Stable limit laws for random walk in a sparse random environment I: moderate sparsity

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Abstract

A random walk in a sparse random environment is a model introduced by Matzavinos et al. [Electron. J. Probab. 21, paper no. 72: 2016] as a generalization of both a simple symmetric random walk and a classical random walk in a random environment. A random walk $(X_n)_{n\in\mathbb{N}\cup\{0\}}$ in a sparse random environment $(S_k,\lambda_k)_{k\in\mathbb{Z}}$ is a nearest neighbor random walk on $\mathbb{Z}$ that jumps to the left or to the right with probability $1/2$ from every point of $\mathbb{Z}\setminus\{\ldots,S_{-1},S_0=0,S_1,\ldots\}$ and jumps to the right (left) with the random probability λ_{k+1} $(1-\lambda_{k+1})$ from the point S_k, $k\in\mathbb{Z}$. Assuming that $(S_k-S_{k-1},\lambda_k)_{k\in\mathbb{Z}}$ are independent copies of a random vector $(\xi,\lambda)\in\mathbb{N}\times(0,1)$ and the mean $\mathbb{E}\xi$ is finite (moderate sparsity) we obtain stable limit laws for X_n, properly normalized and centered, as $n\to\infty$. While the case $\xi\le M$ a.s. for some deterministic $M>0$ (weak sparsity) was analyzed by Matzavinos et al., the case $\mathbb{E}\xi=\infty$ (strong sparsity) will be analyzed in a forthcoming paper.

Keywords: branching process in a random environment with immigration; perpetuity; random difference equation; random walk in a random environment.
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Contents

1 Introduction

Simple random walks on $\mathbb{Z}$ (the set of integers) arise in various areas of classical and modern stochastics. However, their intrinsic homogeneity reduces in some situations applicability of the simple random walks. Solomon [34] eliminated this drawback by introducing a random environment which made a modified random walk space inhomogeneous. In the present article we investigate an intermediate model, called random walk in a sparse random environment (RWSRE), in which homogeneity of an environment is only perturbed on a sparse subset of $\mathbb{Z}$. Since RWSRE is a particular case of a random walk in a random environment (RWRE) we proceed by recalling the definition of the latter.

Set $\Omega = (0,1)^{\mathbb{Z}}$ and $\mathcal{X} = \mathbb{Z}^{\mathbb{N}}$. Let $\mathcal{F}$ be the Borel σ-algebra of subsets of Ω, P a probability measure on $(\Omega, \mathcal{F})$ and $\mathcal{G}$ the σ-algebra generated by the cylinder sets in $\mathcal{X}$. A random environment is a random element $\omega = (\omega_n)_{n\in\mathbb{Z}}$ of the measurable space $(\Omega, \mathcal{F})$ distributed according to P. A quenched (fixed) environment ω provides us with a probability measure $\mathbb{P}_\omega$ on $\mathcal{X}$ whose transition kernel is given by

$$\mathbb{P}_\omega\{X_{n+1} = j | X_n = i\} = \begin{cases} \omega_i, & \text{if } j = i+1, \\ 1-\omega_i, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

With the initial condition $X_0 := 0$ the sequence $X = (X_n)_{n\in\mathbb{N}_0}$ is a Markov chain on $\mathbb{Z}$ (under $\mathbb{P}_\omega$) which is called random walk in the random environment ω. Here and hereafter, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is natural to investigate RWRE from two viewpoints which are different in many aspects: under the quenched measure $\mathbb{P}_\omega$ for almost all (with respect to P) ω, that is, for a typical ω or under an annealed measure. Formally, the annealed measure $\mathbb{P}$ on $(\Omega \times \mathcal{X}, \mathcal{F} \otimes \mathcal{G})$ is defined as a semi-direct product $\mathbb{P} = P \ltimes \mathbb{P}_\omega$ via the formula

$$\mathbb{P}\{F \times G\} = \int_F \mathbb{P}_\omega\{G\} P(\mathrm{d}\omega), \quad F \in \mathcal{F}, \quad G \in \mathcal{G}.$$

Note that in general X is no longer a Markov chain under $\mathbb{P}$. Usually one assumes that an environment ω forms a stationary and ergodic sequence or even a sequence of iid (independent and identically distributed) random variables. In this setting RWRE has attracted a fair amount of attention among probabilistic community resulting in quenched and annealed limit theorems [3, 11, 12, 25, 26, 33, 35] and large deviations [5, 7, 9, 15, 19, 31, 32, 36, 37]. This list of references is far from being complete.

We aim at establishing annealed limit theorems for X (that is, under $\mathbb{P}$) in a so called sparse random environment which corresponds to a particular choice of P which is specified as follows. Let $((\xi_k, \lambda_k))_{k\in\mathbb{Z}}$ be a sequence of independent copies of a random vector (ξ, λ) which satisfies $\lambda \in (0,1)$ and $\xi \in \mathbb{N}$ a.s. For $n \in \mathbb{Z}$, set

$$S_n = \begin{cases} \sum_{k=1}^n \xi_k, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -\sum_{k=n+1}^0 \xi_k, & \text{if } n < 0. \end{cases}$$

The sparse random environment $\omega = (\omega_n)_{n\in\mathbb{Z}}$ is defined by

$$\omega_n = \begin{cases} \lambda_{k+1}, & \text{if } n = S_k \text{ for some } k \in \mathbb{Z}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{1.1}$$

The model (with λ_k in (1.1) replacing λ_{k+1}) was introduced by Matzavinos, Roitershtein and Seol [29]. These authors obtained various results including a recurrence/transience criterion, a strong law of large numbers and limit theorems. However, many results in [29] were proved under quite restrictive conditions including boundedness of ξ, a strong ellipticity condition for the distribution of λ and independence of ξ and λ. In this setting some essential properties of X remain hidden. Our main purpose is to relax the aforementioned assumptions substantially, thereby establishing limit theorems in full generality, and to find out how distributional properties of the vector (ξ,λ) affect the asymptotic behavior of X. It turns out that the asymptotics of X is regulated by the tail behaviors of ξ and $\rho:=(1-\lambda)/\lambda$ which determine sparsity of the environment and the local drift of the environment, respectively. In this paper we investigate the case where $\mathbb{E}\xi<\infty$. We call the corresponding environment 'moderately sparse', whereas in the opposite case where $\mathbb{E}\xi=\infty$ we say that the environment is 'strongly sparse'. The analysis of X in a strongly sparse environment requires completely different techniques and will be carried out in a companion paper [6].

The present article is organized as follows. In Section 2 we formulate our limit theorems for X and the first passage times of X. In Section 3.1 we describe our approach and define a branching process Z in a random environment which is used to analyze the random walk X. In Section 3.2 we introduce necessary notation related to the process Z. In Section 4 we explain a heuristic behind our proof and present a number of important estimates and decompositions used throughout the paper. Among other things, we demonstrate in this section how to reduce the initial problem to the asymptotic analysis of sums of certain iid random variables. The tail behavior of these variables is discussed in Section 5. Section 6 is devoted to the analysis of a particular critical Galton–Watson process with immigration which naturally arises in the context of random walks in the sparse random environment. The proofs of the main results are given in Sections 7.1, 7.2 and 7.3. The proofs of auxiliary lemmas can be found in Section 7.4 and the Appendix.

2 Main results

We focus on the case when X is $\mathbb{P}$-a.s. transient to $+\infty$ and the environment is moderately sparse, that is, $\mathbb{E}\xi<\infty$. Recall the notation

$$\rho=\frac{1-\lambda}{\lambda}.$$

According to Theorem 3.1 in [29], X is $\mathbb{P}$-a.s. transient to $+\infty$ if

$$\mathbb{E}\log\rho\in[-\infty,0)\quad\text{and}\quad\mathbb{E}\log\xi<\infty. \tag{2.1}$$

The first inequality excludes the degenerate case $\rho=1$ a.s. in which X becomes a simple random walk. The second inequality is always true for the moderately sparse environment. We note right away that our standing assumptions $\mathbb{E}\log\rho\in[-\infty,0)$ and $\mathbb{E}\xi<\infty$ hold under the conditions of our main results, Theorems 2.2 and 2.6.

The sequence $(T_n)_{n\in\mathbb{Z}}$ of the first passage times defined by

$$T_n=\inf\{k\geq 0: X_k=n\},\quad n\in\mathbb{Z}$$

is of crucial importance for our arguments. Of course, the observation that the asymptotics of X can be derived from that of (T_n) is not new and has been exploited in many earlier papers in the area of random walks in random environments. Assuming only transience to the right it is shown on p. 12 in [29] that

$$\lim_{n\to\infty}\frac{T_{S_n}}{n}=\mathbb{E}T_{S_1}\quad\mathbb{P}-\text{a.s.}$$

This in combination with Lemma 4.4 in [29] leads to the conclusion that

$$\lim_{n\to\infty}\frac{X_n}{n}=\mathbb{E}\xi/\mathbb{E}T_{S_1}=:v\quad\text{and}\quad\lim_{n\to\infty}\frac{T_n}{n}=\frac{1}{v}\quad\mathbb{P}-\text{a.s.} \tag{2.2}$$

whenever the environment is moderately sparse. Furthermore, under the additional assumption that ξ and λ are independent, Theorem 3.3 in [29] states that

$$v=\frac{(1-\mathbb{E}\rho)\mathbb{E}\xi}{(1-\mathbb{E}\rho)\mathbb{E}\xi^2+2\mathbb{E}\rho(\mathbb{E}\xi)^2} \tag{2.3}$$

provided that $\mathbb{E}\rho<1$ and $\mathbb{E}\xi^2<\infty$, and $v=0$, otherwise.

In Proposition 2.1 we give an explicit formula for v when ξ and λ are allowed to be dependent.

Proposition 2.1. *Assume that* $\mathbb{E}\log\rho \in [-\infty, 0)$ *and* $\mathbb{E}\xi < \infty$. *Then*

$$v = \frac{(1-\mathbb{E}\rho)\mathbb{E}\xi}{(1-\mathbb{E}\rho)\mathbb{E}\xi^2 + 2\mathbb{E}\xi\mathbb{E}\rho\xi}, \qquad \frac{1}{v} = \frac{1}{\mathbb{E}\xi}\left(\mathbb{E}\xi^2 + \frac{2\mathbb{E}\xi\mathbb{E}\rho\xi}{1-\mathbb{E}\rho}\right) \tag{2.4}$$

provided that $\mathbb{E}\rho < 1$, $\mathbb{E}\rho\xi < \infty$ *and* $\mathbb{E}\xi^2 < \infty$, *and* $v = 0$ $(1/v = \infty)$, *otherwise.*

Turning to weak convergence results we first formulate our assumptions on the distribution of ρ. Two different sets of conditions will be used:

(ρ1) for some $\alpha \in (0, 2]$

$$\mathbb{E}\rho^\alpha = 1, \quad \mathbb{E}\rho^\alpha \log^+ \rho < \infty \quad \text{and} \quad \text{the distribution of } \log\rho \text{ is nonarithmetic,}$$

where $\log^+ x := \max(0, \log x)$;

(ρ2) there exists an open interval $\mathcal{I} \subset (0, \infty)$ such that $\mathbb{E}\rho^x < 1$ for all $x \in \mathcal{I}$.

Assuming that (ρ1) holds for some $\alpha > 0$ we further distinguish two cases pertaining to the distribution of ξ:

(ξ1) $\mathbb{E}\xi^{2\alpha\vee 1} < \infty$, where $x \vee y := \max(x, y)$;

(ξ2) there exists a slowly varying function ℓ such that

$$\mathbb{P}\{\xi > t\} \sim t^{-\beta}\ell(t), \quad t \to \infty \tag{2.5}$$

for some $\beta \in (1, 2\alpha]$, and $\mathbb{E}\xi^{2\alpha} = \infty$ if $\beta = 2\alpha$.

Finally, if (ρ2) holds for some open interval $\mathcal{I}$ we assume that either (ξ1) holds for some $\alpha \in \mathcal{I}$ or the regular variation assumption in (ξ2) holds for some β satisfying $\beta/2 \in \mathcal{I}$.

We summarize our results in Table 1 with an emphasis on which component of the environment dominates[1].

	(ξ1)	(ξ2)
(ρ1)	ρ dominates (Thm. 2.2 (A1))	If $\beta < 2\alpha$, see (ρ2) with $\alpha = \beta/2$
		If $\beta = 2\alpha$, $\lim_{t\to\infty}\ell(t) = 0$, then ρ dominates (Thm. 2.2 (A2))
		If $\beta = 2\alpha$, $\lim_{t\to\infty}\ell(t) = C_\ell \in (0,\infty)$, then contributions of ρ and ξ are comparable (Thm. 2.2 (A3))
		If $\beta = 2\alpha$, $\lim_{t\to\infty}\ell(t) = +\infty$, then ξ dominates (Thm. 2.6 (B1))
		If $\beta > 2\alpha \Longrightarrow \mathbb{E}\xi^{2\alpha} < \infty$, see ($\rho$1) and ($\xi$1)
(ρ2)	$2 \in \mathcal{I}$, contributions of ρ and ξ are comparable (Prop. 2.9)	$\beta \in (1,4)$ and $\beta/2 \in \mathcal{I} \Longrightarrow \xi$ dominates (Thm. 2.6 (B2))

Table 1: Influence of the environment and limit theorems for T_n.

In what follows, for $\alpha \in (0, 2)$, we denote by $\mathcal{S}_\alpha$ a random variable with an α-stable distribution defined by

$$-\log\mathbb{E}\exp(-u\mathcal{S}_\alpha) = \Gamma(1-\alpha)u^\alpha, \quad u \geq 0,$$

where $\Gamma(\cdot)$ is the gamma function, if $\alpha \in (0, 1)$;

$$\log\mathbb{E}\exp(\mathrm{i}u\mathcal{S}_1) = -(\pi/2)|u| - \mathrm{i}u\log|u|, \quad u \in \mathbb{R};$$

[1]In some cases we also need additional technical assumptions concerning the joint distribution of ρ and ξ, for instance, $\mathbb{E}(\rho\xi)^\alpha < \infty$. These will be stated explicitly in the corresponding theorems.

$$\log \mathbb{E} \exp(\mathrm{i}u\mathcal{S}_\alpha) = |u|^\alpha \frac{\Gamma(2-\alpha)}{\alpha - 1}(\cos(\pi\alpha/2) - \mathrm{i}\sin(\pi\alpha/2)\mathrm{sign}\, u), \quad u \in \mathbb{R},$$

if $\alpha \in (1, 2)$. Note that $\mathcal{S}_\alpha$ is a positive random variable when $\alpha \in (0, 1)$ and it has a spectrally positive α-stable distribution when $\alpha \in [1, 2)$. Throughout the paper $\xrightarrow{\mathrm{d}}$ and $\xrightarrow{\mathbb{P}}$ will mean convergence in probability and convergence in distribution, respectively.

In Theorem 2.2 and Corollary 2.4 we treat the case ($\rho1$).

Theorem 2.2. *Assume that one of the following sets of assumptions is satisfied:*

(A1) ($\rho1$) *holds for some* $\alpha \in (0, 2]$, ($\xi1$) *holds and* $\mathbb{E}(\rho\xi)^\alpha < \infty$*;*

(A2) ($\rho1$) *holds for some* $\alpha \in (1/2, 2]$ *and* ($\xi2$) *holds with* $\beta = 2\alpha$ *and* $\lim_{t\to\infty} \ell(t) = 0$*, and* $\mathbb{E}(\rho\xi)^\alpha < \infty$*;*

(A3) ($\rho1$) *holds for some* $\alpha \in (1/2, 2)$, ($\xi2$) *holds with* $\beta = 2\alpha$ *and* $\lim_{t\to\infty} \ell(t) = C_\ell \in (0, \infty)$, $\mathbb{E}\rho^{\alpha+\varepsilon} < \infty$ *and* $\mathbb{E}\rho^\alpha\xi^{\alpha+\varepsilon} < \infty$ *for some* $\varepsilon > 0$.

Then there exist absolute constants A_α*,* B_α *and* C_1 *such that the following limit relations hold as* $n \to \infty$*.*

- *If* $\alpha \in (0, 1)$*, then* $\frac{T_n}{B_\alpha n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mathcal{S}_\alpha$.
- *If* $\alpha = 1$*, then* $\frac{T_n - A_1 a(n)}{B_1 n} \xrightarrow{\mathrm{d}} C_1 + \mathcal{S}_1$*, where* $a(n) \sim n \log n$.
- *If* $\alpha \in (1, 2)$*, then* $\frac{T_n - A_\alpha n}{B_\alpha n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mathcal{S}_\alpha$.
- *If* $\alpha = 2$*, then* $\frac{T_n - A_2 n}{B_2 (n \log n)^{1/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1)$*, where* $\mathcal{N}(0, 1)$ *is a standard normal random variable.*

Remark 2.3. See (7.11), (7.12) and (7.14) for explicit forms of the constants A_α, B_α and C_1. In Theorem 2.2 we do not specify the constants by two reasons. First, these involve characteristics of random variables that have not been introduced so far. Second, some of these constants are essentially implicit in the sense that these cannot be calculated.

From Theorem 2.2 we deduce the following corollary.

Corollary 2.4. *Under the assumptions and notation of Theorem 2.2 the following limit relations hold as* $k \to \infty$*.*

- *If* $\alpha \in (0, 1)$*, then* $\frac{X_k}{B_\alpha^{-\alpha} k^\alpha} \xrightarrow{\mathrm{d}} \mathcal{S}_\alpha^{-\alpha}$.
- *If* $\alpha = 1$*, then* $\frac{X_k - A_1^{-1}\widehat{a}(k)}{A_1^{-2} B_1 k (\log k)^{-2}} \xrightarrow{\mathrm{d}} -C_1 - \mathcal{S}_1$*, where* $\widehat{a}(k) \sim k(\log k)^{-1}$.
- *If* $\alpha \in (1, 2)$*, then* $\frac{X_k - A_\alpha^{-1} k}{A_\alpha^{-(1+1/\alpha)} B_\alpha k^{1/\alpha}} \xrightarrow{\mathrm{d}} -\mathcal{S}_\alpha$.
- *If* $\alpha = 2$*, then* $\frac{X_k - A_2^{-1} k}{A_2^{-3/2} B_2 (k \log k)^{1/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1)$.

Remark 2.5. When $\alpha \in (0, 1)$ the distribution of $\mathcal{S}_\alpha^{-\alpha}$ is called the Mittag-Leffler distribution with parameter α. The term stems from the facts that

$$\mathbb{E} \exp(u\Gamma(1-\alpha)\mathcal{S}_\alpha^{-\alpha}) = \sum_{n\geq 0} \frac{u^n}{\Gamma(1 + n\alpha)}, \quad u \in \mathbb{R}$$

and that the right-hand side defines the Mittag-Leffler function with parameter α.

Our next theorem treats weak convergence of T_n in cases where ξ plays a dominant role.

Theorem 2.6. *Assume that one of the following sets of assumptions is satisfied:*

(B1) $(\rho 1)$ *holds for some* $\alpha \in (1/2, 2]$, $(\xi 2)$ *holds with* $\beta = 2\alpha$ *and* $\lim_{t\to\infty} \ell(t) = +\infty$, *and* $\mathbb{E}(\rho\xi)^\alpha < \infty$;

(B2) $(\rho 2)$ *holds and* $(\xi 2)$ *holds with* $\beta \in (1, 4)$ *such that* $\beta/2 \in \mathcal{I}$ *and* $\mathbb{E}(\rho\xi)^{\beta/2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$.

In the case (B2) *put* $\alpha := \beta/2$. *Then there exist the functions* $c_\alpha(t)$ *for* $\alpha \in (1/2, 2)$, $q_1(t)$ *and* $r_2(t)$ *regularly varying at* ∞ *of indices* $1/\alpha$, 1 *and* $1/2$, *respectively, and the absolute constants* A^*_α *and* B^*_α *for* $\alpha \in (1/2, 2]$ *such that the following limit relations hold as* $n \to \infty$.

- *If* $\alpha \in (1/2, 1)$, *then* $\frac{T_n}{B^*_\alpha c_\alpha(n)} \xrightarrow{\mathrm{d}} \mathcal{S}_\alpha$.
- *If* $\alpha = 1$, *then* $\frac{T_n - n - q_1(A^*_1 n)}{B^*_1 c_1(n)} \xrightarrow{\mathrm{d}} \mathcal{S}_1$.
- *If* $\alpha \in (1, 2)$, *then* $\frac{T_n - A^*_\alpha n}{B^*_\alpha c_\alpha(n)} \xrightarrow{\mathrm{d}} \mathcal{S}_\alpha$.
- *If* $\alpha = 2$, *then* $\frac{T_n - A^*_2 n}{B^*_2 r_2(n)} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1)$.

Remark 2.7. This is a counterpart of Remark 2.3. Explicit forms of the normalizing and centering sequences in Theorem 2.6 and Corollary 2.8 given below can be found in (7.16), (7.17), (7.18) and (7.19), and (7.20), (7.21), (7.22) and (7.23), respectively.

Before formulating the corresponding limit theorems for X_k we need to introduce more notation. For $\alpha \in (1/2, 1)$, denote by $c^{\leftarrow}_\alpha(t)$ any positive function satisfying $c_\alpha(c^{\leftarrow}_\alpha(t)) \sim c^{\leftarrow}_\alpha(c_\alpha(t)) \sim t$ as $t \to \infty$. Since $c_\alpha(t)$ is regularly varying at ∞ such $c^{\leftarrow}_\alpha(t)$ do exist by Theorem 1.5.12 in [2].

Corollary 2.8. *Under the assumptions and notation of Theorem 2.6 the following limit relations hold as* $k \to \infty$.

- *If* $\alpha \in (1/2, 1)$, *then* $\frac{X_k}{(B^*_\alpha)^{-\alpha} c^{\leftarrow}_\alpha(k)} \xrightarrow{\mathrm{d}} \mathcal{S}^{-\alpha}_\alpha$.
- *If* $\alpha = 1$, *then* $\frac{X_k - s(k)}{t(k)} \xrightarrow{\mathrm{d}} -\mathcal{S}_1$ *for appropriate sequences* $s(k)$ *and* $t(k)$ *which are specified in formula* (7.21).
- *If* $\alpha \in (1, 2)$, *then* $\frac{X_k - (A^*_\alpha)^{-1} k}{(A^*_\alpha)^{-(1+1/\alpha)} B^*_\alpha c_\alpha(k)} \xrightarrow{\mathrm{d}} -\mathcal{S}_\alpha$.
- *If* $\alpha = 2$, *then* $\frac{X_k - (A^*_2)^{-1} k}{(A^*_2)^{-3/2} B^*_2 r_2(k)} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1)$.

The last result of this section is given for completeness only. It can be derived from a general central limit theorem (Theorem 2.2.1 in [38]) for random walk in a stationary and ergodic random environment. Since the sparse random environment is not stationary in general, to apply this theorem one has to pass to a stationary and ergodic environment. In Theorem 2.1 in [29] it is shown that such a passage is possible whenever $\mathbb{E}\xi < \infty$.

Proposition 2.9. *Assume that* $(\rho 2)$ *and* $(\xi 1)$ *hold for some* $\alpha \geq 2$. *Then there exists* $\sigma_0 \in (0, \infty)$ *such that, as* $n \to \infty$,

$$\frac{T_n - v^{-1} n}{\sigma_0 n^{1/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1)$$

and

$$\frac{X_n - vn}{\sigma_0 v^{3/2} n^{1/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 1),$$

where v *is given in* (2.4).

3 Branching processes in random environment with immigration

The connection between a random walk and a branching process with immigration dates back to Harris [22]. In the context of a random walk in a random environment this connection was successfully used by Kozlov [28] and Kesten, Kozlov and Spitzer [26]. In particular, these authors have shown that the asymptotic behavior of RWRE can be obtained from that of the total progeny of the aforementioned branching process. Since we are going to exploit the same idea we first recall a construction of the latter process. Most of the material in Section 3.1 can be found in [26].

3.1 Branching process with immigration

Throughout the paper the fact that $X_n \to \infty$ $\mathbb{P}$-a.s. plays a crucial role. Let $U_i^{(n)}$ be the number of steps of the process X from i to $i-1$ during the time interval $[0, T_n)$, that is,

$$U_i^{(n)} = \#\big\{k < T_n : X_k = i, X_{k+1} = i-1\big\}, \qquad i \leq n.$$

Since $X_{T_n} = n$ and $X_0 = 0$ we have, for $n \in \mathbb{N}$,

$$\begin{aligned} T_n &= \#\text{ of steps during } [0, T_n) \\ &= \#\text{ of steps to the right during } [0, T_n) + \#\text{ of steps to the left during } [0, T_n) \\ &= n + 2 \cdot \#\text{ of steps to the left during } [0, T_n) \\ &= n + 2 \sum_{i=-\infty}^{n} U_i^{(n)}. \end{aligned}$$

Recalling that the random walk X is transient to the right we infer

$$\sum_{i<0} U_i^{(n)} \leq \text{ total time spent by } X \text{ in } (-\infty, 0) < \infty \quad \text{a.s.} \tag{3.1}$$

In particular, for any $\gamma > 0$,

$$n^{-\gamma} \sum_{i<0} U_i^{(n)} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty.$$

Thus, the asymptotics of T_n as $n \to \infty$ is regulated by that of $n + 2\sum_{i=0}^{n} U_i^{(n)}$.

In what follows, we write $\text{Geom}(p)$ for a geometric distribution with success probability p, that is,

$$\text{Geom}(p)\{\ell\} = p(1-p)^{\ell}, \quad \ell \in \mathbb{N}_0.$$

Claim. Let ω and n be fixed. Then, for $0 \leq j \leq n$, $U_{n-j}^{(n)}$ is equal to the size of the jth generation (excluding the immigrant) of an inhomogeneous branching process with one immigrant in each generation. Under $\mathbb{P}_\omega$, the offspring distribution of the immigrant and the other particles in the $(j-1)$st generation is $\text{Geom}(\omega_{n-j})$.

Proof of the claim. First note that $U_n^{(n)} = 0$ because X cannot reach n before time T_n. Further, $U_{n-1}^{(n)} = V_0^{(n-1)}$, where $V_0^{(n-1)}$ is the number of excursions to the left of $n-1$ made by X before time T_n. Transitivity of X entails that the $\mathbb{P}_\omega$-distribution of $V_0^{(n-1)}$ is $\text{Geom}(\omega_{n-1})$. Finally, for $2 \leq j \leq n-1$, we have

$$U_{n-j}^{(n)} = \sum_{k=1}^{U_{n-j+1}^{(n)}} V_k^{(n-j)} + V_0^{(n-j)} \quad \text{a.s.},$$

where $V_0^{(n-j)}$ denotes the number of excursions to the left from $n-j$ before the first excursion to the left from $n-j+1$ (that is, before the time T_{n-j+1}) and $V_k^{(n-j)}$ denotes the number of excursions to the left from $n-j$ during the kth excursion to the left from $n-j+1$. Under $\mathbb{P}_\omega$, the random variables $(V_k^{(n-j)})_{k\geq 0}$ are iid with distribution $\mathrm{Geom}(\omega_{n-j})$ and also independent of $U_{n-j+1}^{(n)}$. The proof of the claim is complete.

Reversing the order of indices leads to a branching process $Z = (Z_k)_{k\geq 0}$ in a random environment (BPRE) with one immigrant entering the system in each generation. From the very beginning we stress that immigrants in our model are 'artificial', that is, even though they reproduce, they do not belong to any generation and, as such, they are not counted. The evolution of Z can be described as follows. An immigrant enters the 0th generation which is originally empty, that is, $Z_0 = 0$. She gives birth to a random number of offspring with $\mathbb{P}_\omega$-distribution $\mathrm{Geom}(\omega_1)$ which form the first generation. For $n \in \mathbb{N}$, an immigrant enters the nth generation. She and the particles of the nth generation, independently of each other and the particles in the previous generations, give birth to random numbers of offspring with $\mathbb{P}_\omega$ distribution $\mathrm{Geom}(\omega_{n+1})$. The number of these newborn particles which form the $(n+1)$st generation is given by

$$Z_{n+1} = \sum_{k=0}^{Z_n} G_k^{(n)}, \quad n \in \mathbb{N}_0,$$

where $G_0^{(n)}$ is the number of offspring of the $(n+1)$st immigrant and, for $k \in \mathbb{N}$, $G_k^{(n)}$ is the number of offspring of the kth particle in the nth generation (we set $G_k^{(n)} = 0$ if the kth particle in the nth generation does not exist). Observe that, under $\mathbb{P}_\omega$, for each $n \in \mathbb{N}_0$, the random variables $(G_k^{(n)})_{k\geq 0}$ are iid with distribution $\mathrm{Geom}(\omega_n)$ and also independent of Z_n.

Note that when the random environment is sparse (see (1.1)) and fixed, for the most time, the branching process Z behaves like a critical Galton–Watson process with one immigrant and $\mathrm{Geom}(1/2)$ offspring distribution. Only the particles of generation $S_i - 1$ for $i \in \mathbb{N}$ as well as the immigrants arriving in this generation reproduce according to $\mathrm{Geom}(\lambda_i)$ distribution. Averaging over ω and taking into account the structure of the environment we obtain

$$\sum_{j=0}^{S_n} U_j^{(S_n)} \overset{\mathrm{d}}{=} \sum_{k=1}^{S_n} Z_k \quad \text{and} \quad S_n + \sum_{j=0}^{S_n} U_j^{(S_n)} \overset{\mathrm{d}}{=} S_n + \sum_{k=1}^{S_n} Z_k, \quad n \in \mathbb{N} \tag{3.2}$$

under the annealed probability $\mathbb{P}$. This leads to the most important conclusion of the present section

$$T_{S_n} \overset{\mathrm{d}}{=} S_n + 2\sum_{k=1}^{S_n} Z_k + O_{\mathbb{P}}(1), \quad n \in \mathbb{N}, \tag{3.3}$$

where $O_{\mathbb{P}}(1)$ is a term which is bounded in probability. Distributional equality (3.3) will prove useful on many occasions.

3.2 Notation

Before we explain the strategy of our proof some more notation have to be introduced. Denote by $Z(k,n)$ the number of progeny residing in the nth generation of the kth immigrant. In particular, $Z(k,k)$ is the number of offspring of this immigrant. Then

$$Z_n = \sum_{k=1}^{n} Z(k,n).$$

For $n \in \mathbb{N}$ and $1 \leq i \leq n$, let $Y(i,n)$ denote the number of progeny in the generations $i, i+1, \ldots, n$ of the ith immigrant, that is,

$$Y(i,n) = \sum_{k=i}^{n} Z(i,k).$$

Similarly, for $i \in \mathbb{N}$, we denote by Y_i the total progeny of the ith immigrant, that is,

$$Y_i = Y(i,\infty) = \sum_{k\geq i} Z(i,k).$$

We also define W_n to be the total population size in the first n generations, that is,

$$W_n = \sum_{j=1}^{n} Z_j, \quad n \in \mathbb{N}.$$

Motivated by the structure of the environment we shall often divide the population into blocks which include generations $1, \ldots, S_1$; $S_1 + 1, \ldots, S_2$ and so on. As a preparation, we write

$$\mathbb{Z}_n = Z_{S_n}, \quad n \in \mathbb{N}$$

for the number of particles in the generation S_n,

$$\mathbb{W}_n = W_{S_n} - W_{S_{n-1}} = \sum_{j=S_{n-1}+1}^{S_n} Z_j, \quad n \in \mathbb{N}$$

for the total population in the generations $S_{n-1} + 1, \ldots, S_n$ and

$$\mathbb{Y}_n = \sum_{j=S_{n-1}+1}^{S_n} Y_j, \quad n \in \mathbb{N}$$

for the total progeny of immigrants arriving in the generations $S_{n-1}, \ldots, S_n - 1$.

3.3 Analysis of the environment

The asymptotic behavior of the branching process Z depends heavily upon the environment. At the end of this section we specify qualitatively two aspects of this dependence. A random difference equation which arises naturally in the course of our discussion, as well as in [26] and many other papers on RWRE, plays an important role in the subsequent arguments.

We proceed by recalling the definitions of random difference equations and perpetuities. Let $(A_n, B_n)_{n\in\mathbb{N}}$ be a sequence of independent copies of an $\mathbb{R}^2$-valued random vector (A, B). Further, let R_0 be a random variable which is independent of $(A_n, B_n)_{n\in\mathbb{N}}$. The sequence $(R_k)_{k\in\mathbb{N}_0}$, recursively defined by the random difference equation

$$R_k := B_k + A_k R_{k-1}, \quad k \in \mathbb{N},$$

forms a Markov chain which is very well known and well understood. Assuming that $R_0 = 0$ and reversing the indices in an equivalent representation $R_k = A_1 \cdot \ldots \cdot A_{k-1} B_1 + A_2 \cdot \ldots \cdot A_{k-1} B_2 + \ldots + B_k$ leads to the random variable $R_k^* := B_1 + A_1 B_2 + \ldots + A_1 \cdot \ldots \cdot A_{k-1} B_k$ satisfying $R_k^* \overset{\mathrm{d}}{=} R_k$ for all $k \in \mathbb{N}$. Whenever

$$\text{the series } \sum_{j\geq 1} B_j \prod_{l=1}^{j-1} A_l \text{ converges a.s.} \tag{3.4}$$

its infinite version $R_\infty^* := \sum_{j\geq 1} B_j \prod_{l=1}^{j-1} A_l$ is called perpetuity because of a possible actuarial application. The study of the random difference equations and perpetuities has a long history going back to Kesten [24] and Grincevičius [17]. We refer the reader to the recent monographs [4, 23] containing a comprehensive bibliography on the subject.

It is well-known that conditions $\mathbb{E} \log |A| \in [-\infty, 0)$ and $\mathbb{E} \log^+ |B| < \infty$ are sufficient for (3.4) and the distributional convergence $R_k \xrightarrow{\mathrm{d}} R_\infty^*$ as $k \to \infty$. There are numerous results in the literature concerning the tail behavior of R_∞^*. The first assertion of this flavor is the celebrated theorem by Kesten [24] (see also Goldie [16] and Grincevičius [18]), to be referred to as the Kesten-Grincevičius-Goldie theorem. It states that the distribution of R_∞^* has a heavy right tail under the assumptions $A > 0$ a.s., $\mathbb{E} A^s = 1$ for some $s > 0$ and some additional conditions, see formula (7.39) below for more details in the particular case $(A, B) = (\rho, \xi)$. The tail behavior of

R^*_∞ is also well understood in some other cases, in particular, when $\mathbb{P}\{|B| > x\}$ is regularly varying at ∞ (see, for instance, [18], [20] and [8]).

Now we switch attention from the general random difference equations to a particular one which features in the analysis of BPRE Z. Using the branching property one easily obtains the following recurrence

$$\bar{R}_0 := \mathbb{E}_\omega \mathbb{Z}_0 = 0, \quad \bar{R}_k := \mathbb{E}_\omega \mathbb{Z}_k = \mathbb{E}_\omega Z_{S_k} = \rho_k \xi_k + \rho_k \mathbb{E}_\omega Z_{S_{k-1}} = \rho_k \xi_k + \rho_k \bar{R}_{k-1}, \quad k \in \mathbb{N}.$$

This shows, among others, that the Markov chain $(\bar{R}_k)_{k\in\mathbb{N}_0}$ is an instance of the random difference equation which corresponds to $(A, B) = (\rho, \rho\xi)$. Asymptotic distributional properties of a particular perpetuity which corresponds to $(A, B) = (\rho, \xi)$ are essentially used in the proof of Lemma 7.2.

4 Proof strategy

A weak convergence result for T_n, properly normalized and centered, will be derived from the corresponding result for T_{S_n}, again properly normalized and centered. In view of (3.3), the latter may in principle be affected by the asymptotic behavior of S_n, W_{S_n} or both. Fortunately, the contribution of S_n is degenerate in the limit, for it is only regulated by the law of large numbers, fluctuations of S_n around its mean do not come into play. Summarizing, analysis of the asymptotics of W_{S_n} is our dominating task.

While dealing with W_{S_n} our main arguments follow the strategy invented by Kesten et al. [26]. Namely, for large n we decompose W_{S_n} as a sum of random variables which are iid under the annealed probability $\mathbb{P}$. For this purpose we define extinction times

$$\tau_0 := 0, \quad \tau_k := \min\{j > \tau_{k-1} : \mathbb{Z}_j = 0\}, \quad k \in \mathbb{N}. \tag{4.1}$$

Let us emphasize that the extinctions of Z are ignored in the generations other than S_1, $S_2, \dots$ Set

$$\bar{\mathbb{W}}_{\tau_n} := W_{S_{\tau_n}} - W_{S_{\tau_{n-1}}}, \quad n \in \mathbb{N}$$

and note that $(\bar{\mathbb{W}}_{\tau_n}, \tau_n - \tau_{n-1})_{n\in\mathbb{N}}$ are iid random vectors. We have

$$\sum_{k=1}^{\tau_n^*} \bar{\mathbb{W}}_{\tau_k} \le \sum_{k=1}^{S_n} Z_k \le \sum_{k=1}^{\tau_n^*+1} \bar{\mathbb{W}}_{\tau_k}, \tag{4.2}$$

where τ_n^* is the number of extinctions of Z in the generations $S_0, \dots, S_n$, that is,

$$\tau_n^* := \max\{k \ge 0 : \tau_k \le n\}, \quad n \in \mathbb{N}.$$

It turns out that the extinctions occur relatively often as the following lemma confirms.

Lemma 4.1. *Assume that* $\mathbb{E} \log \rho \in [-\infty, 0)$ *and* $\mathbb{E} \log \xi < \infty$*. Then* $\mathbb{E}\tau_1 < \infty$*. If additionally* $\mathbb{E}\rho^\varepsilon < \infty$ *and* $\mathbb{E}\xi^\varepsilon < \infty$ *for some* $\varepsilon > 0$*, then* $\mathbb{E} \exp(\gamma\tau_1) < \infty$ *for some* $\gamma > 0$*.*

The proof of Lemma 4.1 is given in the Appendix.

Under the assumptions of our main results $\mu := \mathbb{E}\tau_1 < \infty$ by Lemma 4.1. The strong law of large numbers for renewal processes makes it plausible that, for large n,

$$W_{S_n} \approx \sum_{k=1}^{\lfloor \mu^{-1} n \rfloor} \bar{\mathbb{W}}_{\tau_k}. \tag{4.3}$$

The right-hand side, properly centered and normalized, converges in distribution if, and only if, the distribution of $\bar{\mathbb{W}}_{\tau_1}$ belongs to the domain of attraction of a stable law. To check the latter, for $i \in \mathbb{N}$, we divide particles residing in the generations $S_{i-1} + 1, \dots, S_i$ into groups:

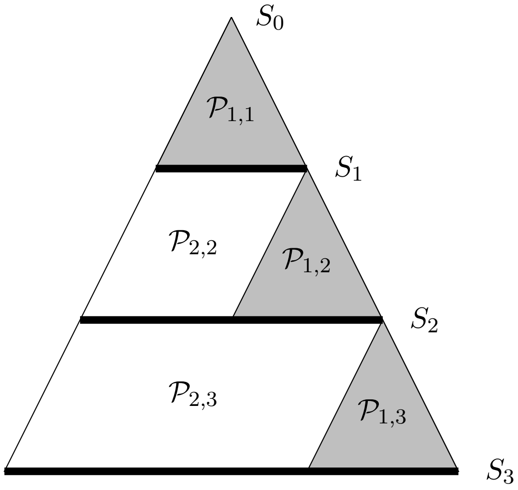

Figure 4.1: The generations 0 through S_3 of the BPRE Z and the partition of the corresponding population into parts $\mathcal{P}_{i,j}$, $i, j = 1, 2, 3$. The bold horizontal lines represent particles in the generations S_1, S_2 and S_3, that is, those comprising the groups $\mathcal{P}_{3,i}$, $i = 1, 2, 3$. By definition, $\mathcal{P}_{2,1} = \oslash$.

- $\mathcal{P}_{1,i}$ – the progeny residing in the generations $S_{i-1}+1, \ldots, S_i - 1$ of the immigrants arriving in the generations $S_{i-1}, \ldots, S_i - 2$, the number of these being

$$\mathbb{W}_i^0 := \sum_{j=S_{i-1}+1}^{S_i-1} \sum_{k=j}^{S_i-1} Z(j,k);$$

- $\mathcal{P}_{2,i}$ – the progeny residing in the generations $S_{i-1}+1, \ldots, S_i - 1$ of the immigrants arriving in the generations $0, 1, \ldots, S_{i-1} - 1$, the number of these being

$$\mathbb{W}_i^{\downarrow} := \sum_{j=1}^{S_{i-1}} \sum_{k=S_{i-1}+1}^{S_i-1} Z(j,k);$$

- $\mathcal{P}_{3,i}$ – particles of the generation S_i, the number of these being $\mathbb{Z}_i$.

The aforementioned partition of the population which is depicted on Figure 4.1 induces the following decompositions

$$\mathbb{W}_i = \mathbb{W}_i^0 + \mathbb{W}_i^{\downarrow} + \mathbb{Z}_i, \quad i \in \mathbb{N} \quad \text{a.s.}$$

and

$$\bar{\mathbb{W}}_{\tau_1} = \sum_{i=1}^{\tau_1} \mathbb{W}_i^0 + \sum_{i=1}^{\tau_1} \mathbb{W}_i^{\downarrow} + \sum_{i=1}^{\tau_1} \mathbb{Z}_i \quad \text{a.s.}$$

which are of primary importance for what follows.

Depending on the assumptions ($\rho 1$), ($\rho 2$), ($\xi 1$) or ($\xi 2$) the random variables $\sum_{i=1}^{\tau_1} \mathbb{W}_i^0$, $\sum_{i=1}^{\tau_1} \mathbb{W}_i^{\downarrow}$ and $\sum_{i=1}^{\tau_1} \mathbb{Z}_i$ may exhibit different tail behaviors. Often, one of the random variables dominates the others thereby determining the tail behavior of the whole sum $\bar{\mathbb{W}}_{\tau_1}$.

5 Tail behavior of $\bar{\mathbb{W}}_{\tau_1}$

In this section we do not assume that $\mathbb{E}\xi < \infty$.

We first analyze the tail behavior of $\sum_{i=1}^{\tau_1} \mathbb{W}_i^0$. Note that by construction $(\mathbb{W}_i^0)_{i\in\mathbb{N}}$ are iid and the random variable τ_1 *does not depend on the future of the sequence* $(\mathbb{W}_i^0)_{i\in\mathbb{N}}$ in the sense of the definition given by Denisov, Foss, Korshunov on p. 987 in [10]. The latter means that, for each $n \in \mathbb{N}$, the collections of random variables $((\mathbb{W}_k^0)_{k\le n}, \mathbf{1}_{\{\tau_1 \le n\}})$ and $(\mathbb{W}_k^0)_{k>n}$ are independent. This observation in combination with Corollary 3 in [10] suggests that it is enough to analyze the tail behavior of just one summand, $\mathbb{W}_1^0$ say, provided that the right tail of $\mathbb{W}_1^0$ is regularly varying and heavier than the right tail of τ_1.

Lemma 5.1. *Assume that* (2.5) *holds with some* $\beta > 0$. *Then*

$$\mathbb{P}\{\mathbb{W}_1^0 > x\} \ \sim \ \mathbb{E}\vartheta^{\beta/2} x^{-\beta/2} \ell(x^{1/2}), \quad x \to \infty,$$

where ϑ *is a random variable with Laplace transform*

$$\mathbb{E}e^{-s\vartheta} = 1/\cosh(s^{1/2}), \quad s \ge 0. \tag{5.1}$$

The proof of Lemma 5.1 is given in Section 6. In the next two lemmas we provide moment estimates for the two other summands $\sum_{i=1}^{\tau_1} \mathbb{W}_i^{\downarrow}$ and $\sum_{i=1}^{\tau_1} \mathbb{Z}_i$.

Lemma 5.2. *Assume that* $\mathbb{E}\log\rho \in [-\infty, 0)$ *and that, for some* $\kappa \le 2$, $\mathbb{E}(\rho\xi)^\kappa$ *and* $\mathbb{E}\xi^\kappa$ *are finite. Then* $\mathbb{E}\mathbb{Z}_1^\kappa < \infty$ *and there exists a positive constant* C *such that, for all* $n \in \mathbb{N}$,

$$\mathbb{E}\mathbb{Z}_n^\kappa \le \begin{cases} C & \text{if } \gamma < 1, \\ Cn & \text{if } \gamma = 1, \\ C\gamma^n & \text{if } \gamma > 1, \end{cases} \tag{5.2}$$

where $\gamma := \mathbb{E}\rho^\kappa$. *If additionally* $\mathbb{E}\xi^{2\kappa} < \infty$, *then*

$$\mathbb{E}\mathbb{W}_1^\kappa < \infty. \tag{5.3}$$

Remark 5.3. Since $\xi \ge 1$ a.s., the assumption $\mathbb{E}(\rho\xi)^\kappa < \infty$ entails $\mathbb{E}\rho^\kappa < \infty$. This explains the absence of the latter condition in Lemma 5.2.

Lemma 5.4. *Assume that, for some* $\kappa \le 2$, $\mathbb{E}\rho^\kappa < 1$, $\mathbb{E}(\rho\xi)^\kappa$ *and* $\mathbb{E}\xi^\kappa$ *are finite. Then, for all* $\kappa_0 \in (0, \kappa)$,

$$\mathbb{E}\left(\sum_{i=1}^{\tau_1} \mathbb{Z}_i\right)^{\kappa_0} < \infty. \tag{5.4}$$

If additionally $\mathbb{E}\xi^{3\kappa/2} < \infty$, *then*

$$\mathbb{E}\left(\sum_{i=1}^{\tau_1} \mathbb{W}_i^{\downarrow}\right)^{\kappa_0} < \infty. \tag{5.5}$$

Lemma 5.5 states that under the assumption (ρ1) the distribution of $\sum_{k=1}^{\tau_1}\left(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\right)$ has a power tail.

Lemma 5.5. *Assume that* (ρ1) *holds for some* $\alpha \in (0, 2]$, $\mathbb{E}\xi^{3\alpha/2} < \infty$ *and* $\mathbb{E}(\rho\xi)^\alpha < \infty$. *Then*

$$\mathbb{P}\left\{\sum_{k=1}^{\tau_1}\left(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\right) > x\right\} \sim C_2(\alpha) x^{-\alpha}, \quad x \to \infty$$

for a positive constant $C_2(\alpha)$.

Lemma 5.6 points out the tail behavior of $\bar{\mathbb{W}}_{\tau_1}$ in the situation where the slowly varying factor in (ξ2) is a constant.

Lemma 5.6. *Assume that* (ρ1) *holds for some* $\alpha \in (0, 2)$, (ξ2) *holds with* $\beta = 2\alpha$ *and* ℓ *such that* $\lim_{t\to\infty} \ell(t) = C_\ell > 0$, $\mathbb{E}\rho^{\alpha+\varepsilon} < \infty$ *and* $\mathbb{E}\rho^\alpha\xi^{\alpha+\varepsilon} < \infty$ *for some* $\varepsilon > 0$. *Then*

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \sim \left((\mathbb{E}\tau_1)(\mathbb{E}\vartheta^\alpha)C_\ell + C_2(\alpha)\right) x^{-\alpha}, \quad x \to \infty,$$

where $C_2(\alpha)$ *is the same constant as in Lemma 5.5.*

The proofs of Lemmas 5.2 through 5.6 are postponed until Section 7.4.

For the ease of reference the tail behavior of $\bar{\mathbb{W}}_{\tau_1}$ is summarized in the following proposition.

Proposition 5.7. *The following asymptotic relations hold.*

(C1) *If* (ρ1) *holds for some* $\alpha \in (0,2]$, *either* $\mathbb{E}\xi^{2\alpha} < \infty$ *or* (ξ2) *holds with* $\beta = 2\alpha$, $\lim_{t\to\infty} \ell(t) = 0$, *and* $\mathbb{E}(\rho\xi)^\alpha < \infty$, *then*

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \sim C_2(\alpha) x^{-\alpha}, \quad x \to \infty,$$

where $C_2(\alpha)$ *is the same constant as in Lemma 5.5.*

(C2) *If* (ρ1) *holds for some* $\alpha \in (0,2)$, (ξ2) *holds with* $\beta = 2\alpha$ *and* $\lim_{t\to\infty} \ell(t) = C_\ell \in (0,\infty)$, $\mathbb{E}\rho^{\alpha+\varepsilon} < \infty$ *and* $\mathbb{E}\rho^\alpha \xi^{\alpha+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *then*

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \sim \left((\mathbb{E}\tau_1)(\mathbb{E}\vartheta^\alpha) C_\ell + C_2(\alpha)\right) x^{-\alpha}, \quad x \to \infty.$$

(C3) *If* (ρ1) *holds for some* $\alpha \in (0,2]$, (ξ2) *holds with* $\beta = 2\alpha$ *and* $\lim_{t\to\infty} \ell(t) = \infty$, *and* $\mathbb{E}(\rho\xi)^\alpha < \infty$, *then*

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \sim (\mathbb{E}\tau_1)(\mathbb{E}\vartheta^\alpha) x^{-\alpha} \ell(x^{1/2}), \quad x \to \infty.$$

(C4) *If* (ρ2) *holds,* (ξ2) *holds for some* $\beta \in (0,4)$ *such that* $\beta/2 \in \mathcal{I}$ *and* $\mathbb{E}(\rho\xi)^{\beta/2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *then*

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \sim (\mathbb{E}\tau_1)(\mathbb{E}\vartheta^{\beta/2}) x^{-\beta/2} \ell(x^{1/2}), \quad x \to \infty.$$

Proof. Under the assumptions (Ci), $i = 1,2,3,4$ τ_1 has some finite exponential moment by Lemma 4.1. This fact will be used when applying Corollary 3 of [10] below.

Proof of (C1). Each of $\mathbb{E}\xi^{2\alpha} < \infty$ and (ξ2) with $\beta = 2\alpha$ implies $\mathbb{E}\xi^{3\alpha/2} < \infty$. Therefore, in view of Lemma 5.5 it is enough to show that

$$\mathbb{P}\Big\{\sum_{i=1}^{\tau_1} \mathbb{W}_i^0 > x\Big\} = o(x^{-\alpha}), \quad x \to \infty. \tag{5.6}$$

Corollary 3 in [10] ensures that

$$\mathbb{P}\Big\{\sum_{i=1}^{\tau_1} \mathbb{W}_i^0 > x\Big\} \sim (\mathbb{E}\tau_1)\mathbb{P}\{\mathbb{W}_1^0 > x\}, \quad x \to \infty \tag{5.7}$$

whenever the right tail of $\mathbb{W}_1^0$ is regularly varying.

If (ξ2) holds with $\beta = 2\alpha$, then according to Lemma 5.1

$$\mathbb{P}\{\mathbb{W}_1^0 > x\} \ \sim\ \mathbb{E}\vartheta^\alpha x^{-\alpha} \ell(x^{1/2}), \quad x \to \infty.$$

This in combination with $\lim_{t\to\infty} \ell(t) = 0$ which holds by assumption and (5.7) proves (5.6).

Assuming that $\mathbb{E}\xi^{2\alpha} < \infty$ we intend to show that

$$\mathbb{E}\Big[\sum_{i=1}^{\tau_1} \mathbb{W}_i^0\Big]^\alpha < \infty \tag{5.8}$$

which, of course, entails (5.6). By Lemma A.1, (5.8) holds provided that $\mathbb{E}[\mathbb{W}_1^0]^\alpha < \infty$. The latter is secured by $\mathbb{E}\xi^{2\alpha} < \infty$ and Lemma 6.3.

Proof of (C2). This is just Lemma 5.6.

Proof of (C3). This follows from Lemma 5.1 in conjunction with Corollary 3 in [10] and Lemma 5.5 because (ξ2) with $\beta = 2\alpha$ entails $\mathbb{E}\xi^{3\alpha/2} < \infty$.

Proof of (C4). Since the interval $\mathcal{I}$ is open, there exists $\varepsilon_1 > 0$ such that $\beta/2 + \varepsilon_1 \in (0,2]$, $\mathbb{E}\rho^{\beta/2+\varepsilon_1} < 1$, $\mathbb{E}\xi^{3\beta/4+3\varepsilon_1/2} < \infty$ and $\mathbb{E}(\rho\xi)^{\beta/2+\varepsilon_1} < \infty$. In view of this Lemma 5.4 applies with $\kappa = \beta/2+\varepsilon_1$ and $\kappa_0 = \beta/2+\varepsilon_1/2$ which gives $\mathbb{E}\big(\sum_{i=1}^{\tau_1} \mathbb{Z}_i\big)^{\beta/2+\varepsilon_1/2} < \infty$ and $\mathbb{E}\big(\sum_{i=1}^{\tau_1} \mathbb{W}_i^{\downarrow}\big)^{\beta/2+\varepsilon_1/2} < \infty$. An appeal to Lemma 5.1 in combination with Corollary 3 in [10] does the rest. $\square$

6 Critical Galton–Watson process with immigration

As has already been mentioned in Section 3, $(Z_n)_{0\leq n\leq \xi_1-1} \overset{\mathrm{d}}{=} (Z_n^{\mathrm{crit}})_{0\leq n\leq \xi_1-1}$, where ξ_1 is assumed independent of $(Z_n^{\mathrm{crit}})_{n\in\mathbb{N}_0}$ a critical Galton–Watson process with unit immigration and Geom(1/2) offspring distribution. In this section we collect some known properties of $(Z_n^{\mathrm{crit}})_{n\in\mathbb{N}_0}$ and prove several auxiliary results which to our knowledge are not available in the literature. The evolution of $(Z_n^{\mathrm{crit}})_{n\in\mathbb{N}_0}$ is the same as that of the BRPE Z with $\omega_n \equiv 1/2$ for all $n \in \mathbb{N}$, see Section 3.1.

For $n \in \mathbb{N}$, let $W_n^{\mathrm{crit}} := \sum_{k=1}^n Z_k^{\mathrm{crit}}$ denote the total progeny in the first n generations. Further, for $n \in \mathbb{N}$ and $1 \leq k \leq n$, write $Z^{\mathrm{crit}}(k, n)$ for the number of the nth generation progeny of the kth immigrant and $Y^{\mathrm{crit}}(k, n)$ for the number of progeny of the kth immigrant which reside in generations k through n, that is,

$$Y^{\mathrm{crit}}(k, n) = \sum_{j=k}^{n} Z^{\mathrm{crit}}(k, j).$$

Here is the main result of this section of which Lemma 5.1 is an immediate consequence because $\mathbb{W}_1^0 \overset{\mathrm{d}}{=} W_{\xi_1-1}^{\mathrm{crit}}$, where ξ_1 is assumed independent of $(W_k^{\mathrm{crit}})_{k\in\mathbb{N}}$.

Proposition 6.1. *Let ς be an integer-valued random variable independent of $(W_n^{\mathrm{crit}})_{n\in\mathbb{N}_0}$ and such that*

$$\mathbb{P}\{\varsigma > x\} \sim x^{-2\alpha}\ell(x), \quad x \to \infty$$

for some $\alpha > 0$ and some ℓ slowly varying at ∞. Then

$$\mathbb{P}\{W_\varsigma^{\mathrm{crit}} > x\} \;\sim\; \mathbb{E}\vartheta^\alpha \mathbb{P}\{\varsigma > x^{1/2}\} \;\sim\; \mathbb{E}\vartheta^\alpha x^{-\alpha}\ell(x^{1/2}), \quad x \to \infty,$$

where ϑ is a random variable with Laplace transform (5.1).

Remark 6.2. For fixed $n \in \mathbb{N}$, $\mathbb{E}W_n^{\mathrm{crit}} = \frac{n(n+1)}{2}$ and the distribution of W_n^{crit} inherits an exponential tail from Geom(1/2) offspring distribution. Thus, for ς which has distribution with a heavy tail and is independent of $(W_n^{\mathrm{crit}})_{n\in\mathbb{N}}$ it is natural to expect that

$$W_\varsigma^{\mathrm{crit}} \approx \mathrm{const}\cdot\varsigma^2.$$

Proposition 6.1 makes this intuition precise.

Lemma 6.3 given next is used in the proof of Proposition 5.7, part (C1).

Lemma 6.3. *Let ς be an integer-valued random variable independent of $(W_n^{\mathrm{crit}})_{n\in\mathbb{N}_0}$ and such that $\mathbb{E}\varsigma^{2\alpha} < \infty$ for some $\alpha > 0$. Then $\mathbb{E}[W_\varsigma^{\mathrm{crit}}]^\alpha < \infty$.*

To prove Proposition 6.1 and Lemma 6.3 we need some auxiliary lemmas. The first one is due to Pakes [30, Theorem 5].

Lemma 6.4. *We have*

$$n^{-2}W_n^{\mathrm{crit}} \overset{\mathrm{d}}{\longrightarrow} \vartheta, \quad n \to \infty, \tag{6.1}$$

where ϑ is a random variable with Laplace transform (5.1).

In the cited article Pakes investigates Galton–Watson processes with general, not necessarily unit, immigration. One of the standing assumptions of that paper is that the probability of having no immigrants is positive. However, a perusal of the proof of Theorem 5 in [30] reveals that the result still holds without this assumption.

With some additional effort one can prove the convergence of all moments in (6.1).

Lemma 6.5. *For each $s > 0$,*

$$\lim_{n\to\infty} \mathbb{E}(n^{-2}W_n^{\mathrm{crit}})^s = \mathbb{E}\vartheta^s. \tag{6.2}$$

Proof. Suppose for the moment that we have verified that

$$\sup_{n\geq n_0} \mathbb{E}\exp(\beta n^{-2}W_n^{\rm crit})<\infty \tag{6.3}$$

for some $\beta>0$ and some $n_0\in\mathbb{N}$. Then in view of

$$\sup_{n\geq n_0} \mathbb{E}(n^{-2}W_n^{\rm crit})^s\leq C(s)\sup_{n\geq n_0} \mathbb{E}\exp(\beta n^{-2}W_n^{\rm crit})<\infty$$

for all $s>0$ and some constant $C(s)$, the Vallée–Poussin criterion for uniform integrability in combination with (6.1) ensures (6.2).

Left with the proof of (6.3) observe that, for fixed $k\in\mathbb{N}$, the process initiated by the kth immigrant $(Z^{\rm crit}(k,n))_{n\geq k}$ is a Galton–Watson process with Geom(1/2) offspring distribution. Moreover, the processes started by different immigrants are iid. Therefore, writing

$$\begin{aligned} W_n^{\rm crit}&=\sum_{k=1}^n Z_k^{\rm crit}=\sum_{k=1}^n\sum_{j=1}^k Z^{\rm crit}(j,k)\\ &=\sum_{j=1}^n\Big(\sum_{k=j}^n Z^{\rm crit}(j,k)\Big)=\sum_{j=1}^n Y^{\rm crit}(j,n)\quad\text{a.s.}\end{aligned}$$

we obtain a representation of $W_n^{\rm crit}$ as the sum of independent random variables. This formula entails

$$\mathbb{E}\exp\big(xW_n^{\rm crit}\big)=\prod_{j=1}^n a_j(x),\qquad x\geq 0 \tag{6.4}$$

(the case that both sides of (6.4) are infinite for some $x>0$ is not excluded), where

$$a_j(x):=\mathbb{E}\exp\big(xY^{\rm crit}(n-j+1,n)\big)=\mathbb{E}\exp\big(xY^{\rm crit}(1,j)\big),\quad 1\leq j\leq n, x\geq 0.$$

We have $a_0(x)=1$ for all $x\geq 0$ and

$$a_1(x)=\mathbb{E}\exp\big(xZ^{\rm crit}(1,1)\big)=\sum_{k\geq 0}e^{kx}2^{-k-1}=(2-e^x)^{-1}$$

for $x\in[0,\log 2)$. Using a decomposition

$$Y^{\rm crit}(1,j)=\sum_{m=1}^{Z^{\rm crit}(1,1)} Y_m^{\rm crit}(1,j-1)+Z^{\rm crit}(1,1),\quad j\geq 2\quad\text{a.s.}, \tag{6.5}$$

where $(Y_m^{\rm crit}(1,j-1))_{m\in\mathbb{N}}$ are independent copies of $Y^{\rm crit}(1,j-1)$ which are also independent of $Z^{\rm crit}(1,1)$ we infer

$$a_j(x)=\frac{1}{2-e^x a_{j-1}(x)},\quad j\in\mathbb{N}.$$

In particular, for every fixed $j\in\mathbb{N}_0$, $a_j(x)<\infty$ for all x from some right vicinity of the origin.

Set $b_j(x)=e^x a_j(x)$ for $j\in\mathbb{N}_0$ and $x\geq 0$, so that

$$b_j(x)=\frac{e^x}{2-b_{j-1}(x)}.$$

By technical reasons, it is more convenient to work with b_j rather than a_j. We intend to show that, for every $\gamma\in(0,1/4)$, there exists $K=K(\gamma)>1$ and $x_0(\gamma)>0$ such that

$$b_j(x)\leq 1+Kx(j+1). \tag{6.6}$$

for $j\in\mathbb{N}_0$ and $x>0$ satisfying $j(1+j)x\leq\gamma$ and $x<x_0(\gamma)$.

Given $\gamma \in (0, 1/4)$ pick $K > 1$ such that $K - K^2\gamma > 1$. This is possible because the largest root of the quadratic equation $\gamma x^2 - x + 1 = 0$ is larger than one. There exists $x_0(\gamma) > 0$ such that

$$e^x \le 1 + (K - K^2\gamma)x, \quad x \in (0, x_0(\gamma)).$$

Moreover, since we assume $j(1+j)x \le \gamma$ we have

$$e^x \le 1 + Kx - K^2x^2 j(j+1) = (1 - Kxj)(1 + Kx(j+1)).$$

Now (6.6) follows by the mathematical induction. While for $j = 0$ we obtain

$$b_0(x) = e^x \le 1 + (K - K^2\gamma)x \le 1 + Kx, \quad x \in (0, x_0(\gamma)),$$

an induction step works as follows

$$b_j(x) = \frac{e^x}{2 - b_{j-1}(x)} \le \frac{e^x}{1 - Kjx} \le 1 + Kx(j+1)$$

for $x \in (0, x_0(\gamma))$ and $j(j+1)x \le \gamma$. The proof of (6.6) is complete.

Armed with (6.6) we can deduce (6.3). Given $\beta \in (0, 1/4)$ take $\gamma \in (\beta, 1/4)$ and pick $n_0 \in \mathbb{N}$ such that $\beta/n^2 < x_0(\gamma)$ and $(n+1)\beta \le n\gamma$ for $n \ge n_0$. Such a choice ensures that $j(j+1)\beta n^{-2} \le \gamma$ for integer $0 \le j \le n$ whenever $n \ge n_0$. Using (6.4) and then (6.6) we arrive at

$$\mathbb{E}\exp(\beta n^{-2} W_n^{\text{crit}}) = \prod_{j=0}^{n} a_j\left(\beta n^{-2}\right) \le \prod_{j=0}^{n} b_j\left(\beta n^{-2}\right) \le \prod_{j=0}^{n}(1 + K\beta n^{-2}(j+1)), \quad n \ge n_0$$

for $\beta \in (0, 1/4)$. It remains to note that

$$\sup_{n \ge n_0} \prod_{j=0}^{n}(1 + K\beta n^{-2}(j+1)) \le \exp(3K\beta) < \infty,$$

thereby finishing the proof of (6.3). □

We are now ready to prove Proposition 6.1 and Lemma 6.3.

Proof of Proposition 6.1. By virtue of (6.1) we infer $W_n^{\text{crit}} \to \infty$ in probability and then $W_n^{\text{crit}} \to \infty$ a.s. by monotonicity. Therefore,

$$\upsilon_x := \inf\{k \in \mathbb{N} : W_k^{\text{crit}} > x\} \in [1, \infty) \quad \text{a.s} \quad \text{for } x > 1.$$

For $x > 1$ we have

$$\mathbb{P}\{W_\varsigma^{\text{crit}} > x\} = \mathbb{P}\{\varsigma \ge \upsilon_x\} = \mathbb{E}h(\upsilon_x),$$

where $h(y) := \mathbb{P}\{\varsigma \ge y\}$. Under the introduced notation, we have to prove that

$$\lim_{x \to \infty} \frac{\mathbb{E}h(\upsilon_x)}{h(x^{1/2})} = \mathbb{E}\vartheta^\alpha. \tag{6.7}$$

By a standard inversion technique à la Feller (see Theorem 7 in [13]) (6.1) entails

$$\frac{\upsilon_x}{x^{1/2}} \xrightarrow{\text{d}} \vartheta^{-1/2}, \quad x \to \infty. \tag{6.8}$$

We claim that the latter implies further that

$$\frac{h(\upsilon_x)}{h(x^{1/2})} \xrightarrow{\text{d}} \vartheta^\alpha, \quad x \to \infty. \tag{6.9}$$

The simplest way to see it is to pass in (6.8) to versions which converge a.s., that is,

$$\lim_{x\to\infty} x^{-1/2}\upsilon_x^* = (\vartheta^*)^{-1/2} \quad \text{a.s.}$$

and then exploit the fact that

$$\lim_{x\to\infty} \frac{h(y(x)x^{1/2})}{h(x^{1/2})} = y^{-2\alpha} \quad \text{whenever} \quad \lim_{x\to\infty} y(x) = y \in (0,\infty)$$

(see Theorem 1.5.2 in [2]). This gives

$$\lim_{x\to\infty} \frac{h((x^{-1/2}\upsilon_x^*)x^{1/2})}{h(x^{1/2})} = (\vartheta^*)^{\alpha} \quad \text{a.s.}$$

because $\vartheta^* > 0$ a.s.

With (6.9) at hand, relation (6.7) follows if we can show that the family $\big(h(\upsilon_x)/h(x^{1/2})\big)_{x\geq x_0}$ is uniformly integrable for some $x_0 > 0$. By Potter's bound for regularly varying functions (Theorem 1.5.6 (iii) in [2]), given $A > 1$ and $\delta > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\frac{h(\upsilon_x)\mathbf{1}_{\{\upsilon_x > n_1\}}}{h(x^{1/2})} \leq A\max((x^{-1/2}\upsilon_x)^{-2\alpha-\delta}, (x^{-1/2}\upsilon_x)^{-2\alpha+\delta}) \quad \text{a.s.}$$

whenever $x \geq n_1^2$. Further, by monotonicity of h,

$$\frac{h(\upsilon_x)\mathbf{1}_{\{\upsilon_x \leq n_1\}}}{h(x^{1/2})} \leq \frac{h(1)}{h(x^{1/2})}\mathbf{1}_{\{\upsilon_x \leq n_1\}} \quad \text{a.s.}$$

Thus, for uniform integrability of $\big(h(\upsilon_x)/h(x^{1/2})\big)_{x\geq x_0}$ it suffices to check two things: first,

$$\sup_{x\geq 4} x^{\beta/2}\mathbb{E}\upsilon_x^{-\beta} < \infty \tag{6.10}$$

for some $\beta > 2\alpha$ and second

$$\sup_{x\geq x_0} \left(\frac{h(1)}{h(x^{1/2})}\right)^{\gamma} \mathbb{P}\{\upsilon_x \leq n_1\} < \infty \tag{6.11}$$

for some $\gamma > 1$.

From the proof of Lemma 6.5 we know that $\mathbb{E}\exp(sW_{n_1}^{\text{crit}}) < \infty$ for some $s > 0$, whence

$$\mathbb{P}\{\upsilon_x \leq n_1\} = \mathbb{P}\{W_{n_1}^{\text{crit}} > x\} = O(e^{-sx}), \quad x \to \infty$$

which proves (6.11).

Now we intend to show that (6.10) holds for all $\beta > 0$. We have for $x \geq 4$

$$\begin{aligned}
\mathbb{E}\upsilon_x^{-\beta} &= \int_0^1 \mathbb{P}\{\upsilon_x^{-\beta} > y\}\mathrm{d}y = \beta\int_1^\infty \mathbb{P}\{\upsilon_x \leq z\}z^{-\beta-1}\mathrm{d}z \leq \beta\sum_{k\geq 2}\mathbb{P}\{\upsilon_x \leq k\}(k-1)^{-\beta-1} \\
&= \beta\sum_{k=2}^{[x^{1/2}]}\mathbb{P}\{W_k^{\text{crit}} > x\}(k-1)^{-\beta-1} + \beta\sum_{k\geq[x^{1/2}]+1}\mathbb{P}\{W_k^{\text{crit}} > x\}(k-1)^{-\beta-1} \\
&\leq \beta\sum_{k=2}^{[x^{1/2}]}\frac{\mathbb{E}(W_k^{\text{crit}})^\beta}{x^\beta(k-1)^{\beta+1}} + \beta\sum_{k\geq[x^{1/2}]+1}\frac{1}{(k-1)^{\beta+1}} \leq \frac{\text{const}}{x^\beta}\sum_{k=1}^{[x^{1/2}]}k^{\beta-1} + O(x^{-\beta/2}) = O(x^{-\beta/2}),
\end{aligned}$$

where the last and penultimate inequalities follow from Lemma 6.5 and Markov's inequality, respectively. The proof of Proposition 6.1 is complete. □

Proof of Lemma 6.3. By Lemma 6.5, $\mathbb{E}[n^{-2}W_n^{\rm crit}]^\alpha \le C$ for all $n \in \mathbb{N}$ and some $C > 0$. This entails

$$\mathbb{E}[W_\varsigma^{\rm crit}]^\alpha = \sum_{n\ge 1} \mathbb{E}[n^{-2}W_n^{\rm crit}]^\alpha n^{2\alpha}\mathbb{P}\{\varsigma = n\} \le C\mathbb{E}\varsigma^{2\alpha} < \infty.$$

The proof of Lemma 6.3 is complete. □

For later use, we note that, for $n \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{E}Z^{\rm crit}(1,n) = 1, \quad {\rm Var}\, Z^{\rm crit}(1,n) = 2n,\\ &\mathbb{E}Y^{\rm crit}(1,n) = n, \quad {\rm Var}\, Y^{\rm crit}(1,n) = \frac{n(n+1)(2n+1)}{3}. \end{aligned} \tag{6.12}$$

The first three of these equalities follow by an elementary calculation. The fourth one can be derived with the help of (6.5) and the mathematical induction.

7 Proofs

7.1 Proof of Proposition 2.1

Recalling that $v = \mathbb{E}\xi/\mathbb{E}T_{S_1}$ it suffices to show that

$$\mathbb{E}T_{S_1} = \begin{cases} \mathbb{E}\xi^2 + \frac{2\mathbb{E}\xi\mathbb{E}\rho\xi}{1-\mathbb{E}\rho}, & \text{if } \mathbb{E}\rho < 1,\ \mathbb{E}\rho\xi < \infty, \mathbb{E}\xi^2 < \infty;\\ \infty, & \text{otherwise.} \end{cases}$$

Using (3.3) yields

$$\frac{T_{S_n}}{n} \overset{\mathbb{P}}{\longrightarrow} \mathbb{E}T_{S_1}, \quad n\to\infty \quad \Longleftrightarrow \quad \frac{\sum_{j=1}^{S_n} Z_j}{n} = \frac{W_{S_n}}{n} \overset{\mathbb{P}}{\longrightarrow} \frac{1}{2}\left(\mathbb{E}T_{S_1} - \mathbb{E}\xi\right), \quad n\to\infty.$$

Let us prove the latter convergence in probability. According to Lemma 4.1, we have $\mathbb{E}\tau_1 < \infty$ whenever $\mathbb{E}\log\rho \in [-\infty, 0)$ and $\mathbb{E}\log^+\xi < \infty$. Recalling from (4.2) that

$$\frac{1}{n}\sum_{k=1}^{\tau_n^*} \bar{\mathbb{W}}_{\tau_k} \le \frac{W_{S_n}}{n} \le \frac{1}{n}\sum_{k=1}^{\tau_n^*+1} \bar{\mathbb{W}}_{\tau_k}$$

we conclude by the strong law of large numbers that

$$\lim_{n\to\infty} \frac{W_{S_n}}{n} = \frac{1}{\mathbb{E}\tau_1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1} \quad \mathbb{P}-\text{a.s.}$$

Hence,

$$\mathbb{E}T_{S_1} = \mathbb{E}\xi + \frac{2}{\mathbb{E}\tau_1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1}.$$

Left with identifying $\mathbb{E}\bar{\mathbb{W}}_{\tau_1}$ we recall that, for $k \in \mathbb{N}$, $\mathbb{Y}_k$ denotes the total progeny of immigrants arriving in the generations $S_{k-1}, \ldots, S_k - 1$, that is,

$$\mathbb{Y}_k = \sum_{j=S_{k-1}+1}^{S_k} Y(j,\infty).$$

Since $\mathbb{Y}_1, \mathbb{Y}_2, \ldots$ are identically distributed and, for $k \in \mathbb{N}$, $\mathbb{Y}_k$ is independent of $\{\tau_1 \ge k\} = \{Z_{S_1} > 0, \ldots, Z_{S_{k-1}} > 0\}$ we infer

$$\mathbb{E}\bar{\mathbb{W}}_{\tau_1} = \mathbb{E}\sum_{k=1}^{\tau_1} \mathbb{Y}_k = \sum_{k\ge 1} \mathbb{E}\mathbb{Y}_k \mathbf{1}_{\{\tau_1\ge k\}} = \sum_{k\ge 1} \mathbb{E}\mathbb{Y}_k\mathbb{P}\{\tau_1 \ge k\} = \mathbb{E}\mathbb{Y}_1\mathbb{E}\tau_1$$

(if $\mathbb{E}\mathbb{Y}_1 = \infty$, the formula just says that $\mathbb{E}\bar{\mathbb{W}}_{\tau_1} = \infty$). To calculate $\mathbb{E}\mathbb{Y}_1$ we note that

$$\mathbb{E}_\omega Y(j,\infty)\mathbf{1}_{\{j\le\xi_1\}} = \Big(\xi_1 - j + \sum_{k\ge 2}\xi_k \prod_{i=1}^{k-1}\rho_i\Big)\mathbf{1}_{\{j\le\xi_1\}} \quad \text{a.s.},$$

whence

$$\mathbb{E}_\omega \mathbb{Y}_1 = \frac{\xi_1(\xi_1-1)}{2} + \xi_1\rho_1\sum_{k\ge 2}\xi_k\prod_{i=2}^{k-1}\rho_i \quad \text{a.s.},$$

where the a.s. convergence of the last series is secured by our assumptions $\mathbb{E}\log\rho \in [-\infty, 0)$ and $\mathbb{E}\xi < \infty$. Taking the expectation with respect to $\mathbb{P}$ yields

$$\mathbb{E}\mathbb{Y}_1 = \begin{cases} \frac{1}{2}\mathbb{E}\xi(\xi-1) + \frac{\mathbb{E}\xi\mathbb{E}\rho\xi}{1-\mathbb{E}\rho}, & \text{if } \mathbb{E}\rho<1, \mathbb{E}\rho\xi<\infty, \mathbb{E}\xi^2<\infty;\\ \infty, & \text{otherwise.}\end{cases}$$

The proof of Proposition 2.1 is complete.

7.2 Proof of Theorem 2.2 and Corollary 2.4

The assumptions of Theorem 2.2 ensure that $\mathbb{E}\xi < \infty$ and that $\mu := \mathbb{E}\tau_1$ and $s^2 := \operatorname{Var}\tau_1$ are finite (for the latter use Lemma 4.1). It is also clear that the distribution of τ_1 is nondegenerate, whence $s^2 > 0$.

From Proposition 5.7 (parts (C1) and (C2)) we know that

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \sim Cx^{-\alpha}, \quad x\to\infty,$$

where $C = C_2(\alpha)$ in the cases (A1) and (A2) and $C = (\mathbb{E}\tau_1)(\mathbb{E}\vartheta^\alpha)C_\ell + C_2(\alpha)$ in the case (A3). Therefore, the distribution of $\bar{\mathbb{W}}_{\tau_1}$ belongs to the domain of attraction of an α-stable distribution. This means that

$$\frac{\sum_{k=1}^n \bar{\mathbb{W}}_{\tau_k} - a(n)}{b(n)} \xrightarrow{\text{d}} \mathcal{S}_\alpha, \quad n\to\infty \tag{7.1}$$

for some $a(t)$ and $b(t)$, where $\mathcal{S}_2 \overset{\text{d}}{=} \mathcal{N}(0,1)$. To find $a(t)$ and $b(t)$ explicitly we use Theorem 3 on p. 580 and formula (8.15) on p. 315 in [14]:
$b(t) = (Ct)^{1/\alpha}$ and $a(t) = 0$ if $\alpha \in (0,1)$;
$b(t) = Ct$ and $a(t) = t\int_0^{Ct}\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x$ if $\alpha = 1$;
$b(t) = (Ct)^{1/\alpha}$ and $a(t) = (\mathbb{E}\bar{\mathbb{W}}_{\tau_1})t$ if $\alpha \in (1,2)$;
$b(t) = (Ct\log t)^{1/2}$ and $a(t) = (\mathbb{E}\bar{\mathbb{W}}_{\tau_1})t$ if $\alpha = 2$.

Our subsequent proof will be based on representation (3.3). In view of this we first analyze the asymptotics of W_{S_n}.

Step 1. Limit theorems for W_{S_n}. We claim that

$$\frac{W_{S_n} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \xrightarrow{\text{d}} \mathcal{S}_\alpha, \quad n\to\infty. \tag{7.2}$$

In view of (4.2) relation (7.2) follows once we have checked that (7.1) entails

$$\frac{\sum_{k=1}^{\tau_n^*}\bar{\mathbb{W}}_{\tau_k} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \xrightarrow{\text{d}} \mathcal{S}_\alpha \quad\text{and}\quad \frac{\sum_{k=1}^{\tau_n^*+1}\bar{\mathbb{W}}_{\tau_k} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \xrightarrow{\text{d}} \mathcal{S}_\alpha, \quad n\to\infty. \tag{7.3}$$

According to the central limit theorem for renewal processes

$$\frac{\tau_n^* - \mu^{-1}n}{s\mu^{-3/2}\sqrt{n}} \xrightarrow{\text{d}} \mathcal{N}(0,1), \quad n\to\infty.$$

This implies that, for $\varepsilon > 0$ small enough, we can pick $z = z(\varepsilon)$ so large that

$$\mathbb{P}\{\tau_n^* \geq t_n\} \geq 1 - \varepsilon,$$

where $t_n := [\mu^{-1}n - s\mu^{-3/2}z\sqrt{n}]$. Note that $n = \mu t_n + O\big(t_n^{1/2}\big)$ and that

$$\lim_{n\to\infty} \frac{a(t_n) - a\big(t_n + O\big(t_n^{1/2}\big)\big)}{b\big(t_n + O\big(t_n^{1/2}\big)\big)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{b\big(t_n + O\big(t_n^{1/2}\big)\big)}{b(t_n)} = 1. \tag{7.4}$$

These can be easily checked with the exception of the case $\alpha = 1$ in which a proof of the first relation is needed: for any $r \in (1, 2]$,

$$\begin{aligned}
&\frac{a\big(t_n + O\big(t_n^{1/r}\big)\big) - a(t_n)}{b(t_n)} \\
= \;&\frac{t_n \int_{Ct_n}^{Ct_n + O(t_n^{1/r})} \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x + O\big(t_n^{1/r}\big)\int_0^{Ct_n + O(t_n^{1/r})} \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x}{Ct_n} \\
\leq \;&\frac{O\big(t_n^{1/r}\big)\log t_n}{t_n} = o(1), \quad n \to \infty.
\end{aligned} \tag{7.5}$$

Motivated by our later needs we have proved this in a slightly extended form with r instead of 2.

To prove the first relation in (7.3) we write, for $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}\left\{\frac{\sum_{k=1}^{\tau_n^*} \bar{\mathbb{W}}_{\tau_k} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \leq x\right\} &\leq \varepsilon + \mathbb{P}\left\{\frac{\sum_{k=1}^{t_n} \bar{\mathbb{W}}_{\tau_k} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \leq x\right\} \\
&= \varepsilon + \mathbb{P}\left\{\frac{\sum_{k=1}^{t_n} \bar{\mathbb{W}}_{\tau_k} - a\big(t_n + O\big(t_n^{1/2}\big)\big)}{b\big(t_n + O\big(t_n^{1/2}\big)\big)} \leq x\right\}.
\end{aligned}$$

Sending $n \to \infty$ in the last inequality and using (7.1) and (7.4) we obtain

$$\limsup_{n\to\infty} \mathbb{P}\left\{\frac{\sum_{k=1}^{\tau_n^*} \bar{\mathbb{W}}_{\tau_k} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \leq x\right\} \leq \varepsilon + \mathbb{P}\{\mathcal{S}_\alpha \leq x\}.$$

Letting now $\varepsilon \to 0+$ yields

$$\limsup_{n\to\infty} \mathbb{P}\left\{\frac{\sum_{k=1}^{\tau_n^*} \bar{\mathbb{W}}_{\tau_k} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \leq x\right\} \leq \mathbb{P}\{\mathcal{S}_\alpha \leq x\}.$$

A symmetric argument leads to

$$\liminf_{n\to\infty} \mathbb{P}\left\{\frac{\sum_{k=1}^{\tau_n^*} \bar{\mathbb{W}}_{\tau_k} - a(\mu^{-1}n)}{b(\mu^{-1}n)} \leq x\right\} \geq \mathbb{P}\{\mathcal{S}_\alpha \leq x\}.$$

The second relation in (7.3) follows in a similar manner.

STEP 2. LIMIT THEOREMS FOR T_{S_n}.

CASE $\alpha > 1$. Since $\mathbb{E}\xi^2 < \infty$ and $\sqrt{n} = o(b(\mu^{-1}n))$ we infer

$$\frac{S_n - (\mathbb{E}\xi)n}{b(\mu^{-1}n)} \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty$$

by the central limit theorem. Now

$$\frac{T_{S_n} - (\mathbb{E}\xi + 2\mu^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})n}{b(\mu^{-1}n)} \overset{\mathrm{d}}{\longrightarrow} 2\mathcal{S}_\alpha, \quad n \to \infty \tag{7.6}$$

follows from (7.2) and (3.3) written in an equivalent form

$$T_{S_n} \overset{\mathrm{d}}{=} (S_n - (\mathbb{E}\xi)n) + (\mathbb{E}\xi)n + 2W_{S_n} + O_{\mathbb{P}}(1), \quad n \to \infty.$$

CASE $\alpha = 1$. Using the weak law of large numbers and (7.2) we arrive at

$$\frac{T_{S_n} - 2a(\mu^{-1}n)}{C\mu^{-1}n} \overset{\mathrm{d}}{\longrightarrow} \frac{\mu\mathbb{E}\xi}{C} + 2\mathcal{S}_1, \quad n \to \infty. \tag{7.7}$$

CASE $\alpha < 1$. Since $n = o(b(\mu^{-1}n))$ we conclude that $\frac{S_n}{b(\mu^{-1}n)} \overset{\mathbb{P}}{\longrightarrow} 0$ as $n \to \infty$ by the weak law of large numbers. This in combination with (7.2) and (3.3) proves

$$\frac{T_{S_n}}{(C\mu^{-1}n)^{1/\alpha}} \overset{\mathrm{d}}{\longrightarrow} 2\mathcal{S}_\alpha, \quad n \to \infty. \tag{7.8}$$

STEP 3. LIMIT THEOREM FOR T_n. Set

$$\nu(t) = \inf\{k \in \mathbb{N} : S_k > t\}, \quad t \geq 0,$$

so that $(\nu(t))_{t\geq 0}$ is the first passage time process associated with the random walk $(S_k)_{k\in\mathbb{N}_0}$. The reason for introducing $\nu(t)$ is justified by

$$T_{S_{\nu(n)-1}} \leq T_n \leq T_{S_{\nu(n)}}, \quad n \in \mathbb{N}. \tag{7.9}$$

CASE $\alpha \geq 1$. Fix any $r \in (1,2)$. Then $\mathbb{E}\xi^r < \infty$ and thereupon

$$\nu(t) - (\mathbb{E}\xi)^{-1}t = o(t^{1/r}), \quad t \to \infty \quad \text{a.s.} \tag{7.10}$$

by Theorem 4.4 on p. 89 in [21].
SUBCASE $\alpha = 1$. Using (7.9) we obtain, for any $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}\Big\{\frac{T_n - 2a((\mu\mathbb{E}\xi)^{-1}n)}{C(\mu\mathbb{E}\xi)^{-1}n} > x\Big\} \leq \mathbb{P}\Big\{\frac{T_{S_{\nu(n)}} - 2a((\mu\mathbb{E}\xi)^{-1}n)}{C(\mu\mathbb{E}\xi)^{-1}n} > x\Big\} \\ \leq\ & \mathbb{P}\big\{\nu(n) > (\mathbb{E}\xi)^{-1}n + \varepsilon n^{1/r}\big\} + \mathbb{P}\Big\{\frac{T_{S_{[(\mathbb{E}\xi)^{-1}n+\varepsilon n^{1/r}]}} - 2a([(\mathbb{E}\xi)^{-1}n + \varepsilon n^{1/r}])}{C(\mu\mathbb{E}\xi)^{-1}n} \\ +\ & \frac{2a([(\mathbb{E}\xi)^{-1}n + \varepsilon n^{1/r}]) - 2a((\mu\mathbb{E}\xi)^{-1}n)}{C(\mu\mathbb{E}\xi)^{-1}n} > x\Big\}. \end{aligned}$$

Letting $n \to \infty$ yields, for $x \in \mathbb{R}$,

$$\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{T_n - 2a((\mu\mathbb{E}\xi)^{-1}n)}{C(\mu\mathbb{E}\xi)^{-1}n} > x\Big\} \leq \mathbb{P}\Big\{\frac{\mu\mathbb{E}\xi}{C} + 2\mathcal{S}_1 > x\Big\}$$

having utilized (7.5), (7.7) and (7.10). Arguing similarly we get the converse inequality for the lower limit, thereby proving that

$$\frac{T_n - 2a((\mu\mathbb{E}\xi)^{-1}n)}{C(\mu\mathbb{E}\xi)^{-1}n} \overset{\mathrm{d}}{\longrightarrow} \frac{\mu\mathbb{E}\xi}{C} + 2\mathcal{S}_1, \quad n \to \infty. \tag{7.11}$$

SUBCASE $\alpha > 1$. An analogous but simpler argument enables us to show that (7.6) entails

$$\frac{T_n - (1 + 2(\mu\mathbb{E}\xi)^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})n}{b((\mu\mathbb{E}\xi)^{-1}n)} \overset{\mathrm{d}}{\longrightarrow} 2\mathcal{S}_\alpha, \quad n \to \infty. \tag{7.12}$$

CASE $\alpha < 1$. The proof given for the case $\alpha \geq 1$ does not work in the case (A1) when $\alpha \leq 1/2$ because it is then not necessarily true that $\mathbb{E}\xi^r < \infty$ for some $r > 1$. In view of this we use the weak law of large numbers

$$\frac{\nu(t)}{t} \overset{\mathbb{P}}{\longrightarrow} \frac{1}{\mu}, \quad t \to \infty \tag{7.13}$$

rather than the Marcinkiewicz-Zygmund strong law (7.10).

Another appeal to (7.9) gives, for any $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$\mathbb{P}\left\{\frac{T_n}{(C(\mu\mathbb{E}\xi)^{-1}n)^{1/\alpha}} > x\right\} \leq \mathbb{P}\left\{\frac{T_{S_{\nu(n)}}}{(C(\mu\mathbb{E}\xi)^{-1}n)^{1/\alpha}} > x\right\}$$
$$\leq \mathbb{P}\{\nu(n) > ((\mathbb{E}\xi)^{-1}+\varepsilon)n\} + \mathbb{P}\left\{\frac{T_{S_{[((\mathbb{E}\xi)^{-1}+\varepsilon)n]}}}{(C(\mu\mathbb{E}\xi)^{-1}n)^{1/\alpha}} > x\right\}.$$

Sending $n \to \infty$ we obtain with the help of (7.8) and (7.13)

$$\limsup_{n\to\infty} \mathbb{P}\left\{\frac{T_n}{(C(\mu\mathbb{E}\xi)^{-1}n)^{1/\alpha}} > x\right\} \leq \mathbb{P}\{2\mathcal{S}_\alpha > x(1+\varepsilon\mathbb{E}\xi)^{-1/\alpha}\}.$$

Letting $\varepsilon \to 0+$ and using continuity of the distribution of $\mathcal{S}_\alpha$ yields

$$\limsup_{n\to\infty} \mathbb{P}\left\{\frac{T_n}{(C(\mu\mathbb{E}\xi)^{-1}n)^{1/\alpha}} > x\right\} \leq \mathbb{P}\{2\mathcal{S}_\alpha > x\}.$$

The converse inequality for the lower limit can be derived analogously. Thus,

$$\frac{T_n}{(C(\mu\mathbb{E}\xi)^{-1}n)^{1/\alpha}} \xrightarrow{\text{d}} 2\mathcal{S}_\alpha, \quad n \to \infty. \tag{7.14}$$

The proof of Theorem 2.2 is complete.

Proof of Corollary 2.4. The forms of limit relations for T_n in our Theorem 2.2 and Theorem on pp. 146–148 in [26] are the same, only the values of constants differ. In view of this the limit relations for X_k in our setting are obtained by copying the corresponding limit relations from the aforementioned theorem in [26]. □

7.3 Proof of Theorem 2.6 and Corollary 2.8

The proof goes the same path as that of Theorem 2.2. However, appearance of nontrivial slowly varying factors leads to minor technical complications. We shall only give the weak convergence results explicitly (recall that in the formulation of Theorem 2.6 normalizing and centering functions were not specified). Also, we shall check several claims wherever we feel it is necessary.

According to Proposition 5.7 (parts (C3) and (C4)),

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \;\sim\; \mathbb{E}\tau_1\mathbb{E}\vartheta^\alpha x^{-\alpha}\ell(x^{1/2}), \quad x \to \infty,$$

where $\alpha = \beta/2$ in case $(B2)$. Therefore, limit relation (7.1) holds with some $a(t)$ and $b(t)$. To identify them we need more notation. For $\alpha \in (1/2, 2)$, let $c_\alpha(t)$ be any positive function satisfying $\lim_{t\to\infty} t\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > c_\alpha(t)\} = 1$. Further, assuming that $\alpha = 2$ let $r_2(t)$ be any positive function satisfying $\lim_{t\to\infty} \int_{[0,\,r_2(t)]} x^2 \mathrm{d}\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} \leq x\}/(r_2(t))^2 = 1$. By Lemma 6.1.3 in [23], $c_\alpha(t)$ and $r_2(t)$ are regularly varying at ∞ of indices $1/\alpha$ and $1/2$, respectively. For the latter, the fact is also needed that the function $t \mapsto \int_{[0,\,r_2(t)]} x^2 \mathrm{d}\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} \leq x\}$ is slowly varying at ∞. Observe that the case $\alpha = 2$ only arises under the assumptions (B1) which then ensure that $\mathbb{E}\xi^2 = \infty$. This in combination with the aforementioned lemma yields

$$\lim_{t\to\infty} t^{-1/2} r_2(t) = \infty. \tag{7.15}$$

Using again Theorem 3 on p. 580 and formula (8.15) on p. 315 in [14] we obtain
$b(t) = c_\alpha(t)$ and $a(t) = 0$ if $\alpha \in (1/2, 1)$;
$b(t) = c_1(t)$ and $a(t) = t\int_0^{c_1(t)} \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x$ if $\alpha = 1$;
$b(t) = c_\alpha(t)$ and $a(t) = (\mathbb{E}\bar{\mathbb{W}}_{\tau_1})t$ if $\alpha \in (1, 2)$;

$b(t) = r_2(t)$ and $a(t) = (\mathbb{E}\bar{\mathbb{W}}_{\tau_1})t$ if $\alpha = 2$.

Case $\alpha \in (1/2, 1)$. Repeating verbatim the proof of Theorem 2.2 we obtain

$$\frac{T_n}{(\mu\mathbb{E}\xi)^{-1/\alpha}c_\alpha(n)} \stackrel{\mathrm{d}}{\longrightarrow} 2\mathcal{S}_\alpha, \quad n \to \infty. \tag{7.16}$$

Case $\alpha = 1$. We need an analogue of relation (7.5): for $r \in (1, 2]$, as $n \to \infty$,

$$\begin{aligned}
&\frac{a\big(t_n + O\big(t_n^{1/r}\big)\big) - a(t_n)}{b(t_n)} \\
&= \frac{t_n \int_{c_1(t_n)}^{c_1(t_n+O(t_n^{1/r}))} \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x + O(t_n^{1/r}) \int_0^{c_1(t_n+O(t_n^{1/r}))} \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x}{c_1(t_n)} \\
&\leq \frac{t_n\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > c_1(t_n)\}(c_1(t_n + O(t_n^{1/r})) - c_1(t_n))}{c_1(t_n)} + \frac{O(t_n^{1/r}) \int_0^{c_1(t_n+O(t_n^{1/r}))} \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x}{c_1(t_n)} = o(1).
\end{aligned}$$

The first summand tends to zero in view of two facts: $\lim_{n\to\infty} t_n\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > c_1(t_n)\} = 1$ by the definition of $c_1(t)$ and $\lim_{n\to\infty} \big(c_1\big(t_n + O\big(t_n^{1/r}\big)\big) - c_1(t_n)\big)/c_1(t_n) = 0$ which is a consequence of regular variation of $c_1(t)$. The second summand tends to zero because $\int_0^{c_1(t)} \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\}\mathrm{d}x$ is slowly varying at ∞ as a superposition of the slowly varying and regularly varying functions.

For Step 2 we need the following modified argument. In view of (ξ2) the function $\mathbb{P}\{\xi > t\}$ is regularly varying at ∞ of index -2 and $\mathbb{E}\xi^2$ can be finite or infinite. Therefore, S_n satisfies the central limit theorem with normalization sequence which is regularly varying at ∞ of index $1/2$. Since $c_1(t)$ is regularly varying at ∞ of order 1 we infer

$$\frac{S_n - (\mathbb{E}\xi)n}{c_1(n)} \stackrel{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty$$

and thereupon

$$\frac{T_{S_n} - (\mathbb{E}\xi)n - 2a(\mu^{-1}n)}{\mu^{-1}c_1(n)} \stackrel{\mathrm{d}}{\longrightarrow} 2\mathcal{S}_1, \quad n \to \infty.$$

To pass from this limit relation to the final result

$$\frac{T_n - n - 2a((\mu\mathbb{E}\xi)^{-1}n)}{(\mu\mathbb{E}\xi)^{-1}c_1(n)} \stackrel{\mathrm{d}}{\longrightarrow} 2\mathcal{S}_1, \quad n \to \infty, \tag{7.17}$$

that is, to realize Step 3, one can mimic the proof of Theorem 2.2.

Case $\alpha \in (1, 2]$. While implementing Step 2 in the case $\alpha = 2$ one uses the fact that according to (7.15) $b(t) = r_2(t)$ satisfies $\sqrt{n} = o(r_2(\mu^{-1}n))$ as $n \to \infty$. Since the other parts of the proof of Theorem 2.2 do not require essential changes we arrive at

$$\frac{T_n - (1 + 2(\mu\mathbb{E}\xi)^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})n}{(\mu\mathbb{E}\xi)^{-1/\alpha}c_\alpha(n)} \stackrel{\mathrm{d}}{\longrightarrow} 2\mathcal{S}_\alpha, \quad n \to \infty, \tag{7.18}$$

when $\alpha \in (1, 2)$, and

$$\frac{T_n - (1 + 2(\mu\mathbb{E}\xi)^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})n}{(\mu\mathbb{E}\xi)^{-1/2}r_2(n)} \stackrel{\mathrm{d}}{\longrightarrow} 2\mathcal{N}(0, 1), \quad n \to \infty, \tag{7.19}$$

when $\alpha = 2$. The proof of Theorem 2.6 is complete.

Proof of Corollary 2.8. Since $(T_n)_{n\in\mathbb{N}_0}$ is an 'inverse' sequence for $(X_k)_{k\in\mathbb{N}_0}$ we can use a standard inversion technique (see, for instance, the proof of Theorem 7 in [13]) to pass from the distributional convergence of T_n, properly centered and normalized, as $n \to \infty$ to that of X_k, again properly centered and normalized, as $k \to \infty$. Additional complications arising in the case $\alpha = 1$ can be handled with the help of arguments given in Section 3 of [1].

Here are the limit relations for X_k, properly normalized and centered, as $k \to \infty$ which correspond to (7.16), (7.17), (7.18) and (7.19):
if $\alpha \in (1/2, 1)$, then

$$\mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > k\} X_k \overset{\mathrm{d}}{\longrightarrow} \mu \mathbb{E}\xi (2S_\alpha)^{-\alpha}; \tag{7.20}$$

if $\alpha = 1$, then

$$\frac{X_k - s(k)}{t(k)} \overset{\mathrm{d}}{\longrightarrow} -\mathcal{S}_1, \tag{7.21}$$

where, with $m(t) := \int_0^t \mathbb{P}\{\bar{\mathbb{W}}_{\tau_1} > x\} \mathrm{d}x$ for $t > 0$ and $b := (\mu \mathbb{E}\xi)^{-1}$,

$$s(k) := \frac{k}{1 + 2bm(c_1(bk/(1 + 2bm(bk))))}, \quad k \in \mathbb{N}$$

and

$$t(k) := \frac{c_1(k/m(k))}{1 + 2bm(k)}, \quad k \in \mathbb{N}$$

(we do not write $2bm(k)$ instead of $1 + 2bm(k)$ because the case $\lim_{t\to\infty} m(t) = \mathbb{E}\bar{\mathbb{W}}_{\tau_1} < \infty$ is not excluded);
if $\alpha \in (1, 2)$, then

$$\frac{X_k - (1 + 2(\mu\mathbb{E}\xi)^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})^{-1} k}{c_\alpha(k)} \overset{\mathrm{d}}{\longrightarrow} -2(\mu\mathbb{E}\xi)^{-1/\alpha}(1 + 2(\mu\mathbb{E}\xi)^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})^{-(1+1/\alpha)}\mathcal{S}_\alpha; \tag{7.22}$$

if $\alpha = 2$, then

$$\frac{X_k - (1 + 2(\mu\mathbb{E}\xi)^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})^{-1} k}{r_2(k)} \overset{\mathrm{d}}{\longrightarrow} 2(\mu\mathbb{E}\xi)^{-1/2}(1 + 2(\mu\mathbb{E}\xi)^{-1}\mathbb{E}\bar{\mathbb{W}}_{\tau_1})^{-3/2}\mathcal{N}(0, 1). \tag{7.23}$$

The proof of Corollary 2.8 is complete. □

7.4 Proof of auxiliary Lemmas 5.2, 5.4, 5.5 and 5.6

7.4.1 Proof of Lemma 5.2

Proof of Lemma 5.2. To prove (5.2) we first represent $Z_{S_n - 1}$ as a sum of independent random variables

$$Z_{S_n - 1} = \sum_{j=1}^{\mathbb{Z}_{n-1}} V_j^{(n)} + \widetilde{V}^{(n)}, \quad n \in \mathbb{N} \quad \text{a.s.}, \tag{7.24}$$

where $V_j^{(n)}$ is the number of progeny residing in the generation $S_n - 1$ of the jth particle in the generation S_{n-1} and $\widetilde{V}^{(n)}$ is the number of progeny residing in the generation $S_n - 1$ of the immigrants arriving in the generations $S_{n-1}, \ldots, S_n - 2$. For later use, we note that, under $\mathbb{P}_\omega$,

$$V_j^{(n)} \overset{\mathrm{d}}{=} Z^{\mathrm{crit}}(1, \xi_n - 1) \quad \text{and} \quad \widetilde{V}^{(n)} \overset{\mathrm{d}}{=} Z^{\mathrm{crit}}_{\xi_n - 1}, \quad n \in \mathbb{N}, \tag{7.25}$$

where ω is assumed independent of $(Z_k^{\mathrm{crit}})_{k \in \mathbb{N}_0}$ a Galton–Watson process with unit immigration and Geom(1/2) offspring distribution.

With the help of (7.24) we now write a standard decomposition for the number of particles in the generation S_n over the particles comprising the generation S_{n-1} and their offspring

$$\mathbb{Z}_n = \sum_{j=1}^{\mathbb{Z}_{n-1}} \sum_{i=1}^{V_j^{(n)}} U_{i,j}^{(n)} + \sum_{i=1}^{\widetilde{V}^{(n)}} \widetilde{U}_i^{(n)} + U_0^{(n)} =: \sum_{j=1}^{\mathbb{Z}_{n-1}} \mathbb{V}_j^{(n)} + \widetilde{\mathbb{V}}^{(n)} + U_0^{(n)}, \quad n \in \mathbb{N} \quad \text{a.s.} \tag{7.26}$$

Here, the notation $U_{i,j}^{(n)}$, $\widetilde{U}_i^{(n)}$, $U_0^{(n)}$ is self-explained. For instance, $U_0^{(n)}$ is the number of offspring of the immigrant arriving in the generation $S_n - 1$. Observe that, under $\mathbb{P}_\omega$, $(U_{i,j}^{(n)})_{i,j\in\mathbb{N}}$, $(\widetilde{U}_i^{(n)})_{i\in\mathbb{N}}$ and $U_0^{(n)}$ are independent with distribution $\mathrm{Geom}(\lambda_n)$. In what follows, for simplicity we omit the superscripts (n): for instance, we write $\mathbb{V}_j$ for $\mathbb{V}_j^{(n)}$ and similarly for the other variables. The following formulas play an important role in the subsequent proof:

$$\begin{aligned} \mathbb{E}_\omega[U_0|\mathbb{Z}_{n-1}] = \mathbb{E}_\omega U_0 = \rho_n, \quad \mathbb{E}_\omega[U_0^2|\mathbb{Z}_{n-1}| = \mathbb{E}_\omega U_0^2 = 2\rho_n^2 + \rho_n \\ \mathbb{E}_\omega\big[\mathbb{V}_i\big|\mathbb{Z}_{n-1}\big] = \mathbb{E}V_i \cdot \rho_n = \rho_n, \quad \mathbb{E}_\omega\big[\widetilde{\mathbb{V}}\big|\mathbb{Z}_{n-1}\big] = (\xi_n - 1)\rho_n. \end{aligned} \tag{7.27}$$

The two cases $\kappa \in (0,1]$ and $\kappa \in (1,2]$ should be treated separately.

Case $\kappa \le 1$. By Jensen's inequality and subadditivity of the function $s \mapsto s^\kappa$ on $[0,\infty)$

$$\begin{aligned} \mathbb{E}_\omega[\mathbb{Z}_n^\kappa|\mathbb{Z}_{n-1}] \le (\mathbb{E}_\omega[\mathbb{Z}_n|\mathbb{Z}_{n-1}])^\kappa = \left[\mathbb{E}_\omega\Big[\sum_{j=1}^{\mathbb{Z}_{n-1}} \mathbb{V}_j + \widetilde{\mathbb{V}} + U_0\Big|\mathbb{Z}_{n-1}\Big]\right]^\kappa \\ \le \Big(\mathbb{Z}_{n-1}\rho_n + (\xi_n - 1)\rho_n + \rho_n\Big)^\kappa \le \mathbb{Z}_{n-1}^\kappa\rho_n^\kappa + \xi_n^\kappa\rho_n^\kappa. \end{aligned}$$

Taking the expectations we obtain

$$\mathbb{E}\mathbb{Z}_n^\kappa \le \gamma\mathbb{E}\mathbb{Z}_{n-1}^\kappa + \mathbb{E}(\rho\xi)^\kappa$$

which entails (5.2).

Case $\kappa \in (1,2]$. An application of conditional Jensen's inequality yields

$$\mathbb{E}_\omega\mathbb{Z}_n^\kappa = \mathbb{E}_\omega\Big[\mathbb{E}_\omega\big[\mathbb{Z}_n^\kappa\big|\mathbb{Z}_{n-1}\big]\Big] \le \mathbb{E}_\omega\Big[\Big(\mathbb{E}_\omega\big[\mathbb{Z}_n^2\big|\mathbb{Z}_{n-1}\big]\Big)^{\kappa/2}\Big]. \tag{7.28}$$

To estimate the conditional second moment we represent it as follows

$$\begin{aligned} \mathbb{E}_\omega\big[\mathbb{Z}_n^2\big|\mathbb{Z}_{n-1}\big] = \mathbb{E}_\omega\Big[\Big(\sum_{j=1}^{\mathbb{Z}_{n-1}} \mathbb{V}_j + \widetilde{\mathbb{V}} + U_0\Big)^2\Big|\mathbb{Z}_{n-1}\Big] = \sum_{1\le i\ne j\le \mathbb{Z}_{n-1}} \mathbb{E}_\omega\big[\mathbb{V}_i\big|\mathbb{Z}_{n-1}\big]\mathbb{E}_\omega\big[\mathbb{V}_j\big|\mathbb{Z}_{n-1}\big] \\ + \sum_{j=1}^{\mathbb{Z}_{n-1}} \mathbb{E}_\omega\big[\mathbb{V}_j^2\big|\mathbb{Z}_{n-1}\big] + 2\mathbb{E}_\omega[\widetilde{\mathbb{V}} + U_0|\mathbb{Z}_{n-1}]\mathbb{E}_\omega\Big[\sum_{j=1}^{\mathbb{Z}_{n-1}} \mathbb{V}_j\Big|\mathbb{Z}_{n-1}\Big] + \mathbb{E}_\omega\big[\widetilde{\mathbb{V}}^2\big|\mathbb{Z}_{n-1}\big] \\ + \mathbb{E}_\omega\big[U_0^2\big|\mathbb{Z}_{n-1}\big] + 2\mathbb{E}_\omega\big[\widetilde{\mathbb{V}}\big|\mathbb{Z}_{n-1}\big]\mathbb{E}_\omega\big[U_0\big|\mathbb{Z}_{n-1}\big]. \end{aligned}$$

Appealing now to (7.27) we conclude that

$$\mathbb{E}_\omega\big[\mathbb{Z}_n^2\big|\mathbb{Z}_{n-1}\big] \le \mathbb{Z}_{n-1}^2\rho_n^2 + \mathbb{Z}_{n-1}\mathbb{E}_\omega\mathbb{V}_1^2 + 2\mathbb{Z}_{n-1}\xi_n\rho_n^2 + \mathbb{E}_\omega\widetilde{\mathbb{V}}^2 + 2\rho_n^2 + \rho_n + 2\xi_n\rho_n^2. \tag{7.29}$$

Plugging the last inequality into (7.28) and using subadditivity once again we obtain

$$\begin{aligned} \mathbb{E}\mathbb{Z}_n^\kappa \le \gamma\mathbb{E}\mathbb{Z}_{n-1}^\kappa + \bigg(\mathbb{E}\mathbb{Z}_{n-1}^{\kappa/2} \cdot \mathbb{E}\Big[\big(\mathbb{E}_\omega\mathbb{V}_1^2\big)^{\kappa/2}\Big] + 2\mathbb{E}\mathbb{Z}_{n-1}^{\kappa/2} \cdot \mathbb{E}\xi^{\kappa/2}\rho^\kappa \\ + \mathbb{E}\Big[\big(\mathbb{E}_\omega\widetilde{\mathbb{V}}^2\big)^{\kappa/2}\Big] + 2\gamma + \mathbb{E}\rho^{\kappa/2} + 2\mathbb{E}\xi^{\kappa/2}\rho^\kappa\bigg). \end{aligned} \tag{7.30}$$

Next, we check that

$$\mathbb{E}\Big[\big(\mathbb{E}_\omega\mathbb{V}_1^2\big)^{\kappa/2}\Big] < \infty \quad \text{and} \quad \mathbb{E}\Big[\big(\mathbb{E}_\omega\widetilde{\mathbb{V}}^2\big)^{\kappa/2}\Big] < \infty. \tag{7.31}$$

With the help of

$$\mathbb{E}_\omega V_i = 1 \quad \text{and} \quad \mathrm{Var}_\omega V_i = 2(\xi_n - 1)$$

which is a consequence of (7.25) and (6.12) we infer

$$\mathbb{E}\Big[\big(\mathbb{E}_\omega \mathbb{V}_1^2\big)^{\kappa/2}\Big] = \mathbb{E}\Bigg[\bigg(\mathbb{E}_\omega\Big(\sum_{j=1}^{V_1} U_{1,j}\Big)^2\bigg)^{\kappa/2}\Bigg] = \mathbb{E}\Bigg[\bigg(\mathbb{E}_\omega\Big[\sum_{1\le j\neq l\le V_1} U_{1,j}U_{1,l} + \sum_{j=1}^{V_1} U_{1,j}^2\Big]\bigg)^{\kappa/2}\Bigg]$$

$$\le \mathbb{E}\Big(\rho_n^2\mathbb{E}_\omega V_1^2 + (2\rho_n^2+\rho_n)\mathbb{E}_\omega V_1\Big)^{\kappa/2} \le 2^{\kappa/2}\mathbb{E}\xi^{\kappa/2}\rho^\kappa + \gamma + \mathbb{E}\rho^{\kappa/2} < \infty.$$

A similar argument in combination with $\mathbb{E}_\omega \widetilde{V} = \xi_n - 1$ leads to the conclusion

$$\mathbb{E}\Big[\big(\mathbb{E}_\omega \widetilde{\mathbb{V}}^2\big)^{\kappa/2}\Big] = \mathbb{E}\Big(\rho_n^2\mathbb{E}_\omega \widetilde{V}^2 + (\rho_n^2+\rho_n)\mathbb{E}_\omega \widetilde{V}\Big)^{\kappa/2} \le \mathbb{E}\Big[\big(\rho_n^2\mathbb{E}_\omega \widetilde{V}^2\big)^{\kappa/2}\Big] + \mathbb{E}\xi^{\kappa/2}\rho^k + \mathbb{E}(\rho\xi)^{\kappa/2}.$$

Left with the proof of finiteness of the first term on the right-hand side we represent $\widetilde{V}$ as a sum of independent random variables

$$\widetilde{V} = \widetilde{V}^{(n)} = \sum_{i=1}^{\xi_n-1} \widetilde{V}_i^{(n)}, \quad n\in\mathbb{N} \quad \text{a.s.},$$

where, for $1\le i\le \xi_n-1$, $\widetilde{V}_i^{(n)}$ is the number of progeny residing in the generation S_n-1 of the immigrant arriving in the generation S_n-i. Under $\mathbb{P}_\omega$, $\widetilde{V}_i^{(n)} \overset{\text{d}}{=} Z^{\text{crit}}(i,\xi_n-1)$, where ω is assumed independent of $(Z^{\text{crit}}(i,k))_{k\ge i}$. With this at hand, an appeal to (6.12) yields

$$\mathbb{E}_\omega \widetilde{V}_i^2 = \mathbb{E}_\omega[Z^{\text{crit}}(i,\xi_n-1)]^2 = \mathbb{E}_\omega[Z^{\text{crit}}(1,\xi_n-i)]^2 = 2(\xi_n-i)+1 \le 2\xi_n$$

and $\mathbb{E}_\omega \widetilde{V}_i = 1$. Here and hereafter, to ease the notation we write $\widetilde{V}_i$ for $\widetilde{V}_i^{(n)}$. Finally,

$$\begin{aligned}\mathbb{E}\Big[\big(\rho_n^2\mathbb{E}_\omega \widetilde{V}^2\big)^{\kappa/2}\Big] &= \mathbb{E}\rho_n^\kappa\bigg(\mathbb{E}_\omega\Big(\sum_{i=1}^{\xi_n-1}\widetilde{V}_i\Big)^2\bigg)^{\kappa/2} = \mathbb{E}\rho_n^\kappa\bigg(\sum_{i=1}^{\xi_n-1}\mathbb{E}_\omega \widetilde{V}_i^2 + \sum_{1\le i\neq j<\xi_n}\mathbb{E}_\omega \widetilde{V}_i \mathbb{E}_\omega \widetilde{V}_j\bigg)^{\kappa/2}\\ &\le (5/2)^{\kappa/2}\mathbb{E}(\rho\xi)^\kappa < \infty\end{aligned}$$

which finishes the proof of (7.31).

Turning to the asymptotic behavior of $\mathbb{E}\mathbb{Z}_{n-1}^{\kappa/2}$ which appears on the right-hand side of (7.30) we consider yet another two cases.

Case $\gamma\le 1$ in which $\mathbb{E}\rho^{\kappa/2}<1$. To see it, observe that when $\gamma=1$ the inequality $\mathbb{E}\rho^{\kappa/2}<\gamma^{1/2}$ is strict because the assumption $\mathbb{E}\log\rho\in[-\infty,0)$ implies that the distribution of ρ is nondegenerate at 1. By the already proved inequality (5.2) for powers ≤ 1

$$\sup_n \mathbb{E}\mathbb{Z}_n^{\kappa/2} < \infty$$

which in combination with (7.31) shows that the expression in the parentheses in (7.30) is bounded. This ensures (5.2).

Case $\gamma>1$. By the already proved inequality (5.2) for powers ≤ 1

$$\mathbb{E}\mathbb{Z}_n^{\kappa/2} \le Ca_n, \quad n\in\mathbb{N},$$

where $a_n=1$ or $=n$ or $=[\mathbb{E}\rho^{\kappa/2}]^n$ depending on whether $\mathbb{E}\rho^{\kappa/2}<1$ or $\mathbb{E}\rho^{\kappa/2}=1$ or $\mathbb{E}\rho^{\kappa/2}>1$. Since in any event $a_n\le\gamma^{n/2}$ for $n\in\mathbb{N}$, (7.30) entails

$$\mathbb{E}\mathbb{Z}_n^\kappa \le \gamma\mathbb{E}\mathbb{Z}_{n-1}^\kappa + C_1\gamma^{n/2}, \quad n\in\mathbb{N}$$

for some $C_1>0$. Iterating this yields $\mathbb{E}\mathbb{Z}_n^\kappa\le C_2\gamma^n$ for some $C_2>1$ and all $n\in\mathbb{N}$, thereby finishing the proof of (5.2) in the case $\gamma>1$ and in general.

To prove (5.3) we use a decomposition $\mathbb{W}_1 = W_{\xi_1-1} + \mathbb{Z}_1$ a.s. Inequality (5.2) tells that we are left with checking that

$$\mathbb{E}W^{\kappa}_{\xi_1-1} < \infty.$$

Since, under $\mathbb{P}_\omega$, $W_{\xi_1-1} \overset{\mathrm{d}}{=} W^{\mathrm{crit}}_{\xi_1-1}$, where ω is assumed independent of $(W^{\mathrm{crit}}_n)_{n\in\mathbb{N}_0}$, an application of Lemma 6.5 yields

$$\mathbb{E}[W^{\kappa}_{\xi_1-1}] = \sum_{j\geq 0}\mathbb{E}\big[\mathbf{1}_{\{\xi_1=j+1\}}(W^{\mathrm{crit}}_j)^{\kappa}\big] \leq C\sum_{j\geq 0}\mathbb{P}\{\xi=j+1\}j^{2\kappa} = C\mathbb{E}(\xi-1)^{2\kappa} < \infty$$

for a positive constant C. The proof of Lemma 5.2 is complete. □

7.4.2 Proof of Lemma 5.4

Proof of Lemma 5.4. We start by proving (5.4). Pick $\kappa_0 \in (0,\kappa)$, put $p = \kappa/\kappa_0$ and choose q such that $1/p + 1/q = 1$. According to Lemma 5.2,

$$\mathbb{E}\mathbb{Z}^{\kappa}_n \leq C, \quad n \in \mathbb{N} \tag{7.32}$$

for a positive constant C, whence

$$\mathbb{E}\Big(\sum_{i=1}^{n}\mathbb{Z}_i\Big)^{\kappa} \leq C\max(n^{\kappa}, n), \quad n\in\mathbb{N}$$

by subadditivity (convexity) of $x \mapsto x^{\kappa}$ when $\kappa \in (0,1]$ ($\kappa \in (1,2]$). By Lemma 4.1, $\mathbb{P}\{\tau_1 = n\} \leq C_1e^{-C_2n}$ for all $n \in \mathbb{N}$ and positive constants C_1 and C_2. With these at hand, an application of Hölder's inequality yields

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\tau_1}\mathbb{Z}_i\Big)^{\kappa_0}\Big] = \sum_{n\geq 1}\mathbb{E}\Big[\Big(\sum_{i=1}^{\tau_1}\mathbb{Z}_i\Big)^{\kappa_0}\mathbf{1}_{\{\tau_1=n\}}\Big]$$
$$\leq \sum_{n\geq 1}\Big(\mathbb{E}\Big(\sum_{i=1}^{n}\mathbb{Z}_i\Big)^{\kappa}\Big)^{1/p}\cdot\mathbb{P}\{\tau_1=n\}^{1/q} \leq C^{1/p}C_1\sum_{n\geq 1}\max(n^{\kappa/p}, n^{1/p})e^{-C_2n/q} < \infty.$$

The proof of (5.4) is complete.

Turning to the proof of (5.5) we shall only show that

$$\mathbb{E}\big(\mathbb{W}^{\downarrow}_n\big)^{\kappa} \leq C, \quad n \geq 2 \tag{7.33}$$

for a positive constant C. Formula (5.5) then follows with the help of the same argument (involving Hölder's inequality) that we used while proving (5.4).

For $i \geq 2$ and $1 \leq j \leq \mathbb{Z}_{i-1} = Z_{S_{i-1}}$, denote by $U^{(i)}_j$ the number of progeny in the generations $S_{i-1}+1,\ldots,S_i-1$ of the jth particle in the generation S_{i-1}. Here is a representation of $\mathbb{W}^{\downarrow}_i$ which is slightly different from the original definition

$$\mathbb{W}^{\downarrow}_i = \sum_{j=1}^{\mathbb{Z}_{i-1}}U^{(i)}_j, \quad i \geq 2.$$

Under $\mathbb{P}_\omega$, $U^{(i)}_j \overset{\mathrm{d}}{=} Y^{\mathrm{crit}}(1,\xi_i-1)$ for $i \geq 2$, where we set $Y^{\mathrm{crit}}(1,0) = 0$ and ω is assumed independent of $(Y^{\mathrm{crit}}(1,k))_{k\in\mathbb{N}}$. In particular, according to (6.12)

$$\mathbb{E}_\omega U^{(i)}_j = \xi_i - 1 \quad\text{and}\quad \mathbb{E}_\omega\big[U^{(i)}_j\big]^2 \leq 3\xi^3_i. \tag{7.34}$$

We shall treat the cases $\kappa \in (0,1]$ and $\kappa \in (1,2]$ separately.

Case $\kappa \in (0,1]$. Under $\mathbb{P}_\omega$, for $1 \le j \le \mathbb{Z}_{i-1}$, $U_j^{(i)}$ is independent of $\mathbb{Z}_{i-1}$. This in combination with (7.34) proves that

$$\mathbb{E}_\omega[\mathbb{W}_i^{\downarrow}|\mathbb{Z}_{i-1}] = \mathbb{Z}_{i-1}(\xi_i - 1), \quad i \ge 2.$$

Therefore, we obtain

$$\mathbb{E}\big(\mathbb{W}_i^{\downarrow}\big)^\kappa \le \mathbb{E}\big[[\mathbb{E}_\omega(\mathbb{W}_i^{\downarrow}|\mathbb{Z}_{i-1})]^\kappa\big] \le \mathbb{E}\xi^\kappa \mathbb{E}\mathbb{Z}_{i-1}^\kappa \le C, \quad i \ge 2$$

having utilized Jensen's inequality, (7.32) and the fact that ξ_i and $\mathbb{Z}_{i-1}$ are independent.
Case $\kappa \in (1,2]$. Another application of Jensen's inequality in combination with (7.34), (7.32) and subadditivity of $x \mapsto x^{\kappa/2}$ on $[0,\infty)$ yields, for $i \ge 2$,

$$\mathbb{E}\big(\mathbb{W}_i^{\downarrow}\big)^\kappa = \mathbb{E}\Big[\mathbb{E}_\omega\Big[\Big(\sum_{j=1}^{\mathbb{Z}_{i-1}} U_j^{(i)}\Big)^\kappa \Big|\mathbb{Z}_{i-1}\Big]\Big] \le \mathbb{E}\Big(\mathbb{E}_\omega\Big[\Big(\sum_{j=1}^{\mathbb{Z}_{i-1}} U_j^{(i)}\Big)^2 \Big|\mathbb{Z}_{i-1}\Big]\Big)^{\kappa/2}$$

$$= \mathbb{E}\Big[\Big(\sum_{1\le l\ne j\le \mathbb{Z}_{i-1}} \mathbb{E}_\omega U_l^{(i)} \mathbb{E}_\omega U_j^{(i)} + \sum_{j=1}^{\mathbb{Z}_{i-1}} \mathbb{E}_\omega\big[U_j^{(i)}\big]^2\Big)^{\kappa/2}\Big] \le \mathbb{E}\mathbb{Z}_{i-1}^\kappa \mathbb{E}\xi^\kappa + 3\mathbb{E}\mathbb{Z}_{i-1}^{\kappa/2}\mathbb{E}\xi^{3\kappa/2} \le C$$

for a positive constant C. The proof of (7.33) is complete. □

7.4.3 Proof of Lemma 5.5

We follow the method invented by Kesten et al. [26]. While some parts of the proofs given in [26] can be directly transferred to our setting, the others require an additional work. We do not present all the details of the proof focussing instead on the main differences.

We begin with a brief overview of the arguments leading to the claim of Lemma 5.5. Given a large positive constant A, put

$$\sigma = \sigma(A) := \min\{i \in \mathbb{N} : Z_j > A \text{ for some } j \le S_i\}.$$

Thus, we observe the process $(Z_n)_{n\in\mathbb{N}_0}$ up to the first time j when it exceeds the level A and then put $\sigma = i$ for the smallest index i satisfying $S_i \ge j$. The following decomposition holds

$$\sum_{k=1}^{\tau_1}\big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big) = \sum_{k=1}^{\tau_1}\big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big)\mathbf{1}_{\{\sigma\ge\tau_1\}} + \Big(\sum_{k=1}^{\sigma-1}\big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big) + \mathbb{S}_\sigma + \sum_{i=\sigma+1}^{\tau_1}\mathbb{Y}_i^{\downarrow}\Big)\mathbf{1}_{\{\sigma<\tau_1\}} \quad \text{a.s.,}$$

where $\mathbb{S}_\sigma$ is the number of particles in the generation S_σ plus their total progeny, and, for $i \in \mathbb{N}$, $\mathbb{Y}_i^{\downarrow}$ is the total progeny in the generations $S_i+1, S_i+2, \ldots$ of the immigrants arriving in the generations $S_{i-1}, \ldots, S_i - 1$.

We intend to prove that the first, second and fourth summands on the right-hand side of this decomposition are negligible in a sense to be made precise, so that

$$\sum_{k=1}^{\tau_1}\big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big) \approx \mathbb{S}_\sigma \mathbf{1}_{\{\sigma<\tau_1\}}.$$

In view of the definition of $\mathbb{S}_\sigma$ and the fact that $\mathbb{Z}_\sigma = Z_{S_\sigma} \approx A$ for A as above one can expect that $\mathbb{S}_\sigma \mathbf{1}_{\{\sigma<\tau_1\}} \approx \mathbb{Z}_\sigma \mathbb{E}_\omega[Y(S_\sigma,\infty)]\mathbf{1}_{\{\sigma<\tau_1\}}$. We shall demonstrate that the variable $\mathbb{E}_\omega[Y(S_\sigma,\infty)]$ is related to a random difference equation whose tail behavior determines that of $\mathbb{S}_\sigma$.

To realize the programme just outlined we need two auxiliary results.

Lemma 7.1. *Assume that the assumptions of Lemma 5.5 hold. Then, for any $A > 0$, as $x \to \infty$,*

$$\mathbb{P}\Big\{\sum_{k=1}^{\tau_1}\big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big) > x,\ \sigma \ge \tau_1\Big\} + \mathbb{P}\Big\{\sum_{k=1}^{\sigma-1}\big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big) > x,\ \sigma < \tau_1\Big\} = o(x^{-\alpha}). \qquad (7.35)$$

Proof. We only give a proof for the first summand in (7.35). The second summand can be treated along similar lines.

The random variable τ_1 has a finite exponential moment by Lemma 4.1. Furthermore, τ_1 does not depend on the future of the sequence $(\xi_i)_{i\in\mathbb{N}}$. Therefore, the assumption $\mathbb{E}\xi^{3\alpha/2}<\infty$ ensures that

$$\mathbb{E}[S_{\tau_1}]^{3\alpha/2}<\infty \tag{7.36}$$

by Lemma A.1.

Write, for $x>0$,

$$\begin{aligned}\mathbb{P}\Big\{\sum_{k=1}^{\tau_1}(\mathbb{Z}_k+\mathbb{W}_k^{\downarrow})>x,\,\sigma\ge\tau_1\Big\}&\le\mathbb{P}\Big\{\sum_{k=1}^{\tau_1-1}(\mathbb{Z}_k+\mathbb{W}_k^{\downarrow})>x/2,\,\sigma\ge\tau_1\Big\}+\mathbb{P}\Big\{\mathbb{W}_{\tau_1}^{\downarrow}>x/2,\,\sigma=\tau_1\Big\}\\&\le\mathbb{P}\{AS_{\tau_1}>x/2\}+\mathbb{P}\{\mathbb{Z}_{\tau_1-1}\le A,\,\mathbb{W}_{\tau_1}^{\downarrow}>x/2\}\end{aligned}$$

and observe that, in view of (7.36), the first summand on the right-hand side is $o(x^{-3\alpha/2})$ as $x\to\infty$. To estimate the second term we use a decomposition

$$\mathbb{W}_{\tau_1}^{\downarrow}=\sum_{i=1}^{\mathbb{Z}_{\tau_1-1}}V_i\quad\text{a.s.},$$

where, for $1\le i\le\mathbb{Z}_{\tau_1-1}$, V_i is the number of progeny in the generations $S_{\tau_1-1}+1,\dots,S_{\tau_1}-1$ of the ith particle in the generation S_{τ_1-1}. We claim that

$$\mathbb{E}V_1^{\alpha}<\infty. \tag{7.37}$$

For the proof, note that $V_1\overset{\mathrm{d}}{=}Y^{\mathrm{crit}}(1,\xi_{\tau_1}-1)$, where ξ_{τ_1} is assumed independent of $(Y^{\mathrm{crit}}(1,n))_{n\in\mathbb{N}}$. Consequently, we obtain with the help of Jensen's inequality and the inequality $\mathbb{E}[Y^{\mathrm{crit}}(1,n)]^2\le 3n^3$ for $n\in\mathbb{N}$ which is a consequence of (6.12)

$$\begin{aligned}\mathbb{E}V_1^{\alpha}=\mathbb{E}[Y^{\mathrm{crit}}(1,\xi_{\tau_1}-1)]^{\alpha}&=\sum_{k\ge0}\mathbb{E}[Y^{\mathrm{crit}}(1,k)]^{\alpha}\mathbb{P}\{\xi_{\tau_1}-1=k\}\\&\le\sum_{k\ge0}\big(\mathbb{E}[Y^{\mathrm{crit}}(1,k)]^2\big)^{\alpha/2}\,\mathbb{P}\{\xi_{\tau_1}-1=k\}\le3\sum_{k\ge0}k^{3\alpha/2}\mathbb{P}\{\xi_{\tau_1}-1=k\}\\&=3\mathbb{E}[\xi_{\tau_1}-1]^{3\alpha/2}\le3\mathbb{E}[S_{\tau_1}]^{3\alpha/2}<\infty,\end{aligned}$$

where the last inequality is secured by (7.36).

With (7.37) at hand, we immediately conclude that

$$\mathbb{P}\{\mathbb{Z}_{\tau_1-1}\le A,\,\mathbb{W}_{\tau_1}^{\downarrow}>x/2\}\le\mathbb{P}\Big\{\sum_{i=1}^{[A]}V_i>x/2\Big\}=o(x^{-\alpha}),\quad x\to\infty$$

because $V_1,\,V_2,\dots$ are identically distributed. The proof of Lemma 7.1 is complete. □

Before formulating another auxiliary result we recall from Section 3.2 the notation $Y_1=\sum_{i\ge1}Z(1,i)$, where $Z(1,i)$ is the number of progeny residing in the ith generation of the first immigrant, so that Y_1 is the total progeny of the first immigrant.

Lemma 7.2. *Suppose that the assumptions of Lemma 5.5 hold. Let $(Y_j^*)_{j\in\mathbb{N}}$ be a sequence of $\mathbb{P}_\omega$-independent copies of Y_1. Then there exists a constant $C>0$ such that*

$$\mathbb{P}\Big\{\sum_{j=1}^{N}Y_j^*>x\Big\}\le CN^{\alpha}x^{-\alpha},\quad N\in\mathbb{N}.$$

Proof. For $k \in \mathbb{N}$, put

$$\widetilde{R}_k = \xi_k + \rho_k \xi_{k+1} + \rho_k \rho_{k+1} \xi_{k+2} + \dots . \qquad (7.38)$$

Recall from Section 3.3 that the so defined random variable is called perpetuity. The Kesten-Grincevičius-Goldie theorem says that if ($\rho 1$) holds and $\mathbb{E}\xi^\alpha < \infty$, then, for all $k \in \mathbb{N}$,

$$\mathbb{P}\{\widetilde{R}_k > x\} \sim C x^{-\alpha}, \quad x \to \infty \qquad (7.39)$$

for some positive constant C which does not depend on k.

Put $Z(1,0) := 1$. For $i \in \mathbb{N}_0$, denote by $Z_1(1,i), Z_2(1,i), \dots$ $\mathbb{P}_\omega$-independent copies of $Z(1,i)$. Our proof will be based on the following decomposition which holds a.s.

$$\sum_{j=1}^N Y_j^* = \sum_{j=1}^N \sum_{i \ge 1} Z_j(1,i) = \sum_{j=1}^N \sum_{k \ge 1} \xi_k Z_j(1, S_{k-1}) + \sum_{j=1}^N \sum_{k \ge 1} \sum_{i=S_{k-1}}^{S_k - 1} \big(Z_j(1,i) - Z_j(1, S_{k-1})\big) =: \mathbb{U}_1 + \mathbb{U}_2.$$

Formula (7.38) implies that, for $k \in \mathbb{N}$, $\xi_k = \widetilde{R}_k - \rho_k \widetilde{R}_{k+1}$, whence

$$\begin{aligned}\mathbb{U}_1 = \sum_{j=1}^N \sum_{k \ge 1} \xi_k Z_j(1, S_{k-1}) &= \sum_{j=1}^N \sum_{k \ge 1} Z_j(1, S_{k-1})(\widetilde{R}_k - \rho_k \widetilde{R}_{k+1}) \\ &= \sum_{k \ge 1} \Big(\sum_{j=1}^N \big(Z_j(1, S_k) - \rho_k Z_j(1, S_{k-1})\big)\Big) \widetilde{R}_{k+1} + N \widetilde{R}_1.\end{aligned}$$

Since

$$\sum_{k \ge 1} 2^{-1} k^{-2} = \pi^2/12 < 1, \qquad (7.40)$$

and $\widetilde{R}_{k+1}$ and $(Z_j(1, S_k), Z_j(1, S_{k-1}), \rho_k)$ are independent for each $j \in \mathbb{N}$ we obtain with the help of (7.39), for $x > 0$,

$$\begin{aligned}&\mathbb{P}\{\mathbb{U}_1 > x\} \\ &\le \sum_{k \ge 1} \mathbb{P}\Big\{ \Big| \sum_{j=1}^N \big(Z_j(1, S_k) - \rho_k Z_j(1, S_{k-1})\big)\Big| \widetilde{R}_{k+1} > x/(4k^2)\Big\} + \mathbb{P}\{N \widetilde{R}_1 > x/2\} \\ &\le \sum_{k \ge 1} \int_{[0,\infty)} \mathbb{P}\Big\{ \Big| \sum_{j=1}^N \big(Z_j(1, S_k) - \rho_k Z_j(1, S_{k-1})\big)\Big| \in {\rm d}s\Big\} \mathbb{P}\{\widetilde{R}_{k+1} > x/(4sk^2)\} + \mathbb{P}\{N \widetilde{R}_1 > x/2\} \\ &\le {\rm const} \cdot x^{-\alpha} \Big(\sum_{k \ge 1} k^{2\alpha} \mathbb{E} \Big| \sum_{j=1}^N \big(Z_j(1, S_k) - \rho_k Z_j(1, S_{k-1})\big)\Big|^\alpha + N^\alpha \Big).\end{aligned}$$

Here and hereafter, const denote constants which may be different on different appearances. To estimate the last term observe that the equality

$$\mathbb{E}_\omega Z(1, S_i) = \rho_1 \cdots \rho_i, \quad i \in \mathbb{N} \qquad (7.41)$$

implies that, under $\mathbb{P}_\omega$, $\sum_{j=1}^N \big(Z_j(1, S_k) - \rho_k Z_j(1, S_{k-1})\big)$ is the sum of iid centered random variables. With this at hand an application of conditional Jensen's inequality yields, for $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E} \Big| \sum_{j=1}^N \big(Z_j(1, S_k) - \rho_k Z_j(1, S_{k-1})\big)\Big|^\alpha &\le \mathbb{E}\Big[\mathbb{E}_\omega \Big(\sum_{j=1}^N \big(Z_j(1, S_k) - \rho_k Z_j(1, S_{k-1})\big)\Big)^2 \Big]^{\alpha/2} \\ &= N^{\alpha/2} \cdot \mathbb{E}\Big(\mathbb{E}_\omega \big(Z(1, S_k) - \rho_k Z(1, S_{k-1})\big)^2 \Big)^{\alpha/2}.\end{aligned}$$

For $k \in \mathbb{N}$ and $1 \le i \le Z(1, S_{k-1})$, take the ith particle among the progeny in the generation S_{k-1} of the first immigrant and denote by $V_i^{(k)}$ the number of progeny residing in the generation S_k of the chosen particle. Then

$$Z(1, S_k) = \sum_{i=1}^{Z(1,S_{k-1})} V_i^{(k)}, \quad k \in \mathbb{N} \quad \text{a.s.},$$

and, under $\mathbb{P}_\omega$, $V_1^{(k)}$, $V_2^{(k)}, \ldots$ are independent copies of $Z(S_{k-1}, S_k)$ which are also independent of $Z(1, S_{k-1})$. Hence,

$$\mathbb{E}_\omega\Big[\big(Z(1,S_k) - \rho_k Z(1,S_{k-1})\big)^2 \big| Z(1,S_{k-1})\Big] = Z(1,S_{k-1})\mathrm{Var}_\omega(V_1^{(k)}), \quad k \in \mathbb{N}.$$

Observe that, under $\mathbb{P}_\omega$,

$$V_1^{(k)} \overset{\mathrm{d}}{=} \sum_{m=1}^{Z^{\mathrm{crit}}(S_{k-1},S_k-1)} U_m^{(k)}, \quad k \in \mathbb{N},$$

where $U_1^{(k)}$, $U_2^{(k)}, \ldots$ are $\mathbb{P}_\omega$-independent random variables with $\mathrm{Geom}(\lambda_k)$ distribution, and ω is assumed independent of $(Z^{\mathrm{crit}}(i,j))_{j \ge i \ge 1}$. This in combination with $Z^{\mathrm{crit}}(i,j) \overset{\mathrm{d}}{=} Z^{\mathrm{crit}}(1, j-i+1)$ for fixed $j \ge i \ge 1$ and (6.12) gives, for $k \in \mathbb{N}$,

$$\begin{aligned}\mathrm{Var}_\omega(V_1^{(k)}) = \mathbb{E}_\omega Z^{\mathrm{crit}}(S_{k-1}, S_k - 1)\mathrm{Var}_\omega(U_1^{(k)}) + (\mathbb{E}_\omega U_1^{(k)})^2 \mathrm{Var}_\omega Z^{\mathrm{crit}}(S_{k-1}, S_k - 1)\\ = (\rho_k + \rho_k^2) + 2\rho_k^2(\xi_k - 1).\end{aligned}$$

Equality (7.41) together with the last formula and subadditivity of $x \mapsto x^{\alpha/2}$ on $[0,\infty)$ enables us to conclude that

$$\begin{aligned}\mathbb{P}\{\mathbb{U}_1 > x\} &\le \frac{\mathrm{const}}{x^\alpha}\bigg(\sum_{k \ge 1} k^{2\alpha} N^{\alpha/2} \mathbb{E}\Big[\big(\mathbb{E}_\omega Z(1,S_{k-1})\big)^{\alpha/2}\big(\rho_k^{\alpha/2} + \rho_k^\alpha + \rho_k^\alpha 2^\alpha (\xi_k - 1)^{\alpha/2}\big)\Big] + N^\alpha\bigg)\\ &\le \frac{\mathrm{const}}{x^\alpha}\bigg(\sum_{k \ge 1} k^{2\alpha} N^{\alpha/2} (\mathbb{E}\rho^{\alpha/2})^{k-1} + N^\alpha\bigg) = \mathrm{const} \cdot N^\alpha x^{-\alpha}.\end{aligned}$$

To obtain the last inequality we have utilized $\mathbb{E}(\rho^\alpha \xi^{\alpha/2}) < \infty$ which is secured by the assumption $\mathbb{E}(\rho\xi)^\alpha < \infty$ and the inequality $\mathbb{E}\rho^{\alpha/2} < 1$ which is a consequence of (ρ1).

To estimate $\mathbb{U}_2$ we proceed similarly but use additionally Markov's inequality

$$\begin{aligned}\mathbb{P}\{\mathbb{U}_2 > x\} &= \mathbb{P}\bigg\{\sum_{j=1}^N \sum_{k \ge 1} \bigg(\sum_{i=S_{k-1}}^{S_k - 1} (Z_j(1,i) - Z_j(1,S_{k-1}))\bigg) > x\bigg\}\\ &= \sum_{k \ge 1} \mathbb{P}\bigg\{\bigg|\sum_{j=1}^N \bigg(\sum_{i=S_{k-1}}^{S_k - 1} (Z_j(1,i) - Z_j(1,S_{k-1}))\bigg)\bigg| > x/(2k^2)\bigg\}\\ &\le \mathrm{const} \cdot x^{-\alpha} \sum_{k \ge 1} k^{2\alpha} \mathbb{E}\bigg|\sum_{j=1}^N \bigg(\sum_{i=S_{k-1}}^{S_k - 1} (Z_j(1,i) - Z_j(1,S_{k-1}))\bigg)\bigg|^\alpha\\ &\le \mathrm{const} \cdot x^{-\alpha} \sum_{k \ge 1} k^{2\alpha} \mathbb{E}\bigg(\mathbb{E}_\omega\bigg(\sum_{j=1}^N \sum_{i=S_{k-1}}^{S_k - 1} (Z_j(1,i) - Z_j(1,S_{k-1}))\bigg)^2\bigg)^{\alpha/2}, \quad x > 0.\end{aligned}$$

For $k \in \mathbb{N}$ and $1 \le r \le Z(1, S_{k-1})$, take the rth particle among the progeny in the generation S_{k-1} of the first immigrant and denote by $W_r^{(k)}$ the number of progeny residing in the generations $S_{k-1}, \ldots, S_k - 1$ of the chosen particle. Then

$$\sum_{i=S_{k-1}}^{S_k - 1} (Z(1,i) - Z(1,S_{k-1})) = \sum_{r=1}^{Z(1,S_{k-1})} (W_r^{(k)} - (\xi_k - 1)), \quad k \in \mathbb{N} \quad \text{a.s.}$$

Furthermore, under $\mathbb{P}_\omega$, $W_1^{(k)}$, $W_2^{(k)}, \ldots$ are independent random variables which are independent of $Z(1, S_{k-1})$ and have the same distribution as $Y^{\rm crit}(1, \xi_k - 1)$. Here, as usual, ω is assumed independent of $(Y^{\rm crit}(1,n))_{n\in\mathbb{N}}$. Invoking (6.12) we infer $\mathrm{Var}_\omega\,(W_r^{(k)}) \leq 2\xi_k^3$ and further

$$\begin{aligned}\mathbb{P}\{\mathbb{U}_2 > x\} &\leq {\rm const}\cdot x^{-\alpha}\sum_{k\geq 1} k^{2\alpha}N^{\alpha/2}\mathbb{E}\big[\big(\mathbb{E}_\omega Z(1,S_{k-1})\mathrm{Var}_\omega(W_1^{(k)})\big)^{\alpha/2}\big]\\ &\leq {\rm const}\cdot x^{-\alpha}\sum_{k\geq 1} k^{2\alpha}N^{\alpha/2}(\mathbb{E}\rho^{\alpha/2})^k\mathbb{E}\xi^{3\alpha/2} \leq {\rm const}\cdot N^{\alpha/2}x^{-\alpha},\quad x>0.\end{aligned}$$

The proof of Lemma 7.2 is complete. $\square$

Proof of Lemma 5.5. Lemma 7.1 implies that the contribution of particles residing in the generations $1, 2, \ldots, S_\sigma - 1$ is negligible in the sense that

$$\mathbb{P}\Big\{\sum_{k=1}^{\tau_1}(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}) > x\Big\} = \mathbb{P}\Big\{\mathbb{S}_\sigma + \sum_{i=\sigma+1}^{\tau_1}\mathbb{Y}_i^{\downarrow} > x,\ \sigma < \tau_1\Big\} + o(x^{-\alpha}),\quad x\to\infty. \tag{7.42}$$

Next we prove that

$$\lim_{A\to\infty}\limsup_{x\to\infty} x^\alpha\mathbb{P}\Big\{\sum_{i=\sigma(A)+1}^{\tau_1}\mathbb{Y}_i^{\downarrow} > x,\ \sigma(A) < \tau_1\Big\} = 0. \tag{7.43}$$

This means that the contribution of the total progeny of immigrants arriving in the generations $S_{\sigma(A)}$, $S_{\sigma(A)}+1, \ldots$ is negligible whenever A is sufficiently large.

The random variables $\mathbb{Y}_1^{\downarrow}$, $\mathbb{Y}_2^{\downarrow}, \ldots$ are identically distributed and, for each $i\in\mathbb{N}$, the random variables $\mathbf{1}_{\{\sigma<i\leq\tau_1\}} = \mathbf{1}_{\{\sigma<i\}}\cdot(1-\mathbf{1}_{\{\tau_1<i\}})$ and $\mathbb{Y}_i^{\downarrow}$ are independent. Therefore,

$$\begin{aligned}\mathbb{P}\Big\{\sum_{i=\sigma(A)+1}^{\tau_1}\mathbb{Y}_i^{\downarrow} > x,\ \sigma(A)<\tau_1\Big\} &\leq \sum_{i\geq 1}\mathbb{P}\big\{\mathbf{1}_{\{\sigma(A)<i\leq\tau_1\}}\mathbb{Y}_i^{\downarrow} > x/(2i^2)\big\}\\ &= \sum_{i\geq 1}\mathbb{P}\{\sigma(A)<i\leq\tau_1\}\mathbb{P}\big\{\mathbb{Y}_1^{\downarrow} > x/(2i^2)\big\}\end{aligned} \tag{7.44}$$

having utilized (7.40). Further, observe that $\mathbb{Y}_1^{\downarrow}$ is the sum of $\mathbb{Z}_1$ $\mathbb{P}_\omega$-independent copies of $Y_1 = Y(1,\infty)$ which are also $\mathbb{P}$-independent of $\mathbb{Z}_1$. Hence, using Lemma 7.2 yields

$$\mathbb{P}\{\mathbb{Y}_1^{\downarrow} > x\} \leq C\mathbb{E}\mathbb{Z}_1^\alpha x^{-\alpha},\quad x>0$$

for some positive constant C. The assumptions $\mathbb{E}\xi^{3\alpha/2}<\infty$ and $\mathbb{E}(\rho\xi)^\alpha<\infty$ guarantee $\mathbb{E}\mathbb{Z}_1^\alpha<\infty$ by Lemma 5.2. Continuing (7.44) we obtain

$$\begin{aligned}\mathbb{P}\Big\{\sum_{i=\sigma(A)+1}^{\tau_1}\mathbb{Y}_i^{\downarrow} > x,\ \sigma(A)<\tau_1\Big\} &\leq C\mathbb{E}\mathbb{Z}_1^\alpha x^{-\alpha}\sum_{i\geq 1} i^{2\alpha}\mathbb{P}\{\sigma(A)<i\leq\tau_1\}\\ &\leq C_1\mathbb{E}\mathbb{Z}_1^\alpha x^{-\alpha}\mathbb{E}\tau_1^{2\alpha+1}\mathbf{1}_{\{\sigma(A)<\tau_1\}}\end{aligned}$$

for a positive constant C_1, and (7.43) follows on letting $A\to\infty$ and recalling that $\mathbb{E}\tau_1^{2\alpha+1}<\infty$ by Lemma 4.1.

Summarizing it remains to show that

$$\mathbb{P}\{\mathbb{S}_{\sigma(A)} > x,\ \sigma(A)<\tau_1\}\ \sim\ C_2(\alpha)x^{-\alpha},\quad x\to\infty,$$

where $C_2(\alpha)$ does not depend on A. This can be accomplished by comparing $\mathbb{S}_{\sigma(A)}$ on the event $\{\sigma(A)<\tau_1\}$ with $\mathbb{Z}_{\sigma(A)}\widetilde{R}_{\sigma(A)+1}$ along the lines of Lemmas 4 and 6 in [26]. We omit the details. $\square$

7.4.4 Proof of Lemma 5.6

Proof of Lemma 5.6. Recall that

$$\bar{\mathbb{W}}_{\tau_1} = W_{S_{\tau_1}} = \sum_{k=1}^{\tau_1} \mathbb{W}_k^0 + \sum_{k=1}^{\tau_1} (\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}) \quad \text{a.s.}$$

According to Lemma 5.5,

$$\mathbb{P}\Big\{ \sum_{k=1}^{\tau_1} \big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big) > x \Big\} \sim C_2(\alpha) x^{-\alpha}, \quad x \to \infty.$$

By the same reasoning as in the proof of Proposition 5.7 (part (C1)), Lemma 5.1 in combination with Lemma 4.1 and Corollary 3 in [10] entails

$$\mathbb{P}\Big\{ \sum_{k=1}^{\tau_1} \mathbb{W}_k^0 > x \Big\} \sim (\mathbb{E}\tau_1)(\mathbb{E}\vartheta^{\alpha}) C_\ell x^{-\alpha}, \quad x \to \infty.$$

Thus to prove the lemma it suffices to check that

$$\mathbb{P}\Big\{ \sum_{k=1}^{\tau_1} \mathbb{W}_k^0 > x, \sum_{k=1}^{\tau_1} \big(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\big) > x \Big\} = o(x^{-\alpha}), \quad x \to \infty, \tag{7.45}$$

see, for example, Lemma B.6.1 in [4].

For the proof of (7.45) we need a number of auxiliary limit relations. First, according to Lemma 4.1 there exists a constant $C_1 > 0$ such that

$$\mathbb{P}\{\tau_1 > C_1 \log x\} = o(x^{-\alpha}), \quad x \to \infty. \tag{7.46}$$

Further, we claim that for any $\delta \in (0,1)$ and large enough x the following inequalities hold uniformly in $k \in \mathbb{N}$

$$\mathbb{P}\big\{\mathbb{W}_k^0 > x/(C_1 \log x),\, \xi_k^2 \le x^{1-\delta}\big\} \le \text{const} \cdot x^{-(\alpha+\varepsilon_1)}; \tag{7.47}$$

$$\mathbb{P}\Big\{ \xi_k^2 > x^{1-\delta}, \sum_{j=1}^{(k-1)\wedge\tau_1} \big(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}\big) > x/2 \Big\} \le \text{const} \cdot x^{-(\alpha+\varepsilon_1)}; \tag{7.48}$$

$$\mathbb{P}\big\{\xi_k^2 > x^{1-\delta},\, \mathbb{Z}_{k-1} > x^{2\delta}\big\} \le \text{const} \cdot x^{-(\alpha+\varepsilon_1)}, \tag{7.49}$$

where $u \wedge v := \min(u,v)$ and $\varepsilon_1 := (\alpha(1-\delta)) \wedge (\alpha\delta/2) > 0$.

Proof of (7.47). Fix any $s > 0$ that satisfies $\delta s > \alpha + \varepsilon_1$. Recall that, under $\mathbb{P}_\omega$, $\mathbb{W}_k^0 \overset{\text{d}}{=} W_{\xi_k - 1}^{\text{crit}}$, where ω is assumed independent of $(W_n^{\text{crit}})_{n \in \mathbb{N}_0}$. This in combination with Markov's inequality yields

$$\begin{aligned}
\mathbb{P}\big\{\mathbb{W}_k^0 > x/(C_1 \log x),\, \xi_k^2 \le x^{1-\delta}\big\} &= \mathbb{P}\big\{W_{\xi_k-1}^{\text{crit}} > x/(C_1 \log x),\, \xi_k^2 \le x^{1-\delta}\big\} \\
&\le \mathbb{P}\big\{W_{[x^{(1-\delta)/2}]}^{\text{crit}} > x/(C_1 \log x)\big\} \\
&\le \frac{\mathbb{E}(W_{[x^{(1-\delta)/2}]}^{\text{crit}})^s}{[x^{(1-\delta)/2}]^{2s}} \frac{(C_1 \log x)^s}{x^{\delta s}} \le \text{const} \cdot x^{-(\alpha+\varepsilon_1)}
\end{aligned}$$

having utilized boundedness of $\mathbb{E}(n^{-2} W_n^{\text{crit}})^s$ for $n \in \mathbb{N}$, see Lemma 6.5.

Proof of (7.48). For fixed $k \in \mathbb{N}$, ξ_k is independent of $\sum_{j=1}^{(k-1)\wedge\tau_1} \big(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}\big)$. Using this, Lemma 5.5 and the assumptions of Lemma 5.6 we conclude that

$$\begin{aligned}
\mathbb{P}\Big\{ \xi_k^2 > x^{1-\delta}, \sum_{j=1}^{(k-1)\wedge\tau_1} \big(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}\big) > x/2 \Big\} &\le \mathbb{P}\{\xi^2 > x^{1-\delta}\} \mathbb{P}\Big\{ \sum_{j=1}^{\tau_1} \big(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}\big) > x/2 \Big\} \\
&\sim 2^{\alpha} C_\ell C_2(\alpha) x^{-\alpha} x^{-\alpha(1-\delta)} \le \text{const} \cdot x^{-(\alpha+\varepsilon_1)}.
\end{aligned}$$

PROOF OF (7.49). Observing that, for every fixed $k \in \mathbb{N}$, ξ_k is independent of $\mathbb{Z}_{k-1}$ and invoking Lemma 5.2 with $\kappa = 3\alpha/4$ we obtain

$$\mathbb{P}\{\xi_k^2 > x^{1-\delta},\, \mathbb{Z}_{k-1} > x^{2\delta}\} = \mathbb{P}\{\xi_k^2 > x^{1-\delta}\}\mathbb{P}\{\mathbb{Z}_{k-1} > x^{2\delta}\} \\ \leq \mathrm{const} \cdot CC_\ell x^{-\alpha(1-\delta)} x^{-(3/2)\alpha\delta} \leq \mathrm{const} \cdot x^{-(\alpha+\varepsilon_1)}.$$

Combining (7.46), (7.47), (7.48) and (7.49) yields, for any $\delta \in (0,1)$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sum_{k=1}^{\tau_1} \mathbb{W}_k^0 > x,\, \sum_{j=1}^{\tau_1}(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}) > x\Big\} \\
&\overset{(7.46)}{\leq} \mathbb{P}\Big\{\sum_{k=1}^{\tau_1} \mathbb{W}_k^0 > x,\, \sum_{j=1}^{\tau_1}(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}) > x,\, \tau_1 \leq C_1 \log x\Big\} + o(x^{-\alpha}) \\
&\leq \sum_{k \leq C_1 \log x} \mathbb{P}\Big\{\mathbb{W}_k^0 > \frac{x}{C_1 \log x},\, \sum_{j=1}^{\tau_1}(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}) > x,\, \tau_1 \leq C_1 \log x\Big\} + o(x^{-\alpha}) \\
&\overset{(7.47)}{\leq} \sum_{k \leq C_1 \log x} \mathbb{P}\Big\{\xi_k^2 > x^{1-\delta},\, \sum_{j=1}^{\tau_1}(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}) > x,\, \tau_1 \leq C_1 \log x\Big\} + o(x^{-\alpha}) \\
&\overset{(7.48)}{\leq} \sum_{k \leq C_1 \log x} \mathbb{P}\Big\{\xi_k^2 > x^{1-\delta},\, \sum_{j=k}^{\tau_1}(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}) > x/2,\, k \leq \tau_1,\, \tau_1 \leq C_1 \log x\Big\} + o(x^{-\alpha}) \\
&\overset{(7.49)}{\leq} \sum_{k \leq C_1 \log x} \mathbb{P}\Big\{\xi_k^2 > x^{1-\delta},\, \sum_{j=k}^{\tau_1}(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}) > x/2,\, k \leq \tau_1,\, \mathbb{Z}_{k-1} \leq x^{2\delta}\Big\} + o(x^{-\alpha}).
\end{aligned}$$

Now (7.45) follows if we can show that for some $\delta \in (0,1)$ the following inequality holds uniformly in k

$$\mathbb{P}\{\xi_k^2 > x^{1-\delta},\, \mathbb{Z}_k + \mathbb{W}_k^{\downarrow} > x/4,\, \mathbb{Z}_{k-1} \leq x^{2\delta}\} \leq \mathrm{const} \cdot x^{-(\alpha+\varepsilon_2)} \tag{7.50}$$

for large enough x and some $\varepsilon_2 > 0$ to be specified below, and that

$$\sum_{k \leq C_1 \log x} \mathbb{P}\Big\{\xi_k^2 > x^{1-\delta},\, \sum_{j=k+1}^{\tau_1}(\mathbb{Z}_j + \mathbb{W}_j^{\downarrow}) > x/4\Big\} = o(x^{-\alpha}), \quad x \to \infty. \tag{7.51}$$

PROOF OF (7.50). Observe that

$$\mathbb{Z}_k + \mathbb{W}_k^{\downarrow} = \sum_{i=1}^{\mathbb{Z}_{k-1}} V_i^{(k)} \quad \text{a.s.},$$

where, for $k \in \mathbb{N}$ and $1 \leq i \leq \mathbb{Z}_{k-1}$, $V_i^{(k)}$ denotes the number of progeny residing in the generations $S_{k-1}+1$ through S_k of the ith particle in the generation S_{k-1}. Clearly, for fixed $k \in \mathbb{N}$, $V_1^{(k)}, \ldots, V_{\mathbb{Z}_{k-1}}^{(k)}$ are independent of $\mathbb{Z}_{k-1}$ and have the same distribution as

$$Y^{\mathrm{crit}}(1, \xi_k - 1) + \sum_{j=1}^{Z^{\mathrm{crit}}(1,\xi_k-1)} U_j^{(k)},$$

where $(Y^{\mathrm{crit}}(1,n))_{n\in\mathbb{N}}$ and $(Z^{\mathrm{crit}}(1,n))_{n\in\mathbb{N}}$ are assumed independent of (ξ_k, ρ_k), $U_1^{(k)}, U_2^{(k)}, \ldots$ have $\mathrm{Geom}(\lambda_k)$ distribution and, given (ξ_k, ρ_k), they are independent of $Z^{\mathrm{crit}}(1, \xi_k - 1)$. In particular,

$\mathbb{E}\big(V_1^{(k)}|(\xi_k,\rho_k)\big) = \xi_k - 1 + \rho_k$ in view of (6.12). With this at hand we obtain

$$\begin{aligned}
&\mathbb{P}\big\{\xi_k^2 > x^{1-\delta}, \mathbb{Z}_k + \mathbb{W}_k^{\downarrow} > x/4, \mathbb{Z}_{k-1} \le x^{2\delta}\big\}\\
&= \mathbb{E}\mathbf{1}_{\{\mathbb{Z}_{k-1}\le x^{2\delta}\}}\mathbb{P}\bigg\{\xi_k^2 > x^{1-\delta}, \sum_{i=1}^{\mathbb{Z}_{k-1}} V_i^{(k)} > x/4\bigg|\mathbb{Z}_{k-1}\bigg\}\\
&\le \mathbb{E}\mathbb{Z}_{k-1}\mathbf{1}_{\{\mathbb{Z}_{k-1}\le x^{2\delta}\}}\mathbb{P}\bigg\{\xi_k^2 > x^{1-\delta}, V_1^{(k)} > x/(4\mathbb{Z}_{k-1})\bigg|\mathbb{Z}_{k-1}\bigg\}\\
&\le x^{2\delta}\mathbb{P}\big\{\xi_k^2 > x^{1-\delta}, V_1^{(k)} > x^{1-2\delta}/4\big\}\\
&\le \text{const}\cdot x^{2\delta}\mathbb{E}\bigg[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}\mathbb{E}\bigg[\frac{(V_1^{(k)})^r}{x^{r(1-2\delta)}}\bigg|(\xi_k,\rho_k)\bigg]\bigg]\\
&\le \text{const}\cdot x^{2\delta-r(1-2\delta)}\mathbb{E}\Big[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}\big(\mathbb{E}[V_1^{(k)}|(\xi_k,\rho_k)]\big)^r\Big]\\
&\le \text{const}\cdot x^{2\delta-r(1-2\delta)}\mathbb{E}\Big[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}(\xi_k+\rho_k)^r\Big]
\end{aligned}$$

for $k\in\mathbb{N}$, large enough x and any $r\in(0,1]$, having utilized conditional Jensen's inequality for the penultimate step. By assumption $\mathbb{E}\rho^\gamma<\infty$ and $\mathbb{E}\xi^\gamma<\infty$ for some $\gamma\in(\alpha,2\alpha)$. Taking $r\in(0,\gamma)$ and applying Hölder's inequality with parameters $\gamma/(\gamma-r)$ and γ/r we arrive at

$$\begin{aligned}
&\mathbb{P}\big\{\xi_k^2 > x^{1-\delta}, \mathbb{Z}_k + \mathbb{W}_k^{\downarrow} > x/4, \mathbb{Z}_{k-1} < x^{2\delta}\big\}\\
&\qquad\qquad\le \text{const}\cdot\big(\mathbb{E}\xi_k^\gamma + \mathbb{E}\rho_k^\gamma\big)^{r/\gamma}x^{2\delta-r(1-2\delta)-(1-\delta)\alpha(1-r/\gamma)}.
\end{aligned}$$

Pick any $\rho\in(0,(1-\alpha/\gamma)/(2+\alpha))$ and then any $r\in(0,\gamma\wedge((1-\alpha/\gamma-\rho(2+\alpha))/(\rho(2-\alpha/\gamma))))$. Setting now $\delta=\rho r$ (so that $\delta\in(0,1)$) we obtain (7.50) with $\varepsilon_2 := -\alpha-2\delta+r(1-2\delta)+(1-\delta)\alpha(1-r/\gamma)$. Throughout the rest of the proof δ always denotes the number chosen above.

Proof of (7.51). For $k\in\mathbb{N}$ and $1\le i\le\mathbb{Z}_k$, denote by $Y_i^{(k)}$ the total progeny of the ith particle in the generation S_k. Further, for $k\in\mathbb{N}$ and $j\ge k+2$, denote by $\mathbb{W}_j^{\downarrow}(k)$ the number of progeny in the generations $S_{j-1}, S_{j-1}+1,\dots,S_j-1$ of the immigrants arriving in the generations $S_k, S_k+1,\dots,S_{j-1}-1$. Then

$$\sum_{j=k+1}^{\tau_1}\big(\mathbb{Z}_j+\mathbb{W}_j^{\downarrow}\big) = \sum_{i=1}^{\mathbb{Z}_k}Y_i^{(k)} + \sum_{j=k+2}^{\tau_1}\mathbb{W}_j^{\downarrow}(k)\quad\text{a.s.}$$

and thereupon, for $x>0$,

$$\begin{aligned}
\mathbb{P}\bigg\{\xi_k^2 > x^{1-\delta}, \sum_{j=k+1}^{\tau_1}\big(\mathbb{Z}_j+\mathbb{W}_j^{\downarrow}\big) > x/4\bigg\} &\le \mathbb{P}\bigg\{\xi_k^2 > x^{1-\delta}, \sum_{i=1}^{\mathbb{Z}_k}Y_i^{(k)} > x/8\bigg\}\\
&\quad+\mathbb{P}\bigg\{\xi_k^2 > x^{1-\delta}, \sum_{j=k+2}^{\tau_1}\mathbb{W}_j^{\downarrow}(k) > x/8\bigg\}\\
&=: I_1(x)+I_2(x).
\end{aligned}$$

Since, for fixed $k\in\mathbb{N}$, $\sum_{i=k+2}^{\tau_1}\mathbb{W}_i^{\downarrow}(k)$ is independent of ξ_k we obtain with the help of a crude estimate

$$\sum_{i=k+2}^{\tau_1}\mathbb{W}_i^{\downarrow}(k) \le \sum_{i=1}^{\tau_1}\big(\mathbb{Z}_i+\mathbb{W}_i^{\downarrow}\big),\quad k\in\mathbb{N}\quad\text{a.s.}$$

and Lemma 5.5

$$I_2(x)\le\mathbb{P}\{\xi_k^2>x^{1-\delta}\}\mathbb{P}\bigg\{\sum_{i=1}^{\tau_1}\big(\mathbb{Z}_i+\mathbb{W}_i^{\downarrow}\big) > x/8\bigg\}\le\text{const}\cdot x^{-\alpha(1-\delta)}x^{-\alpha}$$

for large enough x. Of course, this entails $\sum_{k\le C_1 \log x} I_2(x) = o(x^{-\alpha})$ as $x\to\infty$.

To estimate $I_1(x)$ we note that, for fixed $k\in\mathbb{N}$, under $\mathbb{P}\{\cdot|\omega,\mathbb{Z}_k\}$, $Y_1^{(k)},\dots,Y_{\mathbb{Z}_k}^{(k)}$ are independent copies of $Y(1,\infty)$. Furthermore, these random variables are $\mathbb{P}$-independent of $\mathbb{Z}_k$ and ξ_k. Invoking Lemma 7.2 and conditional Jensen's inequality yields

$$\begin{aligned}\mathbb{P}\Big\{\xi_k^2>x^{1-\delta},\sum_{i=1}^{\mathbb{Z}_k}Y_i^{(k)}>x/8\Big\} &= \mathbb{E}\Big[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}\mathbb{P}\Big[\sum_{i=1}^{\mathbb{Z}_k}Y_i^{(k)}>x/8\Big|\xi_k,\mathbb{Z}_k\Big]\Big]\\ &\le \text{const}\cdot x^{-\alpha}\mathbb{E}\big[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}\mathbb{Z}_k^\alpha\big]\\ &= \text{const}\cdot x^{-\alpha}\mathbb{E}\left[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}\mathbb{E}_\omega\left[\mathbb{Z}_k^\alpha|\mathbb{Z}_{k-1}\right]\right]\\ &\le \text{const}\cdot x^{-\alpha}\mathbb{E}\left[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}\left(\mathbb{E}_\omega\left[\mathbb{Z}_k^2|\mathbb{Z}_{k-1}\right]\right)^{\alpha/2}\right].\end{aligned}$$

Inequality (7.29) was obtained in the proof of Lemma 5.2 under the assumption $\kappa\in(1,2]$. However, by the same reasoning it also holds for $\kappa\in(0,2]$. Using (7.29) in combination with the fact that $\xi\ge 1$ a.s. we infer

$$\left(\mathbb{E}_\omega\left[\mathbb{Z}_k^2|\mathbb{Z}_{k-1}\right]\right)^{\alpha/2}\le \text{const}\cdot\left(\mathbb{Z}_{k-1}^\alpha(\rho_k\xi_k)^\alpha+\mathbb{Z}_{k-1}^{\alpha/2}\big((\rho_k\xi_k)^\alpha+(\rho_k\xi_k)^{\alpha/2}\big)+(\rho_k\xi_k)^\alpha+(\rho_k\xi_k)^{\alpha/2}\right)$$

and thereupon

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{\xi_k^2>x^{1-\delta}\}}\left(\mathbb{E}_\omega\left[\mathbb{Z}_k^2|\mathbb{Z}_{k-1}\right]\right)^{\alpha/2}\right] &\le \text{const}\cdot\left(k\mathbb{E}(\rho\xi)^\alpha\mathbf{1}_{\{\xi^2>x^{1-\delta}\}}+\mathbb{E}(\rho\xi)^{\alpha/2}\mathbf{1}_{\{\xi^2>x^{1-\delta}\}}\right)\\ &\le \text{const}\cdot x^{-\varepsilon(1-\delta)/2}\left(k\mathbb{E}\rho^\alpha\xi^{\alpha+\varepsilon}+\mathbb{E}\rho^{\alpha/2}\xi^{\alpha/2+\varepsilon}\right)\\ &\le \text{const}\cdot kx^{-\varepsilon(1-\delta)/2}\end{aligned}$$

by Lemma 5.2 and the assumption $\mathbb{E}\rho^\alpha\xi^{\alpha+\varepsilon}<\infty$ for some $\varepsilon>0$. The latter entails

$$\sum_{k\le C_1\log x} I_1(x)=o(x^{-\alpha}),\quad x\to\infty.$$

The proof of Lemma 5.6 is complete. □

A Appendix

Lemma A.1 is an important ingredient in the proof of Proposition 5.7, part (C1). In its formulation we use the notion of a random variable which does not depend on the future of a sequence of random variables. The corresponding definition can be found at the beginning of Section 5.

Lemma A.1. *Let $(\theta_i)_{i\in\mathbb{N}}$ be a sequence of iid nonnegative random variables and T a nonnegative integer-valued random variable which does not depend on the future of the sequence $(\theta_i)_{i\in\mathbb{N}}$. Assume that $\mathbb{E}\theta_1^s<\infty$ for some $s>0$ and that $\mathbb{E}e^{\lambda T}<\infty$ for some $\lambda>0$. Then $\mathbb{E}(\sum_{i=1}^T\theta_i)^s<\infty$.*

Proof. Set $R_0:=0$ and $R_i:=\theta_1+\ldots+\theta_i$ for $i\in\mathbb{N}$. By assumption, for fixed $i\in\mathbb{N}$, θ_i is independent of $(R_{i-1},\mathbf{1}_{\{T\ge i\}})$.

The result is trivial when $s\in(0,1]$. Indeed, we use subadditivity of $x\mapsto x^s$ on $[0,\infty)$ together with the aforementioned independence to conclude that

$$\mathbb{E}\Big(\sum_{i=1}^T\theta_i\Big)^s\le\sum_{i\ge1}\mathbb{E}\theta_i^s\mathbf{1}_{\{T\ge i\}}=\mathbb{E}\theta_1^s\mathbb{E}T<\infty.$$

Assume now that $s>1$. Invoking the inequality

$$(x+y)^s\le x^s+sy(x+y)^{s-1},\quad x,y\ge0$$

which is secured by the mean value theorem for differentiable functions we obtain

$$R_{T\wedge i}^s \le R_{T\wedge(i-1)}^s + s\theta_i R_i^{s-1}\mathbf{1}_{\{T\ge i\}}, \quad i\in\mathbb{N}.$$

Iterating this yields

$$R_{T\wedge n}^s \le s\sum_{i=1}^n \theta_i R_i^{s-1}\mathbf{1}_{\{T\ge i\}}, \quad n\in\mathbb{N}.$$

Therefore, it is enough to check that

$$A := \mathbb{E}\sum_{i\ge1}\theta_i R_i^{s-1}\mathbf{1}_{\{T\ge i\}} < \infty.$$

Using once again the aforementioned independence together with the inequality

$$(x+y)^{s-1} \le C_s(x^{s-1}+y^{s-1}), \quad x,y\ge0,$$

where $C_s := \max(2^{s-2},1)$, we infer

$$A \le C_s\mathbb{E}\sum_{i\ge1}\theta_i(R_{i-1}^{s-1}+\theta_i^{s-1})\mathbf{1}_{\{T\ge i\}} = C_s\mathbb{E}\theta_1\sum_{i\ge1}\mathbb{E}R_{i-1}^{s-1}\mathbf{1}_{\{T\ge i\}} + C_s\mathbb{E}\theta_1^s\mathbb{E}T.$$

Left with checking convergence of the series we appeal to Hölder's inequality in conjunction with convexity of $x\mapsto x^s$ on $[0,\infty)$ to get

$$\mathbb{E}R_{i-1}^{s-1}\mathbf{1}_{\{T\ge i\}} \le [\mathbb{E}R_{i-1}^s]^{(s-1)/s}[\mathbb{P}\{T\ge i\}]^{1/s} \le i^{s-1}[\mathbb{E}\theta_1^s]^{(s-1)/s}[\mathbb{P}\{T\ge i\}]^{1/s}.$$

Since $[\mathbb{P}\{T\ge i\}]^{1/s}$ decreases at least exponentially in i, $\mathbb{E}R_{i-1}^{s-1}\mathbf{1}_{\{T\ge i\}}$ is the general term of converging series. The proof of Lemma A.1 is complete. □

The remaining part of the Appendix is concerned with the proof of Lemma 4.1. In essence the lemma follows from the arguments presented by Key [27] who considered a model very similar to ours. For $n\in\mathbb{N}$ and $1\le k\le n$, set

$$\mathbb{Z}(k,n) = \sum_{j=S_{k-1}+1}^{S_k} Z(j,S_n)$$

and observe that, under $\mathbb{P}_\omega$, $\mathbb{Z}(1,n),\dots,\mathbb{Z}(n,n)$ are independent. The following representation holds

$$\mathbb{Z}(0)=0, \quad \mathbb{Z}_n = \sum_{k=1}^{n-1}\mathbb{Z}(k,n) + \mathbb{Z}(n,n), \quad n\in\mathbb{N}$$

which shows that $(\mathbb{Z}_n)_{n\in\mathbb{N}_0}$ is a branching process in a random environment with the random number $\mathbb{Z}(k,k)$ of immigrants in the kth generation. The basic observation for what follows is that $(\mathbb{Z}_n)_{n\ge0}$ has the structure similar to that of the branching process investigated by Key [27]. The main difference manifests in the term $\mathbb{Z}(n,n)$ which is absent in Key's model. It is curious that the branching process in [27] is similar to our $(Z_n)_{n\in\mathbb{N}_0}$ in that the immigrants arriving in the generation n only affect the system by their offspring residing in the generation $n+1$. In particular, neither Key's process nor our $(Z_n)_{n\in\mathbb{N}_0}$ counts immigrants, whereas $(\mathbb{Z}_n)_{n\in\mathbb{N}_0}$ does.

Even though $(\mathbb{Z}_n)_{n\ge0}$ and Key's process are slightly different it is natural to expect that sufficient conditions ensuring finiteness of power and exponential moments of the first extinction time should be similar. While demonstrating that this is indeed the case we shall only point out principal changes with respect to Key's arguments.

Denote by

$$p(n,k) = \mathbb{P}_\omega\{\mathbb{Z}(1,n)=k \mid \mathbb{Z}(1,n-1)=1\}, \quad n\ge2,\ k\in\mathbb{N}_0$$

and

$$a(n,k)=\mathbb{P}_\omega\{\mathbb{Z}(n,n)=k\},\quad n\in\mathbb{N},\ k\in\mathbb{N}_0$$

the quenched reproduction and immigration distribution in the generation n, respectively. It can be checked that the mean of the quenched reproduction distribution is

$$M(n)=\sum_{k\geq 0}kp(n,k)=\mathbb{E}_\omega[\mathbb{Z}(1,n)|\mathbb{Z}(1,n-1)=1]=\rho_n,\quad n\geq 2$$

and that the quenched expected number of immigrants is

$$I(n)=\sum_{k\geq 0}ka(n,k)=\mathbb{E}_\omega[\mathbb{Z}(n,n)]=\rho_n\xi_n,\quad n\in\mathbb{N}.$$

Lemma A.2 is a counterpart of Theorem 3.3 in [27].

Lemma A.2. *Assume that* $\mathbb{E}\log\rho\in[-\infty,0)$ *and* $\mathbb{E}\log^+\xi<\infty$. *Then, for* $k\in\mathbb{N}_0$, $\pi(k)=\lim_{n\to\infty}\mathbb{P}\{\mathbb{Z}_n=k\}$ *exists and defines a probability distribution on* $\mathbb{N}$. *If additionally*

$$\mathbb{P}\{p(2,0)>0,\ a(2,0)>0\}>0,\tag{A.1}$$

then $\pi(0)>0$.

Sketch of proof. As far as the first claim is concerned, the proofs of Lemmas 2.1, 2.2, 3.1, 3.2 in [27] only require inessential changes concerning the range of summation. The second claim follows after a minor alteration, namely the term $q(n,k)$ appearing in the proof of Theorem 3.3 in [27] should be changed to

$$q(n,k)=\mathbb{P}_\omega\{\mathbb{Z}_{n+1}=0\,|\,\mathbb{Z}_n=k\}=p(n+1,0)^k a(n+1,0),\quad n\in\mathbb{N},k\in\mathbb{N}_0.$$

The sequence $(q(1,k))_{k\in\mathbb{N}_0}$ must be positive which justifies condition (A.1). The corresponding condition in [27] is slightly different. ☐

We are ready to prove Lemma 4.1.

Proof of Lemma 4.1. The present proof is very similar to that of Theorem 4.2 in [27]. Put

$$v(n):=\mathbb{P}\{\tau_1>n\},\quad n\in\mathbb{N}_0$$

and

$$V(x):=\sum_{n\geq 1}v(n)x^n,\quad x\geq 0$$

which may be finite or infinite. While finiteness of $\mathbb{E}\tau_1$ is equivalent to $V(1)<\infty$, finiteness of some exponential moment of τ_1 is equivalent to $V(x)<\infty$ for some $x>1$.

For $n\in\mathbb{N}$, put

$$h(k,n):=\mathbb{P}\Big\{\mathbb{Z}(k,n)>0,\ \sum_{j=k+1}^{n}\mathbb{Z}(j,n)=0\Big\},\quad 1\leq k\leq n$$

(with the usual convention that $h(n,n)=\mathbb{P}\{\mathbb{Z}(n,n)>0\}$) and note that $h(k,n)=h(1,n-k+1)$ for $1\leq k\leq n$. Now we use a decomposition

$$\begin{aligned}v(n)=&\mathbb{P}\{\tau_1>n,\ \mathbb{Z}_n>0\}=\mathbb{P}\Big\{\tau_1>n,\ \sum_{k=1}^{n}\mathbb{Z}(k,n)>0\Big\}\\ =&\sum_{k=1}^{n-1}\mathbb{P}\Big\{\tau_1>n,\ \mathbb{Z}(k,n)>0,\ \sum_{j=k+1}^{n}\mathbb{Z}(j,n)=0\Big\}+\mathbb{P}\{\tau_1>n,\ \mathbb{Z}(n,n)>0\}.\end{aligned}$$

in combination with

$$\mathbb{P}\Big\{\tau_1 > n,\ \mathbb{Z}(k,n)>0,\ \sum_{j=k+1}^{n}\mathbb{Z}(j,n)=0\Big\} = \mathbb{P}\Big\{\tau_1 > k-1,\ \mathbb{Z}(k,n)>0,\ \sum_{j=k+1}^{n}\mathbb{Z}(j,n)=0\Big\}$$
$$= \mathbb{P}\{\tau_1 > k-1\}\mathbb{P}\Big\{\mathbb{Z}(k,n)>0,\ \sum_{j=k+1}^{n}\mathbb{Z}(j,n)=0\Big\} = v(k-1)h(k,n) = v(k-1)h(1,n-k+1)$$

which holds for $1 \le k \le n$ to obtain

$$v(n) = \sum_{k=0}^{n-1} v(k)h(1,n-k), \quad n \in \mathbb{N}.$$

This convolution equation is equivalent to

$$V(x) = \frac{H(x)}{1-H(x)}, \quad x \ge 0$$

(the possibility that both sides are infinite is not excluded), where

$$H(x) = \sum_{j\ge 1} h(1,j)x^j, \quad x \ge 0.$$

Now $\mathbb{E}\tau_1 < \infty$ follows from

$$H(1) = \sum_{j\ge 1} h(1,j) = \lim_{n\to\infty} \mathbb{P}\{\mathbb{Z}_n > 0\} = 1 - \pi(0)$$

once we can show that $\pi(0) > 0$. To this end, we recall that $(Z_n)_{n\in\mathbb{N}_0}$ is governed by a geometric distribution, whence

$$p(n,0) \ge \lambda_n \mathbf{1}_{\{\xi_n=1\}} + 2^{-1}\mathbf{1}_{\{\xi_n>1\}} \ge \lambda_n \wedge 1/2, \quad n \ge 2$$

and

$$a(n,0) = \sum_{j\ge 1} \frac{\lambda_n}{j-(j-1)\lambda_n}\mathbf{1}_{\{\xi_n=j\}} \ge \lambda_n \sum_{j\ge 1} j^{-1}\mathbf{1}_{\{\xi_n=j\}}, \quad n \in \mathbb{N}.$$

These inequalities ensure (A.1) and thereupon $\pi(0) > 0$ by Lemma A.2.

To prove finiteness of some exponential moment pick $\delta \in (0,1)$ such that

$$\mathbb{E}(\rho\xi)^\delta < \infty \quad \text{and} \quad r := \mathbb{E}\rho^\delta < 1.$$

Existence of such a δ is justified by assumptions and the Cauchy-Schwarz inequality. In view of

$$h(1,j) \le \mathbb{P}\{\mathbb{Z}(1,j) \ge 1\} \le \mathbb{E}(\mathbb{E}_\omega \mathbb{Z}(1,j))^\delta = \mathbb{E}(\rho\xi)^\delta r^{j-1}$$

we infer that the radius of convergence of H is greater than one. This in combination with $H(1) < 1$ implies that $H(x) < 1$ and thereupon $V(x) < \infty$ for some $x > 1$. □

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