) < \infty, \quad \theta \in \mathbb{R}$$

where here and throughout the paper, E_α is the Mittag-Leffler function with parameter α given by $E_\alpha(z) := \sum_{k \geq 0} \frac{z^k}{\Gamma(k\alpha+1)}$ for $z \in \mathbb{R}$.

Main results for random walks. In what follows we use the notation x_- and x_+ for the negative and positive part of a real number x :

$$x_- := -\min\{x, 0\} \quad \text{and} \quad x_+ := \max\{x, 0\}.$$

Theorem 1.1. *Suppose that either (A1) or (A2) holds, i. e., $\mu < \infty$ and either $\sigma^2 < \infty$ or $\sigma^2 = \infty$ and $\ell_2(t) := \mathbb{E}[\xi^2 \mathbb{1}_{\{\xi \leq t\}}]$ is slowly varying at ∞ . Then*

$$\lim_{t \rightarrow \infty} \mathbb{E}\left[\exp\left(\theta \frac{N(t) - t/\mu}{a(t)}\right)\right] = \mathbb{E}[e^{\theta W}] = e^{\frac{\theta^2}{2}}, \quad \text{for every } \theta \geq 0 \quad (1.3)$$

where W is standard normal, $a(t) = \sqrt{\sigma^2 \mu^{-3} t}$ in the case (A1) and $a(t) = \mu^{-3/2} c(t)$ for a positive function $c(t)$ satisfying $\lim_{t \rightarrow \infty} t\ell_2(c(t))c(t)^{-2} = 1$ in the case (A2). In particular,

$$\lim_{t \rightarrow \infty} \mathbb{E}\left[\left(\frac{N(t) - t/\mu}{a(t)}\right)_+^p\right] = \mathbb{E}[W_+^p] = \frac{2^{p/2-1} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \quad \text{for every } p > 0. \quad (1.4)$$

Further, in the case (A1)

$$\lim_{t \rightarrow \infty} \mathbb{E}\left[\left(\frac{N(t) - t/\mu}{a(t)}\right)_-^p\right] = \mathbb{E}[W_-^p] = \frac{2^{p/2-1} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}}, \quad (1.5)$$

for every $p \in [0, 2]$. In the case (A2) the relation (1.5) holds for $p \in [0, 2)$ and

$$\mathbb{E}[(N(t) - t/\mu)^2] \sim \frac{2t}{\mu^3} \int_0^t \left(\int_x^\infty \mathbb{P}(\xi > z) dz\right) dx, \quad \text{as } t \rightarrow \infty. \quad (1.6)$$

²For $\alpha \in (1, 2)$, $\Gamma(1-\alpha)$ is understood as $-\Gamma(2-\alpha)/(\alpha-1)$.

Remark 1.2. Without further assumptions on the law ξ , the result stated in Theorem 1.1 is best possible in the following sense. There exists a law for ξ such that $\mathbb{E}[\xi^2] < \infty$ and

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\left(\frac{N(t) - t/\mu}{\sqrt{t}} \right)^p \right] = \infty, \quad (1.7)$$

for every $p > 2$. An example is provided at the end of Section 2.1.

⁴⁵ **Remark 1.3.** Convergence (1.5) is well-known in the case (A1) (see, for instance, [6, Theorem 3.8.4]). The asymptotic relation (1.6) follows from [7, Theorems 2.3 and 2.4].

Theorem 1.4. *Suppose that (A3) holds, i. e., $\mathbb{P}(\xi > t) = t^{-\alpha}\ell(t)$ for some $\alpha \in (1, 2)$ and some ℓ slowly varying at ∞ . Let $c(t)$ be a positive function such that $\lim_{t \rightarrow \infty} t\ell(c(t))c(t)^{-\alpha} = 1$ and let W be a random variable with characteristic function given by (1.2). Then, for every $\theta \geq 0$, we have*

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\exp \left(\theta \frac{N(t) - t/\mu}{a(t)} \right) \right] = \mathbb{E}[e^{\theta W}] = e^{-\Gamma(1-\alpha)\theta^\alpha}, \quad (1.8)$$

where $a(t) = \mu^{-(1+\alpha)/\alpha}c(t)$. Further,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[(N(t) - t/\mu)_\pm^p]}{a(t)^p} = \mathbb{E}[W_\pm^p] \quad \text{for all } p > 0, \quad (1.9)$$

where $\mathbb{E}[W_+^p] < \infty$ for all $p > 0$ and $\mathbb{E}[W_-^p] < \infty$ if and only if $p < \alpha$. In particular,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mathbb{E}[|N(t) - t/\mu|^p]}{a(t)^p} &= \mathbb{E}[|W|^p] \\ &= \begin{cases} \frac{2\Gamma(p+1)}{\pi p} \sin\left(\frac{\pi p}{2}\right) \Gamma\left(1 - \frac{p}{\alpha}\right) |\Gamma(1-\alpha)|^{\frac{p}{\alpha}} \cos\left(\frac{\pi p}{2} - \frac{\pi p}{\alpha}\right) & \text{for } 0 < p < \alpha, \\ \infty & \text{for } p \geq \alpha. \end{cases} \end{aligned}$$

Theorem 1.5. *Suppose that (A4) holds, i. e., $\mathbb{P}(\xi > t) = t^{-\alpha}\ell(t)$ for some $\alpha \in [0, 1)$ and some ℓ slowly varying at ∞ . Then*

$$\lim_{t \rightarrow \infty} \mathbb{E}[e^{\theta \mathbb{P}(\xi > t)N(t)}] = \mathbb{E}[e^{\theta W}] = E_\alpha \left(\frac{\theta}{\Gamma(1-\alpha)} \right) < \infty \quad \text{for every } \theta \in \mathbb{R}. \quad (1.10)$$

In particular,

$$\lim_{t \rightarrow \infty} \mathbb{E}[(\mathbb{P}(\xi > t)N(t))^p] = \mathbb{E}[W^p] = \frac{\Gamma(p+1)}{\Gamma(1+\alpha)^p \Gamma(p\alpha+1)} < \infty \quad \text{for every } p \geq 0. \quad (1.11)$$

Main results for subordinators. Let $(X_t)_{t \geq 0}$ denote a subordinator, i. e., a non-decreasing Lévy process, with $X_0 = 0$, drift coefficient $m \geq 0$, no killing and
50 Lévy measure Π that is concentrated on $\mathbb{R}^+ := [0, \infty)$. Notice that compound Poisson processes are not excluded. Put,

$$T_r := \inf\{t \geq 0 : X_t > r\}, \quad r \geq 0.$$

The stochastic process $(T_r)_{r \geq 0}$ is called *first-passage time process* associated with $(X_t)_{t \geq 0}$. The counterpart of (1.1) for $(T_r)_{r \geq 0}$ is

$$\frac{T_r - b(r)}{a(r)} \xrightarrow{d} W \quad \text{as } r \rightarrow \infty \quad (1.12)$$

for suitable constants $b(r) \in \mathbb{R}$ and $a(r) > 0$. Let $N_r := \inf\{k \in \mathbb{N} : X_k > r\}$ for $r \geq 0$. Then $(N_r)_{r \geq 0}$ is the first-passage time process of $(X_n)_{n \in \mathbb{N}_0}$. Clearly,
55 $T_r \leq N_r \leq T_r + 1$. Hence, (1.12) holds if and only if (1.1) holds with $N(t)$ replaced by N_t . Furthermore, convergence of exponential or power moments in (1.12) holds if, and only if, the corresponding convergence for the moments of N_r holds. We summarize these observations in the following corollary.

Corollary 1.6. *Let $(X_t)_{t \geq 0}$ be a subordinator with $X_0 = 0$, drift coefficient
60 $m \geq 0$, no killing and Lévy measure Π concentrated on \mathbb{R}^+ . Define $\xi := X_1$, $\mu := \mathbb{E}[\xi]$ and $\sigma^2 := \mathbb{V}\text{ar}[\xi]$. Then the following assertions hold:*

(a) *If the law of ξ satisfies (A1), equivalently, $\int_{\{|x| \geq 1\}} x^2 \Pi(dx) < \infty$, then*

$$\lim_{r \rightarrow \infty} \mathbb{E} \left[\exp \left(\theta \frac{T_r - r/\mu}{a(r)} \right) \right] = \mathbb{E}[e^{\theta W}] = e^{\frac{\theta^2}{2}} \quad \text{for every } \theta \geq 0 \quad (1.13)$$

where W is standard normal and $a(r) := \sqrt{\sigma^2 \mu^{-3} r}$. In particular,

$$\lim_{r \rightarrow \infty} \mathbb{E} \left[\left(\frac{T_r - r/\mu}{a(r)} \right)_+^p \right] = \mathbb{E}[W_+^p] = \frac{2^{p/2-1} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \quad \text{for every } p > 0. \quad (1.14)$$

Further,

$$\lim_{r \rightarrow \infty} \mathbb{E} \left[\left(\frac{T_r - r/\mu}{a(r)} \right)_-^p \right] = \mathbb{E}[W_-^p] = \frac{2^{p/2-1} \Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \quad \text{for every } p \in [0, 2]. \quad (1.15)$$

(b) *If the law of ξ satisfies (A2), equivalently,*

$$\mu < \infty, \quad \sigma^2 = \infty \quad \text{and} \quad \ell_2^\Pi(t) := \int_{(1,t]} x^2 \Pi(dx) \quad \text{is slowly varying at } \infty, \quad (1.16)$$

then (1.13), (1.14) and (1.15) (the latter only for $0 < p < 2$) hold with $a(r) = \mu^{-3/2} c(r)$ where $c(r)$ is positive and $\lim_{r \rightarrow \infty} r \ell_2^\Pi(c(r)) c(r)^{-2} = 1$.

(c) If the law of ξ satisfies (A3), equivalently, $\Pi((t, \infty)) = t^{-\alpha} \ell^\Pi(t)$ for some $\alpha \in (1, 2)$ and some ℓ^Π slowly varying at ∞ , then (1.13) holds with $a(r) = \mu^{-(1+\alpha)/\alpha} c(r)$ where $c(r)$ is positive and $\lim_{r \rightarrow \infty} r \ell^\Pi(c(r)) c(r)^{-\alpha} = 1$, and W is a random variable with characteristic function given by (1.2). Further,

$$\lim_{r \rightarrow \infty} \frac{\mathbb{E}[(T_r - r/\mu)_\pm^p]}{a(r)^p} = \mathbb{E}[W_\pm^p] \quad \text{for all } p > 0. \quad (1.17)$$

(d) If ξ satisfies (A4), equivalently, $\Pi((t, \infty)) = t^{-\alpha} \ell^\Pi(t)$ for some $\alpha \in [0, 1)$ and a function ℓ^Π which is slowly varying at ∞ , then

$$\lim_{r \rightarrow \infty} \mathbb{E}[e^{\theta \Pi((r, \infty)) T_r}] = \mathbb{E}[e^{\theta W}] = E_\alpha\left(\frac{\theta}{\Gamma(1-\alpha)}\right) < \infty \quad \text{for every } \theta \in \mathbb{R} \quad (1.18)$$

where W has the Mittag-Leffler distribution with parameter α and $E_\alpha(\cdot)$ is the Mittag-Leffler function.

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We close this section with a remark that the asymptotics of $\mathbb{E}[e^{aN(t)}]$ as $t \rightarrow \infty$ as well as exponential moments of the number of visits and the last-exit time was investigated in [8, 9] for random walks with two-sided jumps and in [10] for Lévy processes.

70 2. Proofs of the main results

We denote by φ the Laplace transform of ξ , i.e., $\varphi(\lambda) = \mathbb{E}[e^{-\lambda \xi}]$, $\lambda \geq 0$. Some relevant results about the behavior of φ at 0 which we use in the proofs below are collected in the Appendix.

2.1. Proof of Theorems 1.1, 1.4

Convergence of exponential moments of positive order and power moments of the positive parts. In view of (N1), (N2) and (N3) we have

$$e^{\theta \frac{N(t) - t/\mu}{a(t)}} \xrightarrow{d} e^{\theta W} \quad \text{as } t \rightarrow \infty$$

for every $\theta \geq 0$, where W is standard normal in the cases (A1) and (A2), and W has characteristic function given by (1.2) in the case (A3). Hence, it is enough to show that the family $(\exp(\theta a(t)^{-1}(N(t) - t/\mu)))_{t \geq t_0}$ is uniformly integrable for every $\theta > 0$ and some $t_0 > 0$. To this end, by the Vallée-Poussin criterion of uniform integrability it suffices to check that

$$\sup_{t \geq t_0} \mathbb{E}\left[e^{\theta \frac{N(t) - t/\mu}{a(t)}}\right] < \infty \quad (2.1)$$

for every $\theta > 0$. While doing so, we can neglect the constant factors in the scaling functions $a(t)$ thus working with $c(t)$ in place of $a(t)$. With the help of

Markov's inequality we obtain

$$\begin{aligned}
\mathbb{E}\left[e^{\theta \frac{N(t)-t/\mu}{c(t)}}\right] &= \int_0^\infty \mathbb{P}\left(e^{\theta \frac{N(t)-t/\mu}{c(t)}} > x\right) dx = \int_{-\infty}^\infty e^x \mathbb{P}\left(\theta \frac{N(t)-t/\mu}{c(t)} > x\right) dx \\
&\leq 1 + \int_0^\infty e^x \mathbb{P}(N(t) > xc(t)/\theta + t/\mu) dx \\
&= 1 + \int_0^\infty e^x \mathbb{P}(S_{\lfloor xc(t)/\theta + t/\mu \rfloor} \leq t) dx \\
&\leq 1 + \int_0^\infty e^x e^{\lambda t} (\varphi(\lambda))^{xc(t)/\theta + t/\mu - 1} dx \\
&\leq 1 + (e^{\lambda \mu} \varphi(\lambda))^{t/\mu} \int_0^\infty e^x (\varphi(\lambda))^{xc(t)/\theta - 1} dx
\end{aligned} \tag{2.2}$$

for every $\lambda > 0$. We will demonstrate that (2.1) is a consequence of

$$\sup_{t \geq t_0} \int_0^\infty e^x \varphi(\lambda/c(t))^{xc(t)/\theta - 1} dx < \infty \tag{2.3}$$

75 for some $\lambda > 0$.

CASE (A1) in which $c(t) = \sqrt{t}$. From formula (3.1) in the Appendix we infer

$$\varphi(\lambda/\sqrt{t}) = 1 - \frac{\mu\lambda}{\sqrt{t}} + \frac{\mu^2 + \sigma^2}{2} \frac{\lambda^2}{t} + o\left(\frac{1}{t}\right) \quad \text{as } t \rightarrow \infty$$

whence

$$e^{\lambda\mu/\sqrt{t}} \varphi(\lambda/\sqrt{t}) = 1 + \frac{\sigma^2}{2} \frac{\lambda^2}{t} + o\left(\frac{1}{t}\right) \quad \text{as } t \rightarrow \infty.$$

Thus, substituting λ by λ/\sqrt{t} in (2.2) we see that (2.3) is indeed sufficient for (2.1) to hold.

CASE (A2). From (3.2) and the relation $\lim_{t \rightarrow \infty} t\ell_2(t)c(t)^{-2} = 1$, we infer

$$\varphi(\lambda/c(t)) = 1 - \frac{\mu\lambda}{c(t)} + \frac{1}{2} \frac{\lambda^2}{t} + o\left(\frac{1}{t}\right) \quad \text{as } t \rightarrow \infty$$

and, since $c(t)^{-2} = o(t^{-1})$ as $t \rightarrow \infty$,

$$e^{\lambda\mu/c(t)} \varphi(\lambda/c(t)) = 1 + \frac{1}{2} \frac{\lambda^2}{t} + o\left(\frac{1}{t}\right) \quad \text{as } t \rightarrow \infty.$$

Thus, substituting λ by $\lambda/c(t)$ in (2.2) we conclude that (2.3) is sufficient for (2.1).

CASE (A3). Since $c(t) \rightarrow \infty$ as $t \rightarrow \infty$, we infer from (3.3) in the Appendix that

$$\varphi(\lambda/c(t)) = 1 - \frac{\mu\lambda}{c(t)} + \frac{\lambda^\alpha c_\alpha}{t} + o(1/t) \tag{2.4}$$

whence

$$e^{\lambda\mu/c(t)} \varphi(\lambda/c(t)) = 1 + \frac{\lambda^\alpha c_\alpha}{t} + o(1/t)$$

80 as $t \rightarrow \infty$. This implies the claim.

It remains to prove (2.3). From formulae (3.1), (3.2) and (3.3) we deduce that for every fixed $\varepsilon \in (0, \mu)$, $\lambda > 0$ and sufficiently large t

$$\varphi\left(\frac{\lambda}{c(t)}\right) \leq 1 - \frac{(\mu - \varepsilon)\lambda}{c(t)} \leq e^{-(\mu - \varepsilon)\lambda/c(t)}.$$

Consequently,

$$\int_0^\infty e^x \varphi(\lambda/c(t))^{xc(t)/\theta - 1} dx \leq e^{(\mu - \varepsilon)\lambda/c(t)} \int_0^\infty e^{x(1 - (\mu - \varepsilon)\lambda/\theta)} dx,$$

and the latter integral is finite provided that λ is chosen large enough.

Thus, the first equalities in relations (1.3) and (1.8) are proved. The second equality in (1.3) is a well-known formula for exponential moments of a standard normal law. The second equality in (1.8), namely, $\mathbb{E}[e^{\theta W}] = e^{-\Gamma(1-\alpha)\theta^\alpha}$ for all 85 $\theta \geq 0$, can be found in many sources, see, for instance, [11, Exercise 29.15]. Now the first parts of Theorems 1.1 and 1.4 regarding the exponential moments of positive order are completely proved. Relations (1.4) and (1.9) (the latter only for the positive parts) follow from the inequality $x_+^p \leq e^{px}$ which yields the uniform integrability of the corresponding families.

Convergence of power moments of negative parts. We treat the cases (A2) and (A3) simultaneously. First fix $0 < p < \alpha$ (with $\alpha = 2$ in the case (A2)) and $r \in (p \vee 1, \alpha)$. As before it is enough to show that for some $t_0 > 0$,

$$\sup_{t \geq t_0} \frac{\mathbb{E}[(N(t) - t/\mu)_-^r]}{c(t)^r} < \infty.$$

By the regular variation of c , this is implied by

$$\mathbb{E}[(N(\mu n) - n)_-^r] = O(c(n)^r) \quad \text{as } n \rightarrow \infty. \quad (2.5)$$

We have

$$\begin{aligned} & \mathbb{E}[(N(\mu n) - n)_-^r] \\ &= \sum_{k \geq 1} \mathbb{P}((N(\mu n) - n)_-^r \geq k) = \sum_{k \geq 1} \mathbb{P}(N(\mu n) \leq n - k^{1/r}) \\ &= \sum_{k=1}^{\lfloor n^r \rfloor} \mathbb{P}(S_{\lfloor n - k^{1/r} \rfloor} > \mu n) = \sum_{j=0}^{n-1} \sum_{k \in (j^r, (j+1)^r]} \mathbb{P}(S_{\lfloor n - k^{1/r} \rfloor} > \mu n) \\ &\leq \sum_{j=0}^{n-1} ((j+1)^r - j^r) \mathbb{P}(S_{n-j-1} > \mu n) \leq r \sum_{j=0}^{n-1} (j+1)^{r-1} \mathbb{P}(S_{n-j-1} > \mu n) \\ &= r \sum_{j=1}^n j^{r-1} \mathbb{P}(S_{n-j} - (n-j)\mu > \mu j) \leq r \sum_{j=1}^n j^{r-1} \mathbb{P}\left(\max_{0 \leq i < n} (S_i - i\mu) > \mu j\right) \\ &\leq r + \text{const} \cdot \mathbb{E}\left[\max_{0 \leq i \leq n} |S_i - i\mu|^r\right] \leq r + \text{const} \cdot \mathbb{E}[|S_n - n\mu|^r] = O(c(n)^r) \end{aligned}$$

90 as $n \rightarrow \infty$ where the penultimate step is a consequence of the maximal \mathcal{L}^r -inequality and the last step follows from [5, Lemma 5.2.2]. The formula for $\mathbb{E}[|W|^p]$, $0 < p < \alpha$ in the case (A3) is justified by Lemma 3.1.

Finally, we show that in the case (A3)

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{(N(t) - t/\mu)_-^p}{a(t)^p} \right] = \infty = \mathbb{E}[W_-^p]$$

for $p \geq \alpha$. The second equality follows from the relation $\mathbb{P}\{W_- > x\} \sim cx^{-\alpha}$ as $x \rightarrow \infty$ for a positive constant c . With this at hand the first equality is a consequence of (N3) and Fatou's lemma. The proof of Theorems 1.1 and 1.4 is complete. 95

We close this section with an example showing that convergence of moments of order $p > 2$ may fail in the case of a normal limit.

Example 2.1. If the survival function of ξ is given by

$$\mathbb{P}(\xi > t) = \frac{1}{(t+1)^2 \log^2(t+e)}, \quad t \geq 0,$$

then $\mathbb{E}[\xi^2] < \infty$ and

$$\mathbb{P}(S_n > \gamma n) \geq \mathbb{P}(\max\{\xi_1, \xi_2, \dots, \xi_n\} > \gamma n) = 1 - (1 - \mathbb{P}(\xi > \gamma n))^n \sim n\mathbb{P}(\xi > \gamma n)$$

as $n \rightarrow \infty$ for every fixed $\gamma > 0$. Therefore,

$$\mathbb{E}[(N(2\mu n) - 2n)_-^p] \geq n^p \mathbb{P}(N(2\mu n) \leq n) = n^p \mathbb{P}(S_n > 2\mu n) \geq cn^{p+1} \mathbb{P}(\xi > n)$$

for some $c > 0$ and all sufficiently large n . Hence (1.7) holds.

Alternative proof for the convergence of first absolute moments. There is an alternative elegant proof of the convergence of the first moments in (1.1) for the cases (A1) through (A3) based on the representation

$$\begin{aligned} \mathbb{E}[|S_{N(\mu n)} - S_n|] &= \mathbb{E}[S_{N(\mu n) \vee n} - S_{N(\mu n) \wedge n}] \\ &= \mu \mathbb{E}[(N(\mu n) \vee n) - (N(\mu n) \wedge n)] = \mu \mathbb{E}[|N(\mu n) - n|], \end{aligned}$$

where the second equality follows from Wald's identity. From this one obtains

$$\begin{aligned} \mathbb{E}[|S_n - \mu n| - (S_{N(\mu n)} - \mu n)] &= \mu \mathbb{E}[|N(\mu n) - n|] \\ &\leq \mathbb{E}[|S_n - \mu n| + (S_{N(\mu n)} - \mu n)]. \end{aligned} \quad (2.6)$$

According to [5, Lemma 5.2.2]

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[|S_n - \mu n|]}{c(n)} = \mathbb{E}[|W|]. \quad (2.7)$$

From [7] it is known that, as $t \rightarrow \infty$,

$$\mathbb{E}[S_{N(t)} - t] \sim \begin{cases} \text{const} & \text{in the case (A1),} \\ \text{const} \cdot \ell(t) & \text{in the case (A2),} \\ \text{const} \cdot t^{2-\alpha} \ell(t) & \text{in the case (A3),} \end{cases}$$

100 provided that the law of ξ is non-lattice.

Assume now that the law of ξ is lattice with span $d > 0$. In the case (A1), according to [12, Theorem 9], $\mathbb{E}[S_{N(nd)} - nd]$ tends to a constant as $n \rightarrow \infty$. Hence

$$\mathbb{E}[S_{N(t)} - t] = O(1) \quad \text{as } t \rightarrow \infty.$$

In the cases (A2) and (A3), according to [13, Theorem 6], $\mathbb{E}[S_{N(t)} - t]$ exhibits the same asymptotic behavior as in the non-lattice case.

Since $c(t)$ is regularly varying of index $1/\alpha$ at ∞ (where $\alpha = 2$ in the cases (A1) and (A2)), we conclude that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[S_{N(\mu n)} - \mu n]}{c(n)} = 0.$$

Applying this and (2.7) to (2.6) we infer

$$\lim_{n \rightarrow \infty} \mu \frac{\mathbb{E}[|N(\mu n) - n|]}{c(n)} = \mathbb{E}[|W|].$$

Now we have to check that this relation implies the convergence of the first absolute moments in (1.1). For any $t > 0$ there exists an $n = n(t) \in \mathbb{N}_0$ such that $t \in (\mu n, \mu(n+1)]$. Hence, by subadditivity,

$$\mathbb{E}[N(t) - N(\mu n)] \leq \mathbb{E}[N(\mu(n+1)) - N(\mu n)] \leq \mathbb{E}[N(\mu)].$$

It remains to observe that $\lim_{t \rightarrow \infty} c(\mu n(t)\mu^{-1})/c(t) = \mu^{-1/\alpha}$ by the regular variation of $c(t)$. This implies the asserted convergence of the first absolute
105 moments in (1.1).

2.2. Proof of Theorem 1.5

Arguing as in the proof of Theorems 1.1 and 1.4, we conclude that it suffices to show that

$$\sup_{t \geq t_0} \mathbb{E}[e^{\theta \mathbb{P}(\xi > t)N(t)}] < \infty$$

for every $\theta > 0$ and some $t_0 \geq 0$. Write

$$\begin{aligned} \frac{\mathbb{E}[e^{\theta \mathbb{P}(\xi > t)N(t)}] - 1}{e^{\theta \mathbb{P}(\xi > t)} - 1} &= \sum_{k \geq 0} e^{\theta \mathbb{P}(\xi > t)k} \mathbb{P}(S_k \leq t) \\ &\leq e^{\lambda t} \sum_{k \geq 0} e^{\theta \mathbb{P}(\xi > t)k} \varphi(\lambda)^k = \frac{e^{\lambda t}}{1 - e^{\theta \mathbb{P}(\xi > t)} \varphi(\lambda)} \end{aligned} \quad (2.8)$$

for every $\lambda > 0$ such that $e^{\theta \mathbb{P}(\xi > t)} \varphi(\lambda) < 1$. Pick an arbitrary $c > (\theta/\Gamma(1-\alpha))^{1/\alpha}$ and note that

$$\frac{1 - e^{-\theta \mathbb{P}(\xi > t)}}{1 - \varphi(c/t)} \sim \frac{\theta \mathbb{P}(\xi > t)}{\Gamma(1-\alpha) \mathbb{P}(\xi > t/c)} \rightarrow \frac{\theta c^{-\alpha}}{\Gamma(1-\alpha)} < 1 \quad (2.9)$$

as $t \rightarrow \infty$ where (3.4) has been used. Relation (2.9) entails

$$e^{\theta \mathbb{P}(\xi > t)} \varphi(c/t) < 1$$

for all $t > 0$ large enough. Therefore, choosing $\lambda = c/t$ in (2.8) and using again (2.9) we infer

$$\mathbb{E}[e^{\theta \mathbb{P}(\xi > t) N(t)}] - 1 \leq e^c \frac{e^{\theta \mathbb{P}(\xi > t)} - 1}{1 - e^{\theta \mathbb{P}(\xi > t)} \varphi(c/t)} \rightarrow \frac{e^c \theta}{\Gamma(1 - \alpha) c^\alpha - \theta} \quad \text{as } t \rightarrow \infty$$

which completes the proof of the first equalities in (1.10) and (1.11). While the second equality in (1.10) and the second equality in (1.11) when $\alpha = 0$ are immediate, the second equality in (1.11) when $\alpha \in (0, 1)$ follows from Lemma 3.2. The proof of Theorem 1.5 is complete. 110

2.3. Proof of Corollary 1.6

The claimed asymptotic relations follow almost immediately from Theorems 1.1 to 1.5 and the fact that $T_r \leq N_r \leq T_r + 1$. It remains to check the claimed equivalent reformulations of (A1) through (A4) in terms of the Lévy measure Π and to make sure that we use the right scaling. 115

Proof of (a): $\sigma^2 < \infty$ is equivalent to $\int_{\{|x| \geq 1\}} x^2 \Pi(dx) < \infty$ by standard theory for Lévy processes, see [11, Corollary 25.8].

Proof of (b): In the proof of Lemma 6(a) in [14, 3rd line after (3.10)], it is shown that condition (1.16) implies that

$$\ell_2(t) = \mathbb{E}[\xi^2 \mathbb{1}_{\{\xi \leq t\}}] \sim \int_{(0, t]} x^2 \Pi(dx) = \ell_2^\Pi(t) \quad \text{as } t \rightarrow \infty.$$

Consequently, the asymptotic relation $\lim_{t \rightarrow \infty} t \ell_2(c(t)) c(t)^{-2} = 1$ is equivalent to

$$\lim_{r \rightarrow \infty} r \ell_2^\Pi(c(r)) c(r)^{-2} = 1.$$

(c) and (d): According to [14, Proposition 0], $\mathbb{P}(\xi > t)$ is regularly varying of index $-\alpha$ at ∞ if and only if the same is true for $\Pi((t, \infty))$, and in this case $\mathbb{P}(\xi > t) \sim \Pi((t, \infty))$ as $t \rightarrow \infty$. This proves (d), while (c) follows upon noting that $\ell^\Pi(t) \sim \ell(t)$ which implies that the asymptotic relations $\lim_{t \rightarrow \infty} t \ell(c(t)) c(t)^{-\alpha} = 1$ and $\lim_{r \rightarrow \infty} r \ell^\Pi(c(r)) c(r)^{-\alpha} = 1$ are equivalent. 120

3. Appendix

3.1. Laplace transforms 125

Here, we gather known results on the behavior of Laplace transforms at 0 that play a role in our derivations. Recall that φ denotes the Laplace transform of ξ .

In the case (A1), $\mathbb{E}[\xi^2] = \mu^2 + \sigma^2 < \infty$ and a Taylor expansion of φ at 0 gives

$$\varphi(\lambda) = 1 - \mu\lambda + \frac{\mu^2 + \sigma^2}{2}\lambda^2 + o(\lambda^2) \quad \text{as } \lambda \rightarrow 0+. \quad (3.1)$$

In the case (A2), $\mathbb{E}[\xi^2] = \infty$, $\ell_2(t) = \mathbb{E}[\xi^2 \mathbb{1}_{\{\xi \leq t\}}]$ is slowly varying at ∞ and

$$\varphi(\lambda) - (1 - \mu\lambda) \sim \frac{1}{2}\lambda^2 \ell_2(1/\lambda) \quad \text{as } \lambda \rightarrow 0+ \quad (3.2)$$

by the implication (8.1.11c) \Rightarrow (8.1.9) of [15, Theorem 8.1.6].

In the case (A3), using that $\mathbb{P}(\xi > t)$ is regularly varying of index $-\alpha$ for $\alpha \in (1, 2)$ we infer from [15, Theorem 8.1.6] (with $c_\alpha := \Gamma(2 - \alpha)/(\alpha - 1)$):

$$\varphi(\lambda) - (1 - \mu\lambda) \sim c_\alpha \mathbb{P}(\xi > 1/\lambda) \quad \text{as } \lambda \rightarrow 0+. \quad (3.3)$$

In the case (A4), since $\mathbb{P}(\xi > t)$ is regularly varying of index $-\alpha$ for $\alpha \in [0, 1)$ an application of [15, Corollary 8.1.7] yields

$$1 - \varphi(\lambda) \sim \Gamma(1 - \alpha) \mathbb{P}(\xi > 1/\lambda) \quad \text{as } \lambda \rightarrow 0+. \quad (3.4)$$

3.2. Moment computations

Lemma 3.1. *Let W be a random variable with characteristic function given by (1.2). Then, for $r < \alpha$,*

$$\mathbb{E}[|W|^r] = \frac{2\Gamma(r+1)}{\pi r} \sin\left(\frac{\pi r}{2}\right) \Gamma\left(1 - \frac{r}{\alpha}\right) |\Gamma(1 - \alpha)|^{\frac{r}{\alpha}} \cos\left(\frac{\pi r}{2} - \frac{\pi r}{\alpha}\right).$$

In particular, $\mathbb{E}[|W|] = \frac{2}{\pi} \Gamma(1 - \frac{1}{\alpha}) |\Gamma(1 - \alpha)|^{1/\alpha} \sin(\frac{\pi}{\alpha})$.

Proof. We use the integral representation for the r th absolute moment

$$m_r := \mathbb{E}[|W|^r] = \frac{\Gamma(r+1)}{\pi} \sin\left(\frac{\pi r}{2}\right) \int_{\mathbb{R}} \frac{1 - \operatorname{Re} \mathbb{E}[e^{itW}]}{|t|^{r+1}} dt. \quad (3.5)$$

from [16, Lemma 2]. Set $K(r) := \frac{\Gamma(r+1)}{\pi} \sin(\frac{\pi r}{2})$, $B := \Gamma(1 - \alpha) \cos(\frac{\pi \alpha}{2})$ and $C := \Gamma(1 - \alpha) \sin(\frac{\pi \alpha}{2})$. Using Euler's identity $e^{ix} = \cos x + i \sin x$ in (1.2), we get

$$\operatorname{Re} \mathbb{E}[e^{itW}] = \exp(-B|t|^\alpha) \cos(-C|t|^\alpha \operatorname{sgn}(t)).$$

Substituting this into formula (3.5) and a change of variables ($u := t^\alpha$) yield

$$\begin{aligned} m_r &= 2K(r) \int_0^\infty \frac{1 - e^{-Bt^\alpha} \cos(Ct^\alpha)}{t^{r+1}} dt \\ &= \frac{2K(r)}{\alpha} \int_0^\infty (1 - e^{-Bu}) u^{-1-r/\alpha} du \\ &\quad + \frac{2K(r)}{\alpha} \int_0^\infty e^{-Bu} (1 - \cos(Cu)) u^{-1-r/\alpha} du =: I_1 + I_2. \end{aligned} \quad (3.6)$$

Integration by parts yields:

$$I_1 = \frac{2K(r)B}{r} \int_0^\infty u^{-r/\alpha} e^{-Bu} du = -\frac{2K(r)B^{r/\alpha}}{\alpha} \Gamma\left(-\frac{r}{\alpha}\right).$$

According to [17, Formula (3.945(2))], we have

$$I_2 = \frac{2K(r)}{\alpha} \Gamma\left(-\frac{r}{\alpha}\right) (B^{r/\alpha} - |\Gamma(1 - \alpha)|^{r/\alpha} \cos(\frac{\pi r}{2} - \frac{\pi r}{\alpha})).$$

Now plugging in the values of I_1 and I_2 in (3.6) completes the proof. \square

Lemma 3.2. *Let W be a random variable with*

$$\mathbb{E}[e^{\theta W}] = E_{\alpha}\left(\frac{\theta}{\Gamma(1-\alpha)}\right) \text{ for every } \theta \in \mathbb{R}$$

where E_{α} denotes the Mittag-Leffler function with parameter $\alpha \in (0, 1)$. Then, for any $r > 0$, we have

$$\mathbb{E}[W^r] = \frac{\Gamma(r+1)}{\Gamma(1-\alpha)^r \Gamma(r\alpha+1)}.$$

Proof. For $\alpha \in (0, 1)$, let S_{α} denote a positive α -stable random variable with Laplace transform $\mathbb{E}[e^{-\lambda S_{\alpha}}] = \exp(-\Gamma(1-\alpha)\lambda^{\alpha})$, $\lambda \geq 0$. We shall need the following integration formula for positive random variables X and $s > 0$

$$\mathbb{E}[X^{-s}] = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \mathbb{E}[e^{-tX}] dt \quad (3.7)$$

which follows from the fact that $\mathbb{P}(E/X > t) = \mathbb{E}[e^{-tX}]$ for all $t \geq 0$ where E is an exponential random variable with mean 1 independent of X . Using (3.7) for $X = S_{\alpha}$ and $s = r\alpha$ gives

$$\mathbb{E}[S_{\alpha}^{-r\alpha}] = \frac{1}{\Gamma(r\alpha)} \int_0^{\infty} t^{r\alpha-1} e^{-\Gamma(1-\alpha)t^{\alpha}} dt = \frac{\Gamma(r+1)}{\Gamma(1-\alpha)^r \Gamma(r\alpha+1)}.$$

This implies that the moment generating function of $S_{\alpha}^{-\alpha}$ is the same as that of W , which proves that $S_{\alpha}^{-\alpha}$ has the same law as W . In particular, $\mathbb{E}[W^r] = \mathbb{E}[S_{\alpha}^{-r\alpha}]$ for all $r \geq 0$. This completes the proof. \square

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