On perpetuities with gamma-like tails

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Abstract

An infinite convergent sum of independent and identically distributed random variables discounted by a multiplicative random walk is called perpetuity, because of a possible actuarial application. We give three disjoint groups of sufficient conditions which ensure that the distribution right tail of a perpetuity $\mathbb{P}\{X > x\}$ is asymptotic to $ax^c e^{-bx}$ as $x \to \infty$ for some $a, b > 0$ and $c \in \mathbb{R}$. Our results complement those of Denisov and Zwart [J. Appl. Probab. **44** (2007), 1031–1046]. As an auxiliary tool we provide criteria for the finiteness of the one-sided exponential moments of perpetuities. Several examples are given in which the distributions of perpetuities are explicitly identified.

1 Introduction and results

Let $(A_n, B_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed $\mathbb{R}^2$-valued random vectors with generic copy (A, B). Put $\Pi_0 := 1$ and $\Pi_n := A_1 \cdot \ldots \cdot A_n$ for $n \in \mathbb{N}$. The random discounted sum

$$X := \sum_{k\geq 1} \Pi_{k-1} B_k,$$

provided that $|X| < \infty$ a.s., is called *perpetuity* and is of interest in various fields of applied probability. The term 'perpetuity' stems from the fact that such random series occur in the realm of insurance and finance as sums of discounted payment streams. Detailed information about various aspects of perpetuities, including applications, can be found in the recent monographs [5, 15].

There are a number of papers investigating the asymptotics of $-\log \mathbb{P}\{|X| > x\}$ as $x \to \infty$ in the situations when $\mathbb{P}\{|X| > x\}$ exhibits exponential or superexponential decrease, see [1, 2, 8, 12, 13, 19]. In the present paper we are interested in precise (non-logarithmic) asymptotics of $\mathbb{P}\{X > x\}$ as $x \to \infty$. Specifically, our main concern is: which conditions ensure that $\mathbb{P}\{X > x\} \sim ax^c e^{-bx}$ as $x \to \infty$ for some positive a, b and real c. Distribution tails which exhibit such asymptotics may be called *gamma-like tails*, hence the title of the paper. To our knowledge, works in this direction are rare. We are only aware of [7, 20, 21, 22]. The first three papers are concerned with exponential tails of perpetuities which correspond to nonnegative and

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independent A and B. The results obtained in [22] cover the situation when $A=\gamma\in(0,1)$ a.s., B is not necessarily nonnegative and satisfies $\mathbb{P}\{B>x\}\sim ax^c e^{-x^p}$ as $x\to\infty$ for some positive a, p and real c. Under additional technical assumptions in the case $p>1$ that paper points out the asymptotics of $\mathbb{P}\{X>x\}$ as $x\to\infty$.

The following result was given as Proposition 4.1 in [7] under the assumptions that A and B are a.s. nonnegative and that $\mathbb{P}\{A=1\}=0$ which are partially dispensed with here. For $s\in\mathbb{R}$, define $\psi(s):=\mathbb{E}e^{sX}$ and $\varphi(s):=\mathbb{E}e^{sB}$, finite or infinite.

Proposition 1.1. *Let A and B be independent and $r>0$. Suppose that $\mathbb{P}\{A=1\}\in[0,1)$ and that either*
(a) $\mathbb{P}\{A\in(0,1]\}=1$ or
(b) $\mathbb{P}\{A\in(-1,0)\}=1$, or
(c) $\mathbb{P}\{|A|\in(0,1]\}=1$ and $\mathbb{P}\{A=-1\}\in(0,1)$.
(I) Assume that $\mathbb{P}\{B=0\}<1$.
Let the assumption (a) prevail. If $\mathbb{P}\{A=1\}=0$, then $\mathbb{E}\psi(rA)<\infty$ if, and only if, $\mathbb{E}\varphi(rA)<\infty$. If $\mathbb{P}\{A=1\}\in(0,1)$, then $\mathbb{E}\psi(rA)<\infty$ if, and only if, $\varphi(r)\mathbb{P}\{A=1\}<1$.
Under the assumption (b) $\mathbb{E}\psi(rA)<\infty$ if, and only if,

$$\mathbb{E}e^{rA_1(B_2+A_2B_3)}<\infty. \tag{1}$$

Under the assumption (c) $\mathbb{E}\psi(rA)<\infty$ if, and only if,

$$\mathbb{E}e^{-rB}\mathbb{E}e^{rB}\big[\mathbb{P}\{A=-1\}\big]^2<\big(1-\mathbb{E}e^{-rB}\mathbb{P}\{A=1\}\big)\big(1-\mathbb{E}e^{rB}\mathbb{P}\{A=1\}\big).$$

(II) Suppose that $\mathbb{P}\{B>x\}\sim g(x)e^{-bx}$ as $x\to\infty$ for some $b>0$ and some function g such that $g(\log x)$ is slowly varying at ∞ and that

$$\limsup_{x\to\infty}(\sup_{1\le y\le x}g(y))/g(x)<\infty.$$

Then

$$\mathbb{P}\{X>x\}\sim\mathbb{E}\psi(bA)\mathbb{P}\{B>x\},\quad x\to\infty \tag{2}$$

provided that $\mathbb{E}\psi(bA)<\infty$.

Remark 1.2. Here is a comment on inequality (1). If $B\ge0$ or $B\le0$ a.s., then (1) is equivalent to $\mathbb{E}\varphi(rA_1A_2)<\infty$ and $\mathbb{E}\varphi(rA)<\infty$, respectively. If $A=-\gamma$ a.s. for some $\gamma\in(0,1)$ and B takes values of both signs, then (1) is equivalent to $\varphi(-r\gamma)<\infty$ and $\varphi(r\gamma^2)<\infty$. In the general case, (1) which imposes restrictions on both tails of B entails but is not equivalent to $\mathbb{E}\varphi(rA)<\infty$ and $\mathbb{E}\varphi(rA_1A_2)<\infty$.

The argument of [7] for part (II) remains valid in the extended situation treated here. Our contribution consists in proving part (I), that is, a criterion for $\mathbb{E}\psi(bA)$ to be finite which is actually a consequence of Theorems 3.1, 3.5 and Remark 3.4.

Theorem 1.3. *Suppose that A and B are arbitrarily dependent and that $\mathbb{P}\{A\in(0,1]\}=1$. Further, assume that*

$$\mathbb{P}\{B>x\}\sim ax^c e^{-bx},\quad x\to\infty \tag{3}$$

for some $a,b>0$ and $c<-1$;

$$\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}}<1; \tag{4}$$

$$\mathbb{P}\{Ay+B>x\}\sim f(y)\mathbb{P}\{B>x\},\quad x\to\infty \tag{5}$$

for each $y \in \mathbb{R}$ *and a nonnegative measurable function* f*; and*

$$\mathbb{E}\log(1+B^-) < \infty. \tag{6}$$

Then $\mathbb{E}f(X) < \infty$ *and*

$$\mathbb{P}\{X > x\} \sim \frac{\mathbb{E}f(X)}{1-\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}}}\mathbb{P}\{B > x\}, \quad x \to \infty. \tag{7}$$

Remark 1.4. Recall that the distribution of a nonnegative random variable Y belongs to the class $\mathcal{S}(\alpha)$ for $\alpha > 0$, if
(a) $\lim_{x\to\infty} \frac{\mathbb{P}\{Y>x-y\}}{\mathbb{P}\{Y>x\}} = e^{\alpha y}$ for each $y \in \mathbb{R}$;
(b) $\lim_{x\to\infty} \frac{\mathbb{P}\{Y+Y^*>x\}}{\mathbb{P}\{Y>x\}} = 2\mathbb{E}e^{\alpha Y} < \infty$, where Y^* is an independent copy of Y.

Condition (3) with $c < -1$ ensures that the distribution of B^+ belongs to $\mathcal{S}(b)$. While point (a) above is easily checked, point (b) follows from Lemma 7.1 (iii) in [22]. Our Theorem 1.3 is closely related to Proposition 4.2 in [7] in which an asymptotic result similar to (7) was proved under the assumptions that A and B are independent, that $\mathbb{P}\{A = 1\} = 0$ and $\mathbb{P}\{B \geq 0\} = 1$, and that the distribution of B belongs to the class $\mathcal{S}(b)$. Theorem 3.2 in [20] is another result in this vein. A perusal of the proof given below reveals that (7) remains valid if (3) is replaced with the assumption that the distribution of B^+ belongs to $\mathcal{S}(b)$. However, we refrain from formulating Theorem 1.3 in this way, for our focus here is on the gamma-like tails.

Remark 1.5. Here, we provide more details on functions f arising in (5) assuming that the assumptions of Theorem 1.3 are in force. It is clear that $f(y) = \mathbb{E}e^{byA}$, $y \in \mathbb{R}$ whenever A and B are independent. The last equality is not necessarily true when A and B are dependent. For instance, if $A = \gamma_1\,\mathbb{1}_{\{B>q\}} + \gamma_2\,\mathbb{1}_{\{B\leq q\}}$ for some $\gamma_1, \gamma_2 \in (0, 1)$, $\gamma_1 \neq \gamma_2$ and some $q > 0$, then $f(y) = e^{by\gamma_1} \neq \mathbb{E}e^{byA}$, $y \in \mathbb{R}$.

To proceed we need more notation. Denote by $\gamma(a, b)$ and $\beta(c, d)$ a gamma distribution with parameters $a, b > 0$ and a beta distribution with parameters $c, d > 0$, respectively. Recall that

$$\gamma(a, b)(\mathrm{d}x) = \frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)}\,\mathbb{1}_{(0,\infty)}(x)\mathrm{d}x,$$

where $\Gamma(\cdot)$ is the Euler gamma function, and

$$\beta(c, d)(\mathrm{d}x) = \frac{1}{\mathrm{B}(c, d)} x^{c-1}(1-x)^{d-1}\,\mathbb{1}_{(0,1)}(x)\mathrm{d}x,$$

where $\mathrm{B}(\cdot,\cdot)$ is the Euler beta function. The following example is well-known, see, for instance, Example 3.8.2 in [24].

Example 1.6. Assume that A and B are independent, A has a $\beta(c, 1)$ distribution and B has a $\gamma(1, b)$ (exponential) distribution. Then X has a $\gamma(c + 1, b)$ distribution[1]. In particular,

$$\mathbb{P}\{X > x\} \sim \frac{(bx)^c}{\Gamma(c+1)} e^{-bx}, \quad x \to \infty. \tag{8}$$

Proposition 1.1 and Theorem 1.3 cover the situation where a gamma-like tail of X is inherited from a gamma-like tail of B, the influence of the distribution of A being small, for it is only

[1]This can be checked in several ways, for instance, via the argument given in Example 2.2

seen in the multiplicative constant. Example 1.6 reveals that the distributions of both A and B may give principal contributions to a gamma-like tail of X. Our next result, Theorem 1.7, provides an extension of Example 1.6 in that B is allowed to take values of both signs and that the right-tail of B is approximately, rather than precisely, exponential. Our Theorem 1.7 is close in spirit to Theorem 6.1 in [21] because in both results it is assumed that while one of the independent input random variables A and B obeys a particular distribution (A has a $\beta(1,\lambda)$ distribution in our Theorem 1.7; B has a $\gamma(1,\lambda)$ distribution in Theorem 6.1 [21]), the distribution of the other random variable follows a prescribed tail behavior.

Theorem 1.7. *Assume that A and B are independent; A has a $\beta(\lambda,1)$ distribution for some $\lambda>0$; condition* (6) *holds and*

$$\mathbb{P}\{B>x\}=Ce^{-bx}+r(x) \tag{9}$$

for some $C,b>0$, all $x\geq 0$, and a function r such that

$$\lim_{x\to\infty}e^{bx}r(x)=0, \tag{10}$$

$$\int_1^\infty \frac{e^{by}}{y}r^+(y)\mathrm{d}y<\infty \quad \textit{and} \quad \int_1^\infty \frac{e^{(b+\varepsilon)y}}{y}r^-(y)\mathrm{d}y<\infty \tag{11}$$

for some $\varepsilon>0$, where $r^+(y):=\max(r(y),0)$, $r^-(y):=-\min(r(y),0)$. Then

$$\mathbb{P}\{X>x\}\sim Kx^{\lambda C}e^{-bx}, \quad x\to\infty, \tag{12}$$

where

$$K:=\frac{Cb^{C\lambda}}{\Gamma(C\lambda+1)}\exp\left(\lambda\left[\int_0^\infty\frac{e^{by}-1}{y}r(y)\mathrm{d}y-\int_{-\infty}^0\frac{e^{by}-1}{y}\mathbb{P}\{B\leq y\}\mathrm{d}y\right]\right)<\infty.$$

The remainder of the paper is organized as follows. In Section 2 we give several examples intended to illustrate Proposition 1.1 and Theorem 1.7. Also in this section is a discussion of an interesting connection between perpetuities arising in Theorem 1.7 and certain selfdecomposable distributions. It is exactly this link which makes the proof of Theorem 1.7 relatively simple. In Section 3 we provide criteria for the existence of the one-sided exponential moments of perpetuities, the results which are needed for the proof of Proposition 1.1. The picture is incomplete yet, for a criterion remains a challenge in the case where both A and B take values of both signs with positive probability. All the proofs are given in Sections 4, 5 and 6.

2 Illustrating examples

Here is an example illustrating Proposition 1.1.

Example 2.1. Denote by θ_a and θ_b independent random variables with a $\gamma(1,a)$ and $\gamma(1,b)$ distribution, respectively. Let $A=\gamma\in(0,1)$ a.s. and $\mathbb{E}e^{sB}=\frac{a-\gamma s}{a-s}\frac{b+\gamma s}{b+s}$ for $-b<s<a$. Then $X=\theta_a-\theta_b$ or equivalently $\mathbb{E}e^{sX}=\frac{a}{a-s}\frac{b}{b+s}$. A standard calculation shows that $\mathbb{P}\{X>x\}=\frac{b}{a+b}e^{-ax}$ for $x>0$. The distribution of B is a mixture of the atom at zero with weight γ^2, the distribution of θ_a with weight $\gamma(1-\gamma)$, the distribution of $-\theta_b$ with weight $\gamma(1-\gamma)$ and the distribution of $\theta_a-\theta_b$ with weight $(1-\gamma)^2$. Hence, $\mathbb{P}\{B>x\}=(1-\gamma)\frac{b+a\gamma}{a+b}e^{-ax}$ for $x>0$ in full agreement with Proposition 1.1.

It is well known that the explicit distributions of the perpetuities are rarely available. Below we give several examples of distributions of B satisfying the assumptions of Theorem 1.7 for which distributions of the corresponding perpetuities X can be identified. Among others, this allows us to check validity of formula (12). We start with a trivial observation that the distributions of A and B as given in Example 1.6 satisfy the assumptions of Theorem 1.7 with $\lambda = c$, $C = 1$ and $r(x) \equiv 0$ in which case (12) amounts to (8) as it must be.

Throughout the rest of the section we assume, without further notice, that B is independent of A and that A has a $\beta(1, \lambda)$ distribution. We first point out an interesting connection with special selfdecomposable distributions which enables us to obtain a useful representation

$$\Psi(t) := \mathbb{E}e^{\mathrm{i}tX} = \Phi(t) \exp\left(\lambda \int_0^t \frac{\Phi(u) - 1}{u} \mathrm{d}u\right), \quad t \in \mathbb{R}, \tag{13}$$

where $\Phi(t) := \mathbb{E}e^{\mathrm{i}tB}$, $t \in \mathbb{R}$. The connection is implicit in [23, 24] and perhaps some other works.

The class L of selfdecomposable distributions is comprised of all possible limit distributions for the sums, properly normalized and centered, of independent (not necessarily identically distributed) random variables satisfying an infinitesimality condition. It was proved in [18] that the class L coincides with the class of distributions of the random variables $J := \int_{(0,\infty)} e^{-s}\mathrm{d}Y(s)$, where $(Y(t))_{t\geq 0}$ is a Lévy process with $\mathbb{E}\log(1 + |Y(1)|) < \infty$. It is known (see, for instance, formula (4.4) in [18]) that

$$\log \mathbb{E}e^{\mathrm{i}tJ} = \int_0^t \frac{\log \mathbb{E}e^{\mathrm{i}sY(1)}}{s} \mathrm{d}s, \quad t \in \mathbb{R}.$$

If $(Y(t))_{t\geq 0}$ is a compound Poisson process of intensity λ with jumps B_k satisfying the assumptions of Theorem 1.7, then

$$\log \mathbb{E}e^{\mathrm{i}tJ} = \lambda \int_0^t \frac{\Phi(s) - 1}{s} \mathrm{d}s, \quad t \in \mathbb{R} \tag{14}$$

as a consequence of $\log \mathbb{E}e^{\mathrm{i}tY(1)} = \lambda(\Phi(t) - 1)$ for $t \in \mathbb{R}$. Recalling that the function $x \mapsto \log(1+x)$ is subadditive on $[0, \infty)$ we conclude that conditions (6) and (9) ensure that $\mathbb{E}\log(1 + |B|) \leq \mathbb{E}\log(1 + B^+) + \mathbb{E}\log(1 + B^-) < \infty$, whence $\mathbb{E}\log(1 + |Y(1)|) \leq \mathbb{E}N\mathbb{E}\log(1 + |B|) < \infty$, where N is a Poisson distributed random variable with parameter λ. The latter inequality secures the convergence of the integral in (14). The selfdecomposable distributions with the characteristic functions of form (14) were investigated in [14, 16]. Formula (13) is a consequence of (14) and a representation $X = B_1 + A_1(B_2 + A_2B_3 + \ldots)$ a.s. and the fact that $A_1(B_2 + A_2B_3 + \ldots)$ is independent of B_1 and has the same distribution as J in (14).

Example 2.2. Let $B = \xi/b - \eta/a$ for $a, b > 0$ and independent random variables ξ and η with a $\gamma(1, 1)$ distribution (exponential distribution of unit mean). Then $\mathbb{P}\{B > x\} = \frac{a}{a+b}e^{-bx}$ for $x > 0$ and

$$\mathbb{P}\{B \leq x\} = \frac{b}{a+b}e^{ax} \text{ for } x < 0, \tag{15}$$

so that the assumptions of Theorem 1.7 are satisfied with $C = a/(a + b)$ and $r(x) \equiv 0$. Since

$$\Phi(t) = \mathbb{E}e^{\mathrm{i}tB} = \frac{b}{b - \mathrm{i}t} \frac{a}{a + \mathrm{i}t}, \quad t \in \mathbb{R},$$

we infer

$$\mathbb{E}e^{\mathrm{i}tX} = \left(\frac{b}{b - \mathrm{i}t}\right)^{\frac{a\lambda}{a+b}+1} \left(\frac{a}{a + \mathrm{i}t}\right)^{\frac{b\lambda}{a+b}+1}, \quad t \in \mathbb{R}$$

by using formula (13). Thus, X has the same distribution as $Y-Z$, where Y and Z are independent random variables with $\gamma(a\lambda/(a+b),b)$ and $\gamma(b\lambda/(a+b),a)$ distributions, respectively. Noting that the function $x\mapsto\mathbb{P}\{e^Y>x\}$ is regularly varying at ∞ of index $-b$ and applying Breiman's lemma (Proposition 3 in [4] and Corollary 3.6 (iii) in [6]) we conclude that

$$\mathbb{P}\{X>x\}=\mathbb{P}\{e^Ye^{-Z}>e^x\}\sim\mathbb{E}e^{-bZ}\mathbb{P}\{Y>x\}\sim\left(\frac{a}{a+b}\right)^{\frac{b\lambda}{a+b}+1}\frac{(bx)^{\frac{a\lambda}{a+b}}}{\Gamma(a\lambda/(a+b)+1)}e^{-bx}\quad(16)$$

as $x\to\infty$ having utilized asymptotics (8) for the last step. To check that formula (12) gives the same answer we have to calculate K appearing in that formula. Using (15) we obtain

$$\begin{aligned}\exp\Big(-\lambda\int_{-\infty}^0\frac{e^{by}-1}{y}\mathbb{P}\{B\le y\}\mathrm{d}y\Big)&=\exp\Big(-\frac{b\lambda}{a+b}\int_0^\infty\frac{e^{-ay}-e^{-(a+b)y}}{y}\mathrm{d}y\Big)\\&=\exp\Big(-\frac{b\lambda}{a+b}\log\Big(\frac{a+b}{a}\Big)\Big)=\Big(\frac{a}{a+b}\Big)^{\frac{b\lambda}{a+b}}\quad(17)\end{aligned}$$

having observed that the last integral is a Frullani integral. Thus,

$$K=\frac{\frac{a}{a+b}b^{\frac{a\lambda}{a+b}}}{\Gamma(a\lambda/(a+b)+1)}\left(\frac{a}{a+b}\right)^{\frac{b\lambda}{a+b}}=\left(\frac{a}{a+b}\right)^{\frac{b\lambda}{a+b}+1}\frac{b^{\frac{a\lambda}{a+b}}}{\Gamma(a\lambda/(a+b)+1)}$$

which is in line with (16).

Example 2.3. Put $B:=\xi-\eta$ for independent positive random variables ξ and η. Assume that

$$\mathbb{P}\{\xi>x\}=C_1e^{-bx}+r_1(x),\quad x\ge0\qquad(18)$$

and that r_1 satisfies (10) and (11). Then

$$\begin{aligned}\mathbb{P}\{B>x\}&=\int_0^\infty\mathbb{P}\{\xi>x+y\}\mathbb{P}\{\eta\in\mathrm{d}y\}\\&=C_1(\mathbb{E}e^{-b\eta})e^{-bx}+\mathbb{E}r_1(x+\eta)=:Ce^{-bx}+r(x).\end{aligned}$$

By the Lebesgue dominated convergence theorem $\lim_{x\to\infty}e^{bx}r(x)=0$. Furthermore, by Fubini's theorem

$$\begin{aligned}\int_{1/b}^\infty\frac{e^{by}}{y}r^+(y)\mathrm{d}y&=\mathbb{E}\int_{1/b}^\infty\frac{e^{by}}{y}r_1^+(y+\eta)\mathrm{d}y\\&\le\mathbb{E}\int_{1/b}^\infty\frac{e^{b(y+\eta)}}{y+\eta}r_1^+(y+\eta)\mathrm{d}y\le\int_{1/b}^\infty\frac{e^{by}}{y}r_1^+(y)\mathrm{d}y<\infty\end{aligned}$$

having used the fact that $y\mapsto y^{-1}e^{by}$ is nondecreasing on $[1/b,\infty)$. Analogously,

$$\int_1^\infty\frac{e^{(b+\varepsilon)y}}{y}r^-(y)\mathrm{d}y<\infty.$$

Hence, under (18) the right tail of the distribution of B satisfies the assumptions of Theorem 1.7 with $C:=C_1\mathbb{E}e^{-b\eta}$ and $r(x):=\mathbb{E}r_1(x+\eta)$ whatever the distribution of η.

To give a concrete example let ξ and η be independent with $\mathbb{P}\{\xi > x\} = \mathbb{P}\{\eta > x\} = pe^{-bx} + (1-p)e^{-cx}$ for $x \geq 0$, $c > b > 0$ and $p \in (0,1)$. Condition (18) holds with $C_1 = p$ and $r_1(x) = (1-p)e^{-cx}$ which trivially satisfies (10) and (11). Further,

$$\mathbb{P}\{B > x\} = \mathbb{P}\{B \leq -x\} = \frac{c_1 e^{-bx} + c_2 e^{-cx}}{2}, \quad x \geq 0, \tag{19}$$

where

$$c_1 := p^2 + \frac{2p(1-p)c}{b+c}, \; c_2 := (1-p)^2 + \frac{2p(1-p)b}{b+c},$$

which immediately implies that condition (6) holds and that $B = \xi - \eta$ has the characteristic function

$$\Phi(t) = \mathbb{E}e^{\mathrm{i}tB} = c_1 \frac{b^2}{b^2+t^2} + c_2 \frac{c^2}{c^2+t^2}, \quad t \in \mathbb{R}.$$

Observing that

$$\exp\left(\alpha \int_0^\infty (e^{\mathrm{i}ut} - 1) \frac{e^{-\beta u}}{u} \mathrm{d}u\right) = \left(\frac{\beta}{\beta - \mathrm{i}t}\right)^\alpha, \quad t \in \mathbb{R}$$

for $\alpha, \beta > 0$ we obtain

$$\begin{aligned}
&\exp\left(\lambda \int_0^t \frac{\Phi(u) - 1}{u} \mathrm{d}u\right) \\
= \;&\exp\left(\lambda \int_0^\infty (e^{\mathrm{i}ut} - 1) \frac{\mathbb{P}\{B > u\}}{u} \mathrm{d}u\right) \exp\left(\lambda \int_0^\infty (e^{-\mathrm{i}ut} - 1) \frac{\mathbb{P}\{-B > u\}}{u} \mathrm{d}u\right) \\
= \;&\left(\frac{b^2}{b^2+t^2}\right)^{c_1\lambda/2} \left(\frac{c^2}{c^2+t^2}\right)^{c_2\lambda/2}
\end{aligned}$$

with the help of (19). This entails

$$\mathbb{E}e^{\mathrm{i}tX} = \Phi(t) \left(\frac{b^2}{b^2+t^2}\right)^{c_1\lambda/2} \left(\frac{c^2}{c^2+t^2}\right)^{c_2\lambda/2}$$

from which we conclude that X has the same distribution as $\xi - \eta + Y_1 - Y_2 + Z_1 - Z_2$, where the latter random variables are independent, Y_1 and Y_2 have a $\gamma(c_1\lambda/2, b)$ distribution, and Z_1 and Z_2 have a $\gamma(c_2\lambda/2, c)$ distribution. Note that

$$\mathbb{E}e^{-b\eta} = \frac{p}{2} + \frac{(1-p)c}{b+c} = \frac{c_1}{2p}, \; \mathbb{E}e^{-bY_2} = \left(\frac{1}{2}\right)^{c_1\lambda/2}, \; \mathbb{E}e^{b(Z_1 - Z_2)} = \left(\frac{c^2}{c^2-b^2}\right)^{c_2\lambda/2}$$

and that the exponential moments of order $b + \varepsilon$ for $\varepsilon \in (0, c-b)$ are finite. Invoking Breiman's lemma yields

$$\begin{aligned}
\mathbb{P}\{X > x\} &\sim \mathbb{E}e^{b(-\eta - Y_2 + Z_1 - Z_2)} \mathbb{P}\{\xi + Y_1 > x\} \\
&= \frac{c_1}{2p} \left(\frac{1}{2}\right)^{c_1\lambda/2} \left(\frac{c^2}{c^2-b^2}\right)^{c_2\lambda/2} \mathbb{P}\{\xi + Y_1 > x\} \\
&\sim \frac{c_1}{2p} \left(\frac{1}{2}\right)^{c_1\lambda/2} \left(\frac{c^2}{c^2-b^2}\right)^{c_2\lambda/2} p\,\gamma(c_1(\lambda/2) + 1, b)((x, \infty)) \\
&\sim \frac{c_1}{2} \left(\frac{1}{2}\right)^{c_1\lambda/2} \left(\frac{c^2}{c^2-b^2}\right)^{c_2\lambda/2} \frac{b^{\lambda c_1/2}}{\Gamma(c_1(\lambda/2) + 1)} x^{c_1\lambda/2} e^{-bx}, \quad x \to \infty
\end{aligned} \tag{20}$$

having utilized $\gamma(c_1\lambda/2, b) * \gamma(1, b) = \gamma(c_1\lambda/2 + 1, b)$ and

$$\gamma(c_1\lambda/2, b) * \gamma(1, c)((x, \infty)) = o(\gamma(c_1\lambda/2 + 1, b)((x, \infty))), \quad x \to \infty$$

for the penultimate line and (8) for the last.

Let us show that asymptotics (20) follows from Theorem 1.7 with $C = c_1/2$ and $r(x) = (c_2/2)e^{-cx}$. To this end, we only have to calculate K appearing in (12). Using a formula for Frullani's integrals (see (17)) we obtain

$$\begin{aligned} K &= \frac{c_1}{2}\frac{b^{c_1\lambda/2}}{\Gamma(c_1(\lambda/2)+1)}\exp\Big[\lambda\Big(\int_0^\infty \frac{e^{by}-1}{y}\frac{c_1}{2}e^{-by}\mathrm{d}y - \int_0^\infty \frac{1-e^{-by}}{y}\Big(\frac{c_1}{2}e^{-by}+\frac{c_2}{2}e^{-cy}\Big)\mathrm{d}y\Big)\Big] \\ &= \frac{c_1}{2}\left(\frac{1}{2}\right)^{c_1\lambda/2}\left(\frac{c^2}{c^2-b^2}\right)^{c_2\lambda/2}\frac{b^{\lambda c_1/2}}{\Gamma(c_1(\lambda/2)+1)} \end{aligned}$$

which is in agreement with (20).

Example 2.4. Let B be a positive random variable with the distribution tail

$$\mathbb{P}\{B > x\} = \frac{1}{\lambda}\frac{e^{-bx}(1-e^{-\lambda x})}{1-e^{-x}}, \quad x > 0,$$

where $b, \lambda > 0$ and $2b + \lambda > 1$. The last assumption warrants that the right-hand side is a decreasing function. Writing

$$\mathbb{P}\{B > x\} = \frac{1}{\lambda}e^{-bx} + \frac{1}{\lambda}\frac{e^{-bx}(e^{-x}-e^{-\lambda x})}{1-e^{-x}} =: Ce^{-bx} + r(x)$$

we conclude that if $\lambda > 1$, then $r^+(x) = r(x) \to (\lambda-1)/\lambda$ as $x \to 0+$ and $r^+(x) = O(e^{-(b+1)x})$ as $x \to \infty$, whereas if $\lambda \in (0,1)$, then $r^-(x) = -r(x) \to (1-\lambda)/\lambda$ as $x \to 0+$ and $r^-(x) = O(e^{-(b+\lambda)x})$ as $x \to \infty$. Thus, in both cases conditions (10) and (11) are satisfied. The following representation can be read off from Example 9.2.3 in [3]

$$\begin{aligned} \exp\left(\lambda\int_0^t \frac{\Phi(u)-1}{u}\mathrm{d}u\right) &= \exp\left(\lambda\int_0^\infty \frac{e^{\mathrm{i}ut}-1}{u}\mathbb{P}\{B>u\}\mathrm{d}u\right) \\ &= \exp\left(\int_0^\infty \frac{e^{\mathrm{i}ut}-1}{u}\frac{e^{-bu}(1-e^{-\lambda u})}{1-e^{-u}}\mathrm{d}u\right) \\ &= \frac{\Gamma(b-\mathrm{i}t)\Gamma(b+\lambda)}{\Gamma(b)\Gamma(b+\lambda-\mathrm{i}t)} = \int_0^\infty e^{\mathrm{i}ut}f_{b,\lambda}(u)\mathrm{d}u, \end{aligned}$$

where the penultimate equality follows from formula 3.413(1) in [10], and

$$f_{b,\lambda}(x) = \frac{1}{\mathrm{B}(b,\lambda)}e^{-bx}(1-e^{-x})^{\lambda-1}, \quad x > 0.$$

We note that both the setting and the proof given in [3] are slightly different from ours. Using (13) we conclude that X has the same distribution as $-\log Y + B$, where Y is independent of B and has a $\beta(b,\lambda)$ distribution. This representation enables us to find the asymptotics

$$\begin{aligned} \mathbb{P}\{X > x\} &= \mathbb{P}\{-\log Y > x\} + \mathbb{P}\{-\log Y + B > x, -\log Y \le x\} \\ &= o(xe^{-bx}) + \frac{1}{\lambda\mathrm{B}(b,\lambda)}\int_0^x e^{-b(x-y)}e^{-by}(1-e^{-y})^{\lambda-1}\mathrm{d}y \sim \frac{1}{\lambda\mathrm{B}(b,\lambda)}xe^{-bx}, \quad x \to \infty. \end{aligned}$$

An application of Theorem 1.7 in combination with already used formula 3.413(1) in [10] gives the same asymptotics. We omit details.

3 Criteria for the finiteness of the one-sided exponential moments

Throughout the rest of the paper we shall make a repeated use of the following nondegeneracy conditions:

$$\mathbb{P}\{A = 0\} = 0 \text{ and } \mathbb{P}\{B = 0\} < 1 \tag{21}$$

and

$$\mathbb{P}\{B + Ac = c\} < 1 \text{ for all } c \in \mathbb{R}. \tag{22}$$

Also, for several times, we shall use the following well known decomposition

$$X = B_1 + A_1B_2 + \ldots + A_1 \cdot \ldots \cdot A_{\tau-1}B_\tau + A_1 \cdot \ldots \cdot A_\tau(B_{\tau+1} + A_{\tau+1}B_{\tau+2} + \ldots) =: X_\tau + \Pi_\tau X^{(\tau)}, \tag{23}$$

where $\tau \geq 1$ is either deterministic or a stopping time w.r.t. the filtration generated by $(A_k, B_k)_{k\in\mathbb{N}}$. Observe that $X^{(\tau)} = B_{\tau+1} + A_{\tau+1}B_{\tau+2} + \ldots$ has the same distribution as X and is independent of (Π_τ, X_τ). This particularly shows that X is a perpetuity generated by (Π_τ, X_τ).

A criterion for the finiteness of $\mathbb{E}e^{r|X|}$ was known, see Proposition 7.1. In this section we provide criteria for the finiteness of the one-sided moments $\mathbb{E}e^{rX}$ which is a somewhat more delicate problem.

First we state a criterion for positive A.

Theorem 3.1. *Suppose* (21), (22), $\mathbb{P}\{A > 0\} = 1$, $|X| < \infty$ *a.s., and let* $r > 0$. *The conditions*

$$\mathbb{P}\{A \leq 1\} = 1, \tag{24}$$

$$\mathbb{E}e^{rB} < \infty \quad \textit{and} \quad \mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}} < 1 \tag{25}$$

are sufficient for

$$\mathbb{E}e^{rX} < \infty \tag{26}$$

to hold.

Conversely, if the support of the distribution of X is unbounded from the right, then (26) *entails* (24) *and* (25), *whereas if the support of the distribution of X is bounded from the right, then* $\mathbb{E}e^{sB} < \infty$ *for all* $s > 0$.

Remark 3.2. As far as condition (24) is concerned, the assumption about unboundedness of the support of the distribution of X is indispensable. For a trivial counterexample, just take a.s. nonpositive B, so that $X \in [-\infty, 0]$ a.s. Then $\mathbb{E}e^{rX} < \infty$ for each $r > 0$, irrespective of whether $\mathbb{P}\{A > 1\}$ is positive or equals zero. More interestingly, the support of the distribution of X can be bounded from the right even if $\mathbb{P}\{B > 0, A \neq 1\} > 0$ and $\mathbb{P}\{A > 1\} > 0$. Indeed, assume that the last two inequalities hold true, that $\mathbb{P}\{A > 0\} = 1$ and that

$$\Pi_\tau m + X_\tau = A_1 \cdot \ldots \cdot A_\tau m + B_1 + \ldots + B_\tau \leq m \quad \text{a.s.}$$

for some real m, where $\tau := \inf\{k \in \mathbb{N} : \Pi_k \neq 1\}$ (here, we have used decomposition (23) with the particular τ). Then $X \leq m$ a.s. (see Lemma 2.5.7 and Figure 2.4(c) in [5]) whence $\mathbb{E}e^{rX} < \infty$ for each $r > 0$ yet $\mathbb{P}\{A > 1\} > 0$.

Remark 3.3. A perusal of the proof of Theorem 3.1 reveals that $\mathbb{E}e^{rX} < \infty$ in combination with $\mathbb{P}\{A \in (0, 1]\} = 1$ entails $\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}} < 1$, irrespective of whether the support of the distribution of X is bounded or not.

Remark 3.4. Passing to the case where A is negative with positive probability we first single out a simpler situation in which $\mathbb{P}\{A=-1\}>0$. Then $\mathbb{E}e^{rX}<\infty$ if, and only if, $\mathbb{E}e^{r|X|}<\infty$. Assuming that $\psi(r)=\mathbb{E}e^{rX}<\infty$ we use (32) to obtain

$$\psi(r)=\mathbb{E}e^{rB}\psi(rA)\geq \mathbb{E}e^{rB}\,\mathbb{1}_{\{A=-1\}}\,\psi(-r)$$

which shows that $\psi(-r)<\infty$ whence $\mathbb{E}e^{r|X|}\leq\psi(r)+\psi(-r)<\infty$. This proves the $\Rightarrow$ implication, the implication $\Leftarrow$ being trivial. Thus, whenever $\mathbb{P}\{A=-1\}>0$ a criterion for the finiteness of $\mathbb{E}e^{rX}$ coincides with that for the finiteness $\mathbb{E}e^{r|X|}$. The latter is given in Proposition 7.1, and we refrain from reproducing it here.

When A takes values of both signs and $\mathbb{P}\{A=-1\}=0$ we can only prove a criterion under the additional assumption that B is a.s. nonnegative.

Theorem 3.5. *Suppose* (21), (22), $\mathbb{P}\{A=-1\}=0$, $|X|<\infty$ *a.s., and let* $r>0$.
Assume that $\mathbb{P}\{A<0\}\mathbb{P}\{A>0\}>0$ *and* $\mathbb{P}\{B\geq 0\}=1$. *Then* (26) *holds if, and only if,*

$$\mathbb{P}\{|A|\leq 1\}=1 \tag{27}$$

and condition (25) *holds.*
Assume that $\mathbb{P}\{A<0\}=1$. *Then* (26) *holds if, and only if, condition* (27) *holds and*

$$\mathbb{E}e^{r(B_1+A_1B_2)}<\infty. \tag{28}$$

4 Proofs of Theorems 3.1 and 3.5, and Proposition 1.1

Proof of Theorem 3.1. PROOF OF (24), (25) $\Rightarrow$ (26). Assume first that $A\in(0,1)$ a.s., i.e., $\mathbb{P}\{A=1\}=0$, so that we have to show that $\mathbb{E}e^{rB}<\infty$ entails $\mathbb{E}e^{rX}<\infty$ or, equivalently, that

$$\mathbb{E}e^{rB^+}<\infty\ \Rightarrow\ \mathbb{E}e^{rX^+}<\infty, \tag{29}$$

where $y^+:=\max(y,0)$.

Since the function $x\mapsto x^+$ is subadditive on $\mathbb{R}$ and satisfies $(\alpha x)^+=\alpha x^+$ for $\alpha>0$ and $x\in\mathbb{R}$ we infer

$$\exp\left[rX^+\right]=\exp\left[r\bigg(\sum_{k\geq 1}\Pi_{k-1}B_k\bigg)^+\right]\leq \exp\left[r\sum_{k\geq 1}\Pi_{k-1}B_k^+\right]=:\exp[rX^*].$$

The random variable $X^*\geq 0$ is a perpetuity generated by (A,B^+). Hence, by Proposition 7.1 $\mathbb{E}e^{rB^+}<\infty$ entails $\mathbb{E}e^{rX^*}<\infty$ and thereupon (29).

Assuming that $A\in(0,1]$ a.s. and that $\mathbb{P}\{A=1\}\in(0,1)$ we must check that $\mathbb{E}e^{rB}<\infty$ together with $\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}}<1$ guarantees $\mathbb{E}e^{rX}<\infty$. Putting $\widehat{T}_0:=0$, $\widehat{T}_k:=\inf\{n>\widehat{T}_{k-1}:A_n<1\}$ for $k\in\mathbb{N}$ and then

$$\widehat{A}_k:=A_{\widehat{T}_{k-1}+1}\cdot\ldots\cdot A_{\widehat{T}_k},\quad \widehat{B}_k=B_{\widehat{T}_{k-1}+1}+A_{\widehat{T}_{k-1}+1}B_{\widehat{T}_{k-1}+2}+\ldots+A_{\widehat{T}_{k-1}+1}\cdot\ldots\cdot A_{\widehat{T}_k-1}B_{\widehat{T}_k}$$

for $k\in\mathbb{N}$, so that the vectors $(\widehat{A}_1,\widehat{B}_1)$, $(\widehat{A}_2,\widehat{B}_2),\ldots$ are independent and identically distributed, we infer $X=\widehat{B}_1+\sum_{n\geq 1}\widehat{A}_1\cdot\ldots\cdot\widehat{A}_{n-1}\widehat{B}_n$. Since

$$\begin{aligned}\mathbb{E}e^{r\widehat{B}_1} &= \sum_{n\geq 1}\mathbb{E}e^{r(B_1+A_1B_2+\ldots+A_1\cdot\ldots\cdot A_{n-1}B_n)}\,\mathbb{1}_{\{A_1=\ldots=A_{n-1}=1,A_n<1\}}\\ &= \mathbb{E}e^{rB}\,\mathbb{1}_{\{A<1\}}\sum_{n\geq 1}\left(\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}}\right)^{n-1}=\frac{\mathbb{E}e^{rB}\,\mathbb{1}_{\{A<1\}}}{1-\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}}}<\infty\end{aligned}$$

and $\mathbb{P}\{\widehat{A}_1 = 1\} = 0$ we conclude that $\mathbb{E}e^{rX} < \infty$ by the previous part of the proof.

PROOF OF (26) $\Rightarrow$ (24). Assuming that the support of the distribution of X is unbounded from the right we intend to prove that $\mathbb{P}\{A > 1\} > 0$ entails $\mathbb{E}e^{rX} = \infty$ for any $r > 0$, thereby providing a contradiction.

In view of $\mathbb{P}\{A > 1\} > 0$ there exist positive constants δ, c and $\gamma \in (0,1)$ such that

$$\mathbb{P}\{A > 1+\delta,\ B > -c\} = \gamma. \tag{30}$$

Let $(a_i)_{i\in\mathbb{N}}$ be any sequence satisfying $a_i > 1+\delta$ for all $i \in \mathbb{N}$. Pick now large enough m such that $m/(m-1) \le 1+\delta$. For the subsequent proof we need the following inequality

$$1 + a_1 + a_1a_2 + \ldots + a_1\ldots a_n \le ma_1\ldots a_n \tag{31}$$

which will be proved by the mathematical induction. For $n = 1$ (31) holds because $m-1 \ge 1/(1+\delta) \ge 1/a_1$. Assuming that (31) holds true for $n = k$ let us check it for $n = k+1$:

$$\begin{aligned} 1 + a_1 + a_1a_2 + \ldots + a_1\cdot\ldots\cdot a_k + a_1\cdot\ldots\cdot a_ka_{k+1} &\le a_1\ldots a_k(m + a_{k+1}) \\ &\le ma_1\ldots a_{k+1}(1/a_{k+1} + 1/m) \\ &\le ma_1\ldots a_{k+1}(1/(1+\delta) + 1/m) \\ &\le ma_1\ldots a_{k+1}((m-1)/m + 1/m) \\ &\le ma_1\ldots a_{k+1}, \end{aligned}$$

the penultimate inequality following by our choice of m.

Using (23) with $\tau = n$ gives $X = X_n + \Pi_nX^{(n)}$. By assumption, X takes arbitrarily large values with positive probability which implies that $\mathbb{P}\{X^{(n)} > mc+1\} = \mathbb{P}\{X > mc+1\} = \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. With this at hand, we have for any $n \in \mathbb{N}$ and any $r > 0$

$$\begin{aligned} \mathbb{E}e^{rX} &= \mathbb{E}e^{r(X_n+\Pi_nX^{(n)})} \\ &\ge \mathbb{E}\Big[e^{r(X_n+\Pi_nX^{(n)})}\,\mathbb{1}_{\{A_i>1+\delta,\,B_i>-c\ \text{ for } i=1,\ldots,n\}}\,\mathbb{1}_{\{X^{(n)}>mc+1\}}\Big] \\ &\ge \mathbb{E}\Big[e^{r\left(-c(1+A_1+\ldots+A_1\cdot\ldots\cdot A_{n-1})+\Pi_nX^{(n)}\right)}\,\mathbb{1}_{\{A_i>1+\delta,\ B_i>-c\ \text{ for } i=1,\ldots,n\}}\,\mathbb{1}_{\{X^{(n)}>mc+1\}}\Big] \\ &\ge \mathbb{E}\Big[e^{r\left(-mc\Pi_{n-1}+\Pi_nX^{(n)}\right)}\,\mathbb{1}_{\{A_i>1+\delta,\,B_i>-c\ \text{ for } i=1,\ldots,n\}}\,\mathbb{1}_{\{X^{(n)}>mc+1\}}\Big] \\ &\ge \mathbb{E}\Big[e^{r\left(\Pi_{n-1}(A_nX^{(n)}-mc)\right)}\,\mathbb{1}_{\{A_i>1+\delta,\ B_i>-c\ \text{ for } i=1,\ldots,n\}}\,\mathbb{1}_{\{X^{(n)}>mc+1\}}\Big] \\ &\ge e^{r(1+\delta)^{n-1}}\gamma^n\varepsilon \end{aligned}$$

having utilized (31) for the third inequality. Letting n tend to ∞ we obtain $\mathbb{E}e^{rX} = \infty$.

PROOF OF (26) $\Rightarrow$ (25). Assume that $\psi(r) = \mathbb{E}e^{rX} < \infty$ for some $r > 0$ and that the support of the distribution of X is unbounded from the right. Then $\mathbb{P}\{A \le 1\} = 1$ by the previous part of the proof. Put $c := \min_{0\le t\le r}\psi(t)$ and note that $c > 0$. Decomposition (23) with $\tau = 1$ is equivalent to

$$\psi(r) = \mathbb{E}e^{rB}\psi(rA). \tag{32}$$

Since $\mathbb{E}e^{rB}\psi(rA) \ge c\mathbb{E}e^{rB}$, the proof is complete in the case $\mathbb{P}\{A = 1\} = 0$. Suppose now that $\mathbb{P}\{A = 1\} \in (0,1)$. In order to check the second inequality in (25) we use once again (32) to infer

$$\psi(r) = \mathbb{E}e^{rB}\psi(rA)\,\mathbb{1}_{\{A<1\}} + \psi(r)\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}} > \psi(r)\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}},$$

where the strict inequality follows from $\mathbb{P}\{A<1\}>0$. Now $\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}}<1$ is a consequence of the last centered formula.

It remains to show that $\mathbb{E}e^{sB}<\infty$ for all $s>0$ provided that the support of the distribution of X is bounded from the right. If $X\leq 0$ a.s., then $B\leq 0$ a.s. whence $\mathbb{E}e^{sB}<1$ for all $s>0$. Assume now that $\mathbb{P}\{X>0\}>0$. This implies that $\lim_{s\to\infty}\psi(s)=\infty$. The latter together with log-convexity of ψ and its finiteness for all positive arguments ensures the existence of $s_0\geq 0$ such that $\psi(s_0)=1$ and $\psi(t)>1$ for any $t>s_0$ (note that $s_0=0$ if $\mathbb{P}\{X>0\}=1$, and $s_0>0$ if $\mathbb{P}\{X>0\}\in(0,1)$). Using (32) we obtain for $t>s_0$

$$\psi(t)=\mathbb{E}e^{tB}\psi(tA)\,\mathbb{1}_{\{A\leq 1\}}+\mathbb{E}e^{tB}\psi(tA)\,\mathbb{1}_{\{A>1\}}\geq c_1\mathbb{E}e^{tB}\,\mathbb{1}_{\{A\leq 1\}}+\mathbb{E}e^{tB}\,\mathbb{1}_{\{A>1\}}\geq c_1\mathbb{E}e^{tB},$$

where $c_1:=\min_{0\leq s\leq t}\psi(s)\in(0,1)$. The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.5. We start by showing that $\psi(r)<\infty$ in combination with $\mathbb{P}\{A<0\}>0$ entails (27). Indeed, as a consequence of (32) we infer

$$\psi(r)\geq \mathbb{E}e^{rB}\psi(rA)\,\mathbb{1}_{\{A<0\}},$$

whence $\psi(-s)<\infty$ for some $s\in(0,r]$ and thereupon $\mathbb{E}e^{s|X|}\leq\psi(s)+\psi(-s)<\infty$. Hence, (27) holds true by Proposition 7.1.

Assume now that $\mathbb{P}\{A\in(-1,0)\}=1$. Then $\mathbb{P}\{A_1A_2\in(0,1)\}=1$. Using now decomposition (23) with $\tau=2$ we conclude that $\mathbb{E}e^{rX_2}=\mathbb{E}e^{r(B_1+A_1B_2)}<\infty$ ensures $\mathbb{E}e^{rX}<\infty$ by Theorem 3.1. In the converse direction, assuming merely that A is a.s. negative, so that A_1A_2 is a.s. positive we use again (23) with $\tau=2$ to obtain that $\mathbb{E}e^{rX}<\infty$ entails (28).

Throughout the rest of the proof we assume that A takes values of both signs and that B is a.s. nonnegative.

Proof of (25) and (27) $\Rightarrow$ (26). We shall use representation (23) with

$$\tau:=\inf\{k\in\mathbb{N}:\Pi_k>0\}.$$

Observe that $\mathbb{P}\{\tau=1\}=\mathbb{P}\{A>0\}=:p$ and $\mathbb{P}\{\tau=k\}=p^{k-2}(1-p)^2$ for $k\geq 2$, whence $\tau<\infty$ a.s. In view of the first condition in (25)

$$\begin{aligned}\mathbb{E}e^{rX_\tau}&=\mathbb{E}e^{r(B_1+\Pi_1B_2+\ldots+\Pi_{\tau-1}B_\tau)}\\&=\mathbb{E}e^{rB_1}\,\mathbb{1}_{\{A_1>0\}}+\sum_{n\geq 2}\mathbb{E}e^{r(B_1+\ldots+\Pi_{n-1}B_n)}\,\mathbb{1}_{\{A_1<0,A_2>0,\ldots,A_{n-1}>0,A_n<0\}}\\&\leq\mathbb{E}e^{rB}+\mathbb{E}e^{rB}\sum_{n\geq 2}\mathbb{P}\{A_2>0,\ldots,A_{n-1}>0,A_n<0\}\leq 2\mathbb{E}e^{rB}<\infty.\end{aligned}$$

Further, $\mathbb{E}e^{rX_\tau}\,\mathbb{1}_{\{\Pi_\tau=1\}}=\mathbb{E}e^{rB_1}\,\mathbb{1}_{\{A_1=1\}}<1$ according to the second condition in (25). Since $\mathbb{P}\{\Pi_\tau\in(0,1]\}=1$ we conclude that (26) holds true by Theorem 3.1 which applies because X is also the perpetuity generated by (Π_τ,X_τ).

Proof of (26) $\Rightarrow$ (25). We shall use τ as above. Recall that we have already proved that (26) ensures (27) and thereupon $\mathbb{P}\{\Pi_\tau\in(0,1]\}=1$. Hence, $\mathbb{E}e^{rX}<\infty$ entails $\mathbb{E}e^{rB_1}\,\mathbb{1}_{\{A_1=1\}}=\mathbb{E}e^{rX_\tau}\,\mathbb{1}_{\{\Pi_\tau=1\}}<1$ by Remark 3.3 and $\mathbb{E}e^{rX_\tau}<\infty$ by Theorem 3.1. In particular,

$$\infty>\mathbb{E}e^{rX_\tau}\,\mathbb{1}_{\{\tau=1\}}=\mathbb{E}e^{rB}\,\mathbb{1}_{\{A>0\}}$$

and

$$\infty>\mathbb{E}e^{rX_\tau}\,\mathbb{1}_{\{\tau=2\}}=\mathbb{E}e^{r(B_1+A_1B_2)}\,\mathbb{1}_{\{A_1<0,A_2<0\}}\geq\mathbb{E}e^{rB}\,\mathbb{1}_{\{A<0\}}\,\mathbb{E}e^{-B}\,\mathbb{1}_{\{A<0\}}$$

whence $\mathbb{E}e^{rB}<\infty$. The proof of Theorem 3.5 is complete. □

Proof of Proposition 1.1. In view of our remark in the introduction we only prove part (I).

For $k \in \mathbb{N}$, put $(A_k^*, B_k^*) := (A_k, A_k B_{k+1})$. The vectors $(A_1^*, B_1^*), (A_2^*, B_2^*), \ldots$ are independent and identically distributed, and

$$A_1 B_2 + A_1 A_2 B_3 + \ldots = B_1^* + A_1^* B_2^* + A_1^* A_2^* B_3^* + \ldots =: X^*$$

which shows that the left-hand side is a perpetuity generated by $(A_k^*, B_k^*)_{k \in \mathbb{N}}$. This implies $\mathbb{E}\psi(rA) = \mathbb{E}e^{r(A_1B_2 + A_1A_2B_3 + \ldots)} = \mathbb{E}e^{rX^*}$.
Case (a). By Theorem 3.1 $\mathbb{E}e^{rX^*} < \infty$ if, and only if, $\infty > \mathbb{E}e^{rB_1^*} = \mathbb{E}\varphi(rA)$ and $1 > \mathbb{E}e^{rB_1^*} \mathbb{1}_{\{A_1^*=1\}} = \mathbb{E}e^{rB}\mathbb{P}\{A = 1\}$. If $\mathbb{P}\{A = 1\} = 0$, the last inequality holds automatically, whereas if $\mathbb{P}\{A = 1\} \in (0, 1)$ it entails $\varphi(r) < \infty$ and thereupon $\mathbb{E}\varphi(rA) < \infty$ because $A \in (0, 1]$ a.s.
Case (b). By Theorem 3.5, $\mathbb{E}e^{rX^*} < \infty$ if, and only if, $\infty > \mathbb{E}e^{r(B_1^* + A_1^* B_2^*)} = \mathbb{E}e^{rA_1(B_2 + A_2 B_3)}$.
Case (c). According to Remark 3.4, $\mathbb{E}e^{rX^*} < \infty$ if, and only if, $\infty > \mathbb{E}e^{r|B_1^*|} = \mathbb{E}e^{r|A_1B_2|}$ and

$$\mathbb{E}e^{-rB_1^*} \mathbb{1}_{\{A_1^*=-1\}} \mathbb{E}e^{rB_1^*} \mathbb{1}_{\{A_1^*=-1\}} < (1 - \mathbb{E}e^{-rB_1^*} \mathbb{1}_{\{A_1^*=1\}})(1 - \mathbb{E}e^{rB_1^*} \mathbb{1}_{\{A_1^*=1\}}).$$

The latter is equivalent to

$$\mathbb{E}e^{-rB}\mathbb{E}e^{rB}[\mathbb{P}\{A = -1\}]^2 < (1 - \mathbb{E}e^{-rB}\mathbb{P}\{A = 1\})(1 - \mathbb{E}e^{rB}\mathbb{P}\{A = 1\}) \tag{33}$$

which entails

$$\mathbb{E}e^{r|A_1B_2|} \leq \mathbb{E}e^{r|B|} \leq \mathbb{E}e^{-rB} + \mathbb{E}e^{rB} < \infty.$$

Thus, $\mathbb{E}e^{rX^*} < \infty$ if, and only if, (33) holds. □

5 Proof of Theorem 1.3

Our proof of Theorem 1.3 is based on two auxiliary results.

Lemma 5.1. *Suppose* (3) *with* $c < -1$, (4), (5) *and* $\mathbb{P}\{A \in (0, 1]\} = 1$. *Let* Y *be a random variable independent of* (A, B) *which satisfies*

$$\mathbb{P}\{Y > x\} \ \sim \ c_Y \mathbb{P}\{B > x\}, \quad x \to \infty \tag{34}$$

for some constant $c_Y > 0$. *Then* $\mathbb{E}f(Y) < \infty$ *and*

$$\mathbb{P}\{AY + B > x\} \ \sim \ \big(c_Y \mathbb{E}e^{bB} \mathbb{1}_{\{A=1\}} + \mathbb{E}f(Y)\big)\mathbb{P}\{B > x\}, \quad x \to \infty.$$

Proof. Fix $\delta \in (0, 1)$. In view of

$$\mathbb{P}\{B > (1 - \delta)x, Y > \delta x\} \ \sim \ a^2 c_Y e^{-bx} x^{2c} (\delta(1 - \delta))^c = o(e^{-bx} x^c), \quad x \to \infty$$

and

$$\{AY + B > x, B \leq (1 - \delta)x\} \subseteq \{AY > \delta x\} \subseteq \{Y > \delta x\} \quad \text{a.s.}$$

we have

$$\begin{aligned} \mathbb{P}\{AY + B > x\} &= \mathbb{P}\{AY + B > x, Y \leq \delta x\} + \mathbb{P}\{AY + B > x, B \leq (1 - \delta)x\} \\ &+ o(e^{-bx} x^c) =: I_1(x) + I_2(x) + o(e^{-bx} x^c), \quad x \to \infty. \end{aligned}$$

We claim that

$$\frac{I_1(x)}{\mathbb{P}\{B>x\}} = \int_{(-\infty,\,\delta x]} \frac{\mathbb{P}\{B>x-Ay\}}{\mathbb{P}\{B>x\}}\mathbb{P}\{Y\in \mathrm{d}y\} \ \to\ \mathbb{E}f(Y)<\infty, \quad x\to\infty.$$

Indeed, this is a consequence of (5) and the Lebesgue dominated convergence theorem in combination with the following two facts: (i)

$$\frac{\mathbb{P}\{B>x-Ay\}}{\mathbb{P}\{B>x\}} \le \frac{\mathbb{P}\{B>x-y\}}{\mathbb{P}\{B>x\}} \le Me^{by}\left(\frac{x-y}{x}\right)^c \le Me^{by}(1-\delta)^c$$

for large enough x, $y\in[0,\delta x]$ and an appropriate $M>0$;

$$\frac{\mathbb{P}\{B>x-Ay\}}{\mathbb{P}\{B>x\}} \le 1$$

for all $x>0$ and all $y<0$, and (ii) $\mathbb{E}e^{bY}<\infty$ which is an easy consequence of (3) and (34).

Passing to the analysis of $I_2(x)$ we observe that

$$\lim_{x\to\infty}\frac{\mathbb{P}\{uY>x-v\}}{\mathbb{P}\{B>x\}} = c_Y e^{bv}\,\mathbb{1}_{\{u=1\}}$$

for $u\in(0,1]$ and $v\in\mathbb{R}$. Furthermore,

$$\frac{\mathbb{P}\{uY>x-v\}}{\mathbb{P}\{B>x\}} \le \frac{\mathbb{P}\{Y>x-v\}}{\mathbb{P}\{B>x\}} \le Me^{bv}\left(\frac{x-v}{x}\right)^c \le Me^{bv}\delta^c$$

for large enough x, all $u\in[0,1]$, $v\in[0,(1-\delta)x]$ and some appropriate $M>0$, and

$$\frac{\mathbb{P}\{uY>x-v\}}{\mathbb{P}\{B>x\}} \le M_1$$

for large enough x, all $u\in[0,1]$, $v<0$ and appropriate $M_1>0$.

Recalling that $\mathbb{E}e^{bB}<\infty$ we infer

$$\frac{I_2(x)}{\mathbb{P}\{B>x\}} = \int_{[0,1]\times(-\infty,(1-\delta)x]} \frac{\mathbb{P}\{uY>x-v\}}{\mathbb{P}\{B>x\}}\mathbb{P}\{A\in\mathrm{d}u, B\in\mathrm{d}v\} \ \to\ c_Y\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}}, \quad x\to\infty$$

by the dominated convergence theorem.

Combining pieces together finishes the proof of the lemma. □

Apart from Lemma 5.1 we shall use a technique of stochastic bounds which is a quite commonly used method nowadays. In the area of perpetuities this approach, as far as we know, originates from [11]. For random variables U and V we shall write $U\le_{\mathrm{st}} V$ to indicate that V stochastically dominates U, that is, $\mathbb{P}\{U>x\}\le\mathbb{P}\{V>x\}$ for all $x\in\mathbb{R}$.

Lemma 5.2. *Suppose* (3) *with* $c<-1$, (4), (5) *and* $\mathbb{P}\{A\in(0,1]\}=1$. *On a possibly enlarged probability space there exists a nonnegative random variable* Z *independent of* (A,B) *such that*

$$\mathbb{P}\{Z>x\} \ \sim\ c_Z\mathbb{P}\{B>x\}, \quad x\to\infty \tag{35}$$

for a positive constant c_Z *and* $AZ+B\le_{\mathrm{st}} Z$.

Proof. Pick large enough $q > 0$ satisfying

$$\mathbb{E}e^{bB}\, \mathbb{1}_{\{A=1\}} + \mathbb{E}e^{bB}\, \mathbb{1}_{\{B>q\}} < 1$$

and then large enough $d > 0$ satisfying

$$e^{bd} > \frac{\mathbb{P}\{B \le q\}}{1 - \mathbb{E}e^{bB}\, \mathbb{1}_{\{A=1\}} - \mathbb{E}e^{bB}\, \mathbb{1}_{\{B>q\}}}.$$

Let B' be a copy of B independent of (A, B). Setting $Y := (B' + d)\, \mathbb{1}_{\{B'>q\}}$ we infer

$$\mathbb{P}\{Y > x\} \sim e^{bd}\mathbb{P}\{B > x\}, \quad x \to \infty.$$

Using Lemma 5.1 with $c_Y = e^{bd}$ yields

$$\mathbb{P}\{AY + B > x\} \sim \big(e^{bd}\mathbb{E}e^{bB}\, \mathbb{1}_{\{A=1\}} + \mathbb{E}f(Y)\big)\mathbb{P}\{B > x\}, \quad x \to \infty. \tag{36}$$

Since for each $y \ge 0$

$$\frac{\mathbb{P}\{B > x - Ay\}}{\mathbb{P}\{B > x\}} \le \frac{\mathbb{P}\{B > x - y\}}{\mathbb{P}\{B > x\}} \to e^{by}, \quad x \to \infty$$

in view of (3) we conclude that

$$f(y) \le e^{by}, \quad y \ge 0. \tag{37}$$

This implies that

$$\mathbb{E}f(Y) \le \mathbb{E}e^{bY} = e^{bd}\mathbb{E}e^{bB}\, \mathbb{1}_{\{B>q\}} + \mathbb{P}\{B \le q\},$$

whence

$$e^{bd}\mathbb{E}e^{bB}\, \mathbb{1}_{\{A=1\}} + \mathbb{E}f(Y) \le e^{bd}\mathbb{E}e^{bB}\, \mathbb{1}_{\{A=1\}} + e^{bd}\mathbb{E}e^{bB}\, \mathbb{1}_{\{B>q\}} + \mathbb{P}\{B \le q\} < e^{bd} \tag{38}$$

by the choice of d and q. Now (36) and (38) together imply that there exists $x_0 > 0$ such that $\mathbb{P}\{AY + B > x\} \le \mathbb{P}\{Y > x\}$ whenever $x \ge x_0$.

Let Z be a random variable independent of (A, B) with the distribution

$$\mathbb{P}\{Z > x\} = \mathbb{P}\{Y > x \,|\, Y \ge x_0\}.$$

For $x \ge x_0$ we have

$$\mathbb{P}\{AZ + B > x\} = \mathbb{P}\{AY + B > x \,|\, Y \ge x_0\} \le \frac{\mathbb{P}\{AY + B > x\}}{\mathbb{P}\{Y \ge x_0\}} \le \frac{\mathbb{P}\{Y > x\}}{\mathbb{P}\{Y \ge x_0\}} = \mathbb{P}\{Z > x\}.$$

For $x < x_0$ $\mathbb{P}\{Z > x\} = 1$, so that $\mathbb{P}\{AZ + B > x\} \le \mathbb{P}\{Z > x\}$ holds for all $x \in \mathbb{R}$. The proof of Lemma 5.2 is complete. □

Proof of Theorem 1.3. Let X_0 be a nonnegative random variable independent of $(A_n, B_n)_{n\in\mathbb{N}}$. The sequence $(X_n)_{n\in\mathbb{N}_0}$, recursively defined by the random difference equation

$$X_n = A_n X_{n-1} + B_n, \quad n \in \mathbb{N}, \tag{39}$$

forms a Markov chain. Occasionally, we write $X_n(X_0)$ for X_n to bring out the dependence on X_0.

While condition (3) entails $\mathbb{E}\log(1+B^+)<\infty$ which in combination with (6) ensures that $\mathbb{E}\log(1+|B|)<\infty$ (see the paragraph following formula (14)), condition $\mathbb{P}\{A\in(0,1]\}=1$ together with (4) guarantees that $\mathbb{E}\log A\in[-\infty,0)$. Further, condition (22) obviously holds. Invoking now Theorem 3.1 (c) in [9] we conclude that X_n converges in distribution to the a.s. finite $X=\sum_{k\geq 1}\Pi_{k-1}B_k$ as $n\to\infty$ whatever the distribution of X_0. Our plan is to approach the distribution of X from above and from below by the distributions of $X_n(X_0^{(i)})$, $n\in\mathbb{N}_0$, $i=1,2$. By picking appropriate distributions of $X_0^{(i)}$ we shall be able to provide tight bounds on the distribution tail of X.

Upper bound. Put $X_0=Z$ for a random variable Z as defined in Lemma 5.2 which is also independent of $(A_n,B_n)_{n\in\mathbb{N}}$. Then

$$X_1=A_1X_0+B_1\leq_{\rm st}X_0$$

and thereupon

$$X_{n+1}=A_{n+1}X_n+B_{n+1}\leq_{\rm st}A_nX_{n-1}+B_n=X_n,\quad n\in\mathbb{N}$$

because $A_k>0$ a.s. for $k\in\mathbb{N}$.

Define a sequence $(c_{X_n})_{n\in\mathbb{N}_0}$ recursively by

$$c_{X_0}=c_Z,\quad c_{X_{n+1}}=c_{X_n}\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}}+\mathbb{E}f(X_n),\quad n\in\mathbb{N}_0.$$

Note that

$$\mathbb{E}f(X)\leq\mathbb{E}f(X_n)\leq\mathbb{E}f(Z)\leq\mathbb{E}e^{bZ}<\infty,$$

where the first two inequalities hold true because f is nondecreasing and $(X_n)_{n\in\mathbb{N}_0}$ is a stochastically nonincreasing sequence, the third inequality is a consequence of (37), and the fourth inequality follows from (35) and (3). Starting with

$$\mathbb{P}\{X_0>x\}\ \sim\ c_{X_0}\mathbb{P}\{B>x\},\quad x\to\infty$$

we use the mathematical induction to obtain

$$\mathbb{P}\{X_n>x\}=\mathbb{P}\{A_nX_{n-1}+B_n>x\}\ \sim\ c_{X_n}\mathbb{P}\{B>x\},\quad x\to\infty$$

with the help of Lemma 5.1. The latter limit relation together with the stochastic monotonicity implies that $(c_{X_n})_{n\in\mathbb{N}_0}$ is a nonincreasing sequence of positive numbers which must have a limit c_X, say, given by

$$c_X=\frac{\mathbb{E}f(X)}{1-\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}}}.$$

The form of the limit is justified by the fact that the distributional convergence of X_n to X together with continuity of the distribution of X (see Theorem 2.1.2 in [15] or Theorem 1.3 in [2]) ensures that $f(X_n)$ converges in distribution to $f(X)$ as $n\to\infty$ whence $\lim_{n\to\infty}\mathbb{E}f(X_n)=\mathbb{E}f(X)$ by the Lévy monotone convergence theorem.

Since $X\leq_{\rm st}X_n$ for each $n\in\mathbb{N}_0$, we infer

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{X>x\}}{\mathbb{P}\{B>x\}}\leq\limsup_{x\to\infty}\frac{\mathbb{P}\{X_n>x\}}{\mathbb{P}\{B>x\}}=c_{X_n}$$

for each $n\in\mathbb{N}_0$ and thereupon

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{X>x\}}{\mathbb{P}\{B>x\}}\leq c_X.\tag{40}$$

Lower bound. We start by noting that

$$\mathbb{P}\{X>x\}=\mathbb{P}\{AX+B>x\}\geq \mathbb{P}\{AX+B>x, X>0\}\geq \mathbb{P}\{X>0\}\mathbb{P}\{B>x\},\quad x\in\mathbb{R}.$$

Therefore, denoting by X_0 a random variable which is independent of $(A_n,B_n)_{n\in\mathbb{N}_0}$ and has distribution $\mathbb{P}\{X_0>x\}=\mathbb{P}\{X>0\}\mathbb{P}\{B>x\}$ for $x\geq 0$ and $\mathbb{P}\{X_0>x\}=\mathbb{P}\{X>x\}$ for $x<0$, and arguing in the same way as in the previous part of the proof we obtain a sequence $(X_n)_{n\in\mathbb{N}_0}$ approaching X in distribution such that $X_n\leq_{\rm st} X$ for $n\in\mathbb{N}_0$. It is worth stating explicitly that $(X_n)_{n\in\mathbb{N}_0}$ is not necessarily stochastically monotone.

Define a sequence $(c'_{X_n})_{n\in\mathbb{N}_0}$ recursively by

$$c'_{X_0}=\mathbb{P}\{X>0\},\quad c'_{X_{n+1}}=c'_{X_n}\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}}+\mathbb{E}f(X_n),\quad n\in\mathbb{N}_0.$$

We claim that

$$\lim_{n\to\infty}\mathbb{E}f(X_n)=\mathbb{E}f(X)<\infty, \tag{41}$$

where the finiteness follows from the previous part of the proof. Mimicking the argument given in the treatment of the upper bound we conclude that $f(X_n)$ converges in distribution to $f(X)$ as $n\to\infty$. Therefore, $\liminf_{n\to\infty}\mathbb{E}f(X_n)\geq \mathbb{E}f(X)$ by Fatou's lemma. On the other hand, we have $\mathbb{E}f(X_n)\leq \mathbb{E}f(X)$ for $n\in\mathbb{N}_0$, and (41) follows.

Now (41) together with $c'_{X_n}=(\mathbb{P}\{X>0\}\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}})^n+\sum_{k=0}^{n-1}(\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}})^{n-k-1}\mathbb{E}f(X_k)$ for $n\in\mathbb{N}$ ensures that $c'_X:=\lim_{n\to\infty}c'_{X_n}$ exists and

$$c'_X=\frac{\mathbb{E}f(X)}{1-\mathbb{E}e^{bB}\,\mathbb{1}_{\{A=1\}}}=c_X.$$

The same argument as in the previous part of the proof enables us to conclude that

$$\liminf_{x\to\infty}\frac{\mathbb{P}\{X>x\}}{\mathbb{P}\{B>x\}}\geq c'_{X_n}$$

for each $n\in\mathbb{N}_0$, whence

$$\liminf_{x\to\infty}\frac{\mathbb{P}\{X>x\}}{\mathbb{P}\{B>x\}}\geq c_X. \tag{42}$$

A combination of (40) and (42) yields (7). The proof of Theorem 1.3 is complete. □

6 Proof of Theorem 1.7

Recall that $\Psi(t)=\mathbb{E}e^{{\rm i}tX}$, $t\in\mathbb{R}$ satisfies (13). Using

$$\begin{aligned}\int_0^\infty & e^{{\rm i}ty}\mathbb{P}\{B>y\}{\rm d}y-\int_{-\infty}^0 e^{{\rm i}ty}\mathbb{P}\{B\leq y\}{\rm d}y\\ &=\mathbb{E}\left(\left[\int_0^B e^{{\rm i}ty}{\rm d}y\right]\mathbb{1}_{\{B>0\}}\right)-\mathbb{E}\left(\left[\int_B^0 e^{{\rm i}ty}{\rm d}y\right]\mathbb{1}_{\{B\leq 0\}}\right)\\ &=\mathbb{E}\left(\frac{e^{{\rm i}tB}-1}{{\rm i}t}\mathbb{1}_{\{B>0\}}\right)-\mathbb{E}\left(\frac{1-e^{{\rm i}tB}}{{\rm i}t}\mathbb{1}_{\{B\leq 0\}}\right)=\frac{\Phi(t)-1}{{\rm i}t},\end{aligned}$$

we obtain an equivalent form of (13)

$$\begin{aligned}\Psi(t)&=\Phi(t)\exp\left({\rm i}\lambda\int_0^t\left[\int_0^\infty e^{{\rm i}uy}\mathbb{P}\{B>y\}{\rm d}y-\int_{-\infty}^0 e^{{\rm i}uy}\mathbb{P}\{B\leq y\}{\rm d}y\right]{\rm d}u\right)\\ &=\Phi(t)\exp\left(\lambda\left[\int_0^\infty\frac{e^{{\rm i}ty}-1}{y}\mathbb{P}\{B>y\}{\rm d}y-\int_{-\infty}^0\frac{e^{{\rm i}ty}-1}{y}\mathbb{P}\{B\leq y\}{\rm d}y\right]\right)\end{aligned}$$

for $t\in\mathbb{R}$, where $\Phi(t)=\mathbb{E}e^{\mathrm{i}tB}$. In view of (9) this can be further represented as

$$\begin{aligned}&\Psi(t)\exp\left(\lambda\int_0^\infty\frac{e^{\mathrm{i}ty}-1}{y}r^-(y)\mathrm{d}y\right) \qquad (43)\\ =\quad &\Phi(t)\left(\frac{b}{b-\mathrm{i}t}\right)^{C\lambda}\exp\left(\lambda\int_0^\infty\frac{e^{\mathrm{i}ty}-1}{y}r^+(y)\mathrm{d}y\right)\exp\left(\lambda\int_0^\infty\frac{e^{-\mathrm{i}ty}-1}{y}\mathbb{P}\{-B\geq y\}\mathrm{d}y\right)\end{aligned}$$

for $t\in\mathbb{R}$. Let Y_1, Y_2 and Y_3 be infinitely divisible nonnegative random variables with zero drifts and the Lévy measures $\nu_1(\mathrm{d}y)=y^{-1}r^-(y)\,\mathbb{1}_{(0,\infty)}(y)\mathrm{d}y$, $\nu_2(\mathrm{d}y)=y^{-1}r^+(y)\,\mathbb{1}_{(0,\infty)}(y)\mathrm{d}y$ and $\nu_3(\mathrm{d}y)=y^{-1}\mathbb{P}\{-B\geq y\}\,\mathbb{1}_{(0,\infty)}(y)\mathrm{d}y$, respectively. Let Z be a random variable with a $\gamma(C\lambda,b)$ distribution. Assume that Y_1 is independent of X and that B, Z, Y_2 and Y_3 are mutually independent. Equality (43) tells us that $X+Y_1$ has the same distribution as $B+Z+Y_2-Y_3$. We claim that

$$\mathbb{P}\{X+Y_1>x\}=\mathbb{P}\{B+Z+Y_2-Y_3>x\}\ \sim\ C\mathbb{E}e^{b(Y_2-Y_3)}\frac{(bx)^{C\lambda}}{\Gamma(C\lambda+1)}e^{-bx},\quad x\to\infty,\qquad (44)$$

where $\mathbb{E}e^{b(Y_1-Y_2)}\leq\mathbb{E}e^{bY_1}<\infty$ by virtue of the first condition in (11).

Proof of (44). By (9) and (10), $\mathbb{P}\{B>x\}\sim Ce^{-bx}$ as $x\to\infty$. Hence,

$$\mathbb{P}\{B+Z>x\}\sim\frac{C(bx)^{C\lambda}}{\Gamma(C\lambda+1)}e^{-bx},\quad x\to\infty$$

by Lemma 7.1 (iii) in [22] (in the notation of [22] we set $Y_1:=bB$ and $Y_2:=bZ$). According to an extension of Breiman's lemma (Proposition 2.1 in [7]) relation (44) follows provided that the following conditions hold: (a) $\mathbb{E}e^{b(Y_1-Y_2)}<\infty$; (b) $\mathbb{P}\{e^{Y_1-Y_2}>x\}=o(h(x))$, where $h(x):=x^b\mathbb{P}\{e^{B+Z}>x\}$ for $x\geq 0$; (c) $\limsup_{x\to\infty}\frac{\sup_{1\leq y\leq x}h(y)}{h(x)}<\infty$.

We already know that (a) holds which particularly implies that $\lim\limits_{x\to\infty}x^b\mathbb{P}\{e^{Y_1-Y_2}>x\}=0$. While this in combination with $h(x)\sim(Cb^{C\lambda}/\Gamma(C\lambda+1))(\log x)^{C\lambda}$ proves (b), the last asymptotic relation alone secures (c). The proof of (44) is complete.

With (44) at hand we infer

$$\mathbb{P}\{X>x\}\sim\frac{\mathbb{E}e^{b(Y_2-Y_3)}}{\mathbb{E}e^{bY_1}}\frac{C(bx)^{\lambda C}}{\Gamma(\lambda C+1)}e^{-bx}=Kx^{\lambda C}e^{-bx},\quad x\to\infty$$

by Corollary 4.3 (ii) in [17] which is applicable because the distribution of Y_1 is infinitely divisible and $\mathbb{E}e^{(b+\varepsilon)Y_1}<\infty$ by the second part of (11). The proof of Theorem 1.7 is complete.

7 Appendix

Parts (a) and (b) of Proposition 7.1 are Theorems 1.6 and 1.7 in [2], respectively.

Proposition 7.1. *(a) Suppose* (21), (22) *and* $\mathbb{P}\{|A|=1\}=0$, *and let* $r>0$. *Then* $\mathbb{E}e^{r|X|}<\infty$ *if, and only if,*

$$\mathbb{P}\{|A|<1\}=1\ \textit{and}\ \mathbb{E}e^{r|B|}<\infty.$$

(b) Suppose (21), (22) *and* $\mathbb{P}\{|A|=1\}\in(0,1)$, *and let* $r>0$. *Then* $\mathbb{E}e^{r|X|}<\infty$ *if, and only if,*

$$\mathbb{P}\{|A|\leq 1\}=1,\quad \mathbb{E}e^{r|B|}<\infty$$

and

$$\mathbb{E}e^{-rB}\,\mathbb{1}_{\{A=-1\}}\,\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=-1\}}<(1-\mathbb{E}e^{-rB}\,\mathbb{1}_{\{A=1\}})(1-\mathbb{E}e^{rB}\,\mathbb{1}_{\{A=1\}}).$$

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