RANDOM WALKS IN A STRONGLY SPARSE RANDOM ENVIRONMENT

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Abstract. The integer points (sites) of the real line are marked by the positions of a standard random walk. We say that the set of marked sites is weakly, moderately or strongly sparse depending on whether the jumps of the standard random walk are supported by a bounded set, have finite or infinite mean, respectively. Focussing on the case of strong sparsity we consider a nearest neighbor random walk on the set of integers having jumps ± 1 with probability $1/2$ at every nonmarked site, whereas a random drift is imposed at every marked site. We prove new distributional limit theorems for the so defined random walk in a strongly sparse random environment, thereby complementing results obtained recently in Buraczewski et al. (2018+) for the case of moderate sparsity and in Matzavinos et al. (2016) for the case of weak sparsity. While the random walk in a strongly sparse random environment exhibits either the diffusive scaling inherent to a simple symmetric random walk or a wide range of subdiffusive scalings, the corresponding limit distributions are non-stable.

1. Introduction

A simple random walk (SRW) on the set $\mathbb{Z}$ of integers is one of the most fundamental objects in both classical and modern probability. We consider a slightly perturbed version of SRW obtained by imposing a random drift at some randomly chosen (marked) sites. Allowance is made for occasional huge gaps between the marked sites which thus form a rather sparse subset of $\mathbb{Z}$. Our main purpose is to reveal new effects generated by this extreme sparsity which are absent in [23] and [6, 26]. While the former article treats the nonsparse case in which random drifts are imposed at each site, the latter two papers are concerned with a sparse situation like here, the only difference being that enormous gaps between the marked sites are prohibited. Now we define the model in focus more precisely, as a particular random walk in a random environment (RWRE). Also, we discuss its relation to a multi-skewed Brownian motion and a one-dimensional trap model.

We first recall the definition of a general RWRE. Let $\Omega = (0,1)^{\mathbb{Z}}$ be the set of all possible environments equipped with the corresponding Borel σ-algebra $\mathcal{F}$ and a probability measure P. A random element $\omega = (\omega_n)_{n\in\mathbb{Z}}$ defined on $(\Omega, \mathcal{F}, P)$ is called *random environment*. A random walk in a random environment ω is a nearest neighbor random walk $X = (X_n)_{n\in\mathbb{N}_0}$ (here and hereafter, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) on $\mathbb{Z}$. To define its transition probabilities we first set $\mathcal{X} = \mathbb{Z}^{\mathbb{N}_0}$ and equip it with a Borel σ-algebra $\mathcal{G}$. Plainly, $\mathcal{X}$ can be thought of as the set of trajectories of X. Now, given ω, let $\mathbb{P}_\omega$ be a (quenched) probability measure on $\mathcal{X}$ such that $X_0 = 0$ $\mathbb{P}_\omega$- almost surely (a.s.) and

$$\mathbb{P}_\omega\{X_{n+1} = j | X_n = i\} = \begin{cases} \omega_i, & \text{if } j = i+1, \\ 1-\omega_i, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, under $\mathbb{P}_\omega$, X is a time-homogeneous Markov chain on $\mathbb{Z}$. The randomness of the environment ω influences significantly various properties of X. In view of this, it is quite

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natural to investigate the behavior of X under the annealed measure $\mathbb{P}$ which is defined as a unique probability measure on $(\Omega \times \mathcal{X}, \mathcal{F} \otimes \mathcal{G})$ satisfying

$$\mathbb{P}\{F \times G\} = \int_F \mathbb{P}_\omega\{G\} P(\mathrm{d}\omega), \quad F \in \mathcal{F}, \quad G \in \mathcal{G}.$$

After these preparations we are ready to define the object of our interest in the present paper. Denote by $((\xi_k, \lambda_k))_{k\in\mathbb{Z}}$ a sequence of independent copies of a random vector (ξ, λ), where $\lambda \in (0,1)$ and $\xi \in \mathbb{N}$ $\mathbb{P}$-a.s. Setting

$$S_n = \begin{cases} \sum_{k=1}^n \xi_k, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -\sum_{k=n+1}^0 \xi_k, & \text{if } n < 0. \end{cases}$$

we define a specific random environment $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Omega$ by

$$\omega_n = \begin{cases} \lambda_{k+1}, & \text{if } n = S_k \text{ for some } k \in \mathbb{Z}, \\ 1/2, & \text{otherwise.} \end{cases} \tag{1.1}$$

Thus, the sequence $(S_n)_{n\in\mathbb{Z}}$ determines the marked sites in which the random drifts λ_{k+1} are placed. Since for the nonmarked sites n (that is, for most of sites) $\omega_n = 1/2$, it is natural to call ω a *sparse random environment*. Following [26] we use the term *random walk in sparse random environment* (RWSRE) for X as defined above with ω being a sparse random environment. We call the environment ω *moderately sparse* or *strongly sparse* depending on whether $\mathbb{E}\xi < \infty$ or

$$\mathbb{E}\xi = \infty. \tag{1.2}$$

While the case of moderate sparsity was analyzed in the recent article [6], the case of strong sparsity is investigated in the present work. In particular, (1.2) is our main standing assumption.

The behavior of any RWRE is affected by both randomness of the environment and randomness of the walk given the environment. The earlier works [6, 23, 26] demonstrate that in the nonsparse case randomness of the environment has dominating effect. A remarkable feature of the sparse case is that randomness of the environment and randomness of the walk may contribute to a comparable extent. Another source of the new effects arising in the sparse case are the properties of the environment alone which are essentially different from those in the nonsparse case.

Since the early work [36] a general RWRE has been attracting a fair amount of attention among probabilistic community. We refer to [43] for a classical introduction to the topic. In the literature one usually treats the cases where the environment ω forms a stationary ergodic sequence or just a collection of independent identically distributed (iid) random variables (note that in the present article we go beyond these settings). There are numerous articles which prove, under these assumptions, quenched and annealed distributional limit theorems [4, 9, 11, 22, 23, 35, 38] and investigate large deviations [5, 7, 8, 14, 17, 29, 30, 39, 42]. The list above is far from being complete.

Further, we discuss a relation of RWSRE to two other models. Following [31] we recall that a multi-skewed Brownian motion evolves like a standard Brownian motion with the exception of some deterministic sites (interfaces) $(s_k)_{k\in\mathbb{Z}}$ in which some additional skewness (perturbation) is imposed. Thus, the RWSRE can be seen as a discrete analogue of a multi-skewed Brownian motion with random interfaces and random perturbations. To demonstrate a connection to the other model we only observe the walker at the marked sites. To be more precise, set

$$\widehat{X}_0 := 0, \quad \widehat{X}_n := \begin{cases} \widehat{X}_{n-1}, & \text{if } X_n \notin \{S_k : k \in \mathbb{Z}\}, \\ k, & \text{if } X_n = S_k \text{ for some } k \in \mathbb{Z}; \end{cases} \quad n \in \mathbb{N}. \tag{1.3}$$

and note that $\widehat{X}_n$ is the index of the last marked site visited by the walker up to time n. The sequence $\widehat{X} = (\widehat{X}_k)_{k\in\mathbb{N}_0}$ is a nearest-neighbor random walk on $\mathbb{Z}$ in a random environment which has a positive probability to stay immobile at any time. Thus, $\widehat{X}$ is a discrete variant of a one-dimensional trap model [1, 13, 44]. The setting of [44] is closely related to that of the present article. To justify the claim, assume that the distribution of ξ is heavy-tailed in the sense that $\mathbb{P}\{\xi > t\} \sim t^{-\beta}$ as $t \to \infty$ for some $\beta \in (0, 1)$. Using the solution to gambler's ruin problem enables us to calculate the transition probabilities of $\widehat{X}$ explicitly and then conclude that $\mathbb{P}\{\tau > t\}$ is proportional to $\mathbb{P}\{\xi^2 > t\}$, where τ is the time of the first jump from a given state (trapping time). One-dimensional trap models with heavy-tailed trapping times are analyzed in [44]. In particular, it is shown that the corresponding nearest-neighbor continuous-time Markov process, properly scaled, converges weakly in the Skorokhod space to an inverse β- stable subordinator. On the other hand, let us note right away that the assertions of the present article cannot be derived from those in [44]. The explanation is simple: the evolution of X between the marked sites is extremely important; thus, restricting attention to the marked sites only leads to an essential loss of information.

The article is organized as follows. In Section 2 we present our main results and review some earlier results which are particularly relevant to ours. In Section 3 we recall the construction of a branching process associated with X. In Section 4 we explain our proof strategy. In Section 5 several important auxiliary facts are established. Finally, Section 6 contains the proofs of our main results.

2. Main results

2.1. Preliminaries. In the paper [26] the authors address the question of transience and recurrence of RWSRE and prove a strong law of large numbers and some distributional limit theorems for X. Put

$$\rho = \frac{1-\lambda}{\lambda}.$$

According to Theorem 3.1 in [26], X is $\mathbb{P}$-a.s. transient to $+\infty$ whenever

$$\mathbb{E}\log\rho \in [-\infty, 0) \quad \text{and} \quad \mathbb{E}\log\xi < \infty \tag{2.1}$$

(note that the first inequality in (2.1) excludes the degenerate case $\rho = 1$ a.s. in which X becomes a simple random walk). Under (2.1), the RWSRE also satisfies a strong law of large numbers, that is,

$$\frac{X_n}{n} \to v \quad \mathbb{P}-a.s. \tag{2.2}$$

where

$$v = \begin{cases} \frac{(1-\mathbb{E}\rho)\mathbb{E}\xi}{(1-\mathbb{E}\rho)\mathbb{E}\xi^2+2\mathbb{E}\rho\xi\mathbb{E}\xi} & \text{if } \mathbb{E}\rho < 1,\ \mathbb{E}\rho\xi < \infty \text{ and } \mathbb{E}\xi^2 < \infty \\ 0 & \text{otherwise} \end{cases},$$

see Theorem 3.3 in [26] and Proposition 2.1 in [6].

We note right away that conditions (1.2) and (2.1) are satisfied under the conditions of our main results. Thus, the random walks in a sparse random environment that we treat here are transient to the right with zero asymptotic speed ($v = 0$).

2.2. Notation. To state our main results we need more notation. Set

$$T_n = \inf\{k \geq 0 : X_k = n\}, \quad n \in \mathbb{Z}.$$

We are going to derive limit theorems for X_n from those for T_n via a standard inversion technique. It will become clear in Section 3 that the stopping times T_n are easier to deal with, for, unlike X_n, these can be analyzed with the help of an auxiliary branching process.

Now we formulate an assumption concerning the distribution of ξ:

(ξ) there exists $\beta \in (0,1]$ and a slowly varying function ℓ such that

$$\mathbb{P}\{\xi > t\} \sim t^{-\beta}\ell(t), \quad t \to \infty. \tag{2.3}$$

and $\mathbb{E}\xi = \infty$ when $\beta = 1$ ($\mathbb{E}\xi = \infty$ holds automatically when $\beta \in (0,1)$).

Further, we point out two sets of assumptions regarding the distribution of ρ:

($\rho1$) $\mathbb{E}\rho^\alpha = 1$ for some $\alpha > 0$, $\mathbb{E}\rho^\gamma < \infty$ for some $\gamma > \alpha$ and the distribution $\log \rho$ is nonarithmetic;

($\rho2$) there exists an open interval $\mathcal{I} \subset (0,\infty)$ such that $\mathbb{E}\rho^x < 1$ for all $x \in \mathcal{I}$.

Note that ($\rho1$) and ($\rho2$) are not disjoint because ($\rho1$) implies ($\rho2$) with $\mathcal{I} \subset (0,\alpha)$.

To ease the presentation we shall state separately our results for $\beta \in (0,1)$ and $\beta = 1$ because the latter case is technically more involved.

2.3. **Results for $\beta \in (0,1)$.** We shall need two assumptions concerning the joint distribution of (ξ,ρ):

($\xi\rho1$) $\lim_{t\to\infty} \frac{\mathbb{P}\{\xi > t^{1/2},\, \rho > c_1(t)\}}{\mathbb{P}\{\xi > t\}} = 0$, where $c_1(t) = t$;

($\xi\rho2$) $\lim_{t\to\infty} \frac{\mathbb{P}\{\xi > t^{\alpha/\beta},\, \rho > c_2(t)\}}{\mathbb{P}\{\xi > t\}} = 0$, where $c_2(t) := t^{-1}\mathbb{P}\{\xi > t\}^{-1/\alpha}$. For the most part, α is supposed to be the same as in ($\rho1$). But occasionally we allow α to be any positive number satisfying $\alpha \le \beta/2$.

An application of Markov's inequality reveals that under ($\rho1$) with $\alpha = \beta/2$ or ($\rho2$) with $\beta/2 \in \mathcal{I}$ condition ($\xi\rho1$) holds whenever ξ and ρ are independent. Similarly, under ($\rho1$) with $\alpha \in (0,\beta/2]$ condition ($\xi\rho2$) holds provided that ξ and ρ are independent. Further, it is clear that, for $i = 1,2$, $\mathbb{P}\{\rho > c_i(t)\} = o(\mathbb{P}\{\xi > t\})$ is a sufficient condition for ($\xi\rho\,$i) which is far from being necessary.

Denote by ϑ a positive random variable with Laplace transform

$$\mathbb{E}\exp(-s\vartheta) = \frac{1}{\cosh\sqrt{s}}, \quad s \ge 0. \tag{2.4}$$

Define the measure μ on $\mathbb{K} := [0,\infty]^2 \setminus \{(0,0)\}$ by

$$\mu\{(u,v) \in \mathbb{K} : u > x_1 \text{ or } v > x_2\} = x_1^{-\beta} + \mathcal{C}_\mu x_2^{-\beta/2} - \mathbb{E}\min(x_1^{-\beta}, x_2^{-\beta/2}\vartheta^{\beta/2}) \tag{2.5}$$

for $x_1, x_2 > 0$, where $\mathcal{C}_\mu > 0$ is a constant to be specified later. Let $N := \sum_k \delta_{(t_k,\mathbf{j}_k)}$ be a Poisson random measure on $[0,\infty) \times \mathbb{K}$ with intensity measure LEB $\otimes\, \mu$. Here, $\delta_{(t,\mathbf{x})}$ is the probability measure concentrated at $(t,\mathbf{x}) \in [0,\infty) \times \mathbb{K}$, and LEB is the Lebesgue measure on $[0,\infty)$. Set

$$\mathbf{L}(t) := (L_1(t), L_2(t)) = \sum_{k\,:\,t_k \le t} \mathbf{j}_k, \quad t \ge 0. \tag{2.6}$$

Lemma 6.4 given in Section 6 secures

$$\int_{|x|\neq 0} (|\mathbf{x}| \wedge 1)\,\mu(\mathrm{d}\mathbf{x}) < \infty, \tag{2.7}$$

where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1,x_2) \in \mathbb{R}^2$. This ensures that the series on the right-hand side of (2.6) converges a.s. Furthermore, $(\mathbf{L}(t))_{t\ge0}$ is a two-dimensional (non-stable) Lévy process with the Lévy measure μ. Its components L_1 and L_2 are dependent drift-free stable subordinators with parameters β and $\beta/2$, respectively. Put

$$L_1^{\leftarrow}(t) := \inf\{s \ge 0 : L_1(s) > t\}, \quad t \ge 0.$$

The process $(L_1^{\leftarrow}(t))_{t\ge0}$ is known in the literature as the inverse β-stable subordinator. Set

$$\chi := L_2(L_1^{\leftarrow}(1)-) + \vartheta(1 - L_1(L_1^{\leftarrow}(1)-))^2,$$

where ϑ is assumed to be independent of $\mathbf{L}$.

Our first result, Theorem 2.1, provides a distributional limit theorem for X in the situation where the distribution of ξ plays a dominant role, a contribution of the distribution of ρ being small.

Theorem 2.1. *Let $\beta \in (0,1)$ and assume that one of the following sets of conditions is satisfied:*

- *(A) (ξ) and ($\rho 2$) with $\beta/2 \in \mathcal{I}$ hold;*
- *(B1) (ξ) holds with ℓ such that $\lim_{t\to\infty} \ell(t) = \infty$, and ($\rho 1$) holds with $\alpha = \beta/2$;*
- *(B2) (ξ) holds with ℓ such that $\mathcal{C}_\ell := \lim_{t\to\infty} \ell(t) \in (0,\infty)$, and ($\rho 1$) holds with $\alpha = \beta/2$.*

Further, assume that condition ($\xi\rho 1$) holds and that $\mathbb{E}(\rho\xi)^{\beta/2+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then

$$\frac{T_n}{n^2} \xrightarrow{\mathrm{d}} 2\chi, \quad n \to \infty \tag{2.8}$$

and

$$\frac{X_n}{n^{1/2}} \xrightarrow{\mathrm{d}} (2\chi)^{-1/2}, \quad n \to \infty. \tag{2.9}$$

The constant $\mathcal{C}_\mu$ in (2.5) is given as follows:

$$\mathcal{C}_\mu = \begin{cases} \mathbb{E}\vartheta^{\beta/2} & \text{in the cases (A) and (B1)} \\ \mathcal{C}_\ell \mathbb{E}\vartheta^{\beta/2} + \mathcal{C}_Z(\beta/2) & \text{in the case (B2)} \end{cases},$$

where the constant $\mathcal{C}_Z(\beta/2)$ is specified in Lemma 5.5.

In Theorem 2.1 the condition $\mathbb{E}\rho^\alpha = 1$ may hold for any $\alpha > 0$. In the situations (B1) and (B2) (as well as in Theorem 2.2 given below) it holds for $\alpha \in (0, \beta/2]$. Assuming that it holds for $\alpha > \beta/2$ we conclude that ($\rho 2$) with $\beta/2 \in \mathcal{I}$ holds, so that the situation (A) prevails.

Here is a very informal explanation of why n^2 should be the correct normalization for T_n. Since the distribution of ξ dominates that of ρ, it is tempting to assume, at least as a first approximation, that $\rho = 1$ $\mathbb{P}$-a.s. Then X is a SRW, and the fact that T_n/n^2 converges in distribution is well-known.

When ξ and ρ are independent, the conditions ($\xi\rho 1$) and $\mathbb{E}(\rho\xi)^{\beta/2+\varepsilon} < \infty$ for some $\varepsilon > 0$ are secured by the other assumptions of Theorem 2.1 (as for ($\xi\rho 1$), see the discussion at the beginning of Section 2.3).

Theorem 2.2 given next is a counterpart of Theorem 2.1 in which the distributions of ξ and ρ play comparable roles. To formulate it we need some additional notation. Pick any $\alpha \in (0, \beta/2]$ and denote by $\widehat{L}_1 := (\widehat{L}_1(t))_{t\geq 0}$ and $(\widehat{L}_2(t))_{t\geq 0}$ *independent* drift-free β- and α-stable subordinators with the Lévy measures ν_1 and ν_2 given by

$$\nu_1((x,\infty)) = x^{-\beta}, \quad \nu_2((x,\infty)) = \mathcal{C}_Z(\alpha) x^{-\alpha}, \quad x > 0,$$

respectively (see Lemma 5.5 for the definition of $\mathcal{C}_Z(\alpha)$). Also, let $(\widehat{L}_1^{\leftarrow}(t))_{t\geq 0}$ denote an inverse β-stable subordinator which corresponds to $\widehat{L}_1$. Finally, whenever (ξ) holds we denote by λ an asymptotic inverse function for $s \mapsto \mathbb{P}\{\xi > s\}^{-1/\alpha}$. This means that λ satisfies

$$\lim_{t\to\infty} \frac{\mathbb{P}\{\xi > \lambda(t)\}^{-1/\alpha}}{t} = \lim_{t\to\infty} \frac{\lambda(\mathbb{P}\{\xi > t\}^{-1/\alpha})}{t} = 1.$$

Such a function λ is uniquely determined up to asymptotic equivalence by Theorem 1.5.12 in [3]. Moreover, it is regularly varying of index α/β.

Theorem 2.2. *Let $\beta \in (0,1)$ and assume that one of the following sets of conditions is satisfied:*

- *(B3) (ξ) holds with ℓ such that $\lim_{t\to\infty}\ell(t)=0$, and $(\rho 1)$ holds with $\alpha=\beta/2$;*
- *(C) (ξ) holds, and $(\rho 1)$ holds with $\alpha\in(0,\beta/2)$.*

Further, assume that condition $(\xi\rho 2)$ holds and that $\mathbb{E}(\rho\xi)^{\alpha+\varepsilon}<\infty$ for some $\varepsilon>0$. Then

$$\mathbb{P}\{\xi>n\}^{1/\alpha}T_n \overset{\mathrm{d}}{\longrightarrow} 2\widehat{L}_2(\widehat{L}_1^{\leftarrow}(1)), \quad n\to\infty \tag{2.10}$$

and

$$\frac{X_n}{\lambda(n)} \overset{\mathrm{d}}{\longrightarrow} (2\widehat{L}_2(\widehat{L}_1^{\leftarrow}(1)))^{-\alpha/\beta}, \quad n\to\infty. \tag{2.11}$$

An informal but well-justified explanation of why the normalization in Theorem 2.2 is plausible inevitably requires introducing a new notation that we prefer to avoid at this stage. Thus, we only note, without going into details, that the normalization $\mathbb{P}\{\xi>n\}^{-1/\alpha}$ which is different from that in Theorem 2.1 is given by the composition $(f\circ g)(n)$, where $f(x)=x^{1/\alpha}$ and $g(x)=\mathbb{P}\{\xi>x\}^{-1}$. The functions g and f represent the contributions of the distributions of ξ and ρ, respectively.

When ξ and ρ are independent, the conditions $(\xi\rho 2)$ and $\mathbb{E}(\rho\xi)^{\alpha+\varepsilon}<\infty$ for some $\varepsilon>0$ are implied by the other assumptions of Theorem 2.2 (as for $(\xi\rho 2)$, see the discussion at the beginning of Section 2.3).

2.4. **Results for $\beta=1$.** The boundary case $\beta=1$ is essentially simpler but technically more involved than the case $\beta\in(0,1)$.

Whenever (2.3) holds (for $\beta\in(0,1]$) we denote by a any positive measurable function satisfying

$$\lim_{t\to\infty} t\mathbb{P}\{\xi>a(t)\}=1. \tag{2.12}$$

Further, we put, for $t>0$,

$$m(t)=\int_0^t \mathbb{P}\{\xi>u\}\mathrm{d}u, \quad \text{and} \quad \pi(t):=m(a(t)) \tag{2.13}$$

and define a positive measurable function π^* such that

$$\lim_{t\to\infty}\pi(t)\pi^*(t\pi(t))=\lim_{t\to\infty}\pi^*(t)\pi(t\pi^*(t))=1.$$

Since $\beta=1$ and $\mathbb{E}\xi=\infty$, m and π are slowly varying and unbounded, and π^* is a de Bruijn conjugate function for π, see Theorem 1.5.13 in [3]. In the present case $\beta=1$ we shall use conditions $(\xi\rho 1)$ and $(\xi\rho 2)$ introduced in Section 2.3 with

$$c_1(t):=a(t\pi^*(t)), \qquad \text{and} \qquad c_2(t):=t^{-1}(t\pi^*(t))^{1/\alpha}. \tag{2.14}$$

These functions may be well-defined for large t only which is sufficient for our purposes. Although we are not going to use this observation, let us note that the so defined c_i are asymptotically equivalent, up to multiplicative constants, to the c_i defined in the conditions $(\xi\rho\,\mathrm{i})$ in Section 2.3 for the case $\beta\in(0,1)$.

When $\beta=1$, the two-dimensional subordinator $\mathbf{L}$ defined in (2.6) does not exist simply because (2.7) does not hold any longer. However, its second component L_2 is still well-defined. Actually, it is a drift-free stable subordinator with parameter $1/2$ and the Lévy measure μ_2 given by $\mu_2((x,\infty))=\mathcal{C}_\mu x^{-1/2}$ for $x>0$. As a final preparation, denote by w and κ asymptotic inverse functions for $s\mapsto a(s\pi^*(s))^2$ and $s\mapsto(s\pi^*(s))^{1/\alpha}$, respectively, where $\alpha>0$. Since $s\mapsto a(s\pi^*(s))^2$ and $s\mapsto(s\pi^*(s))^{1/\alpha}$ are regularly varying of indices 2 and $1/\alpha$, such w and κ are regularly varying functions of indices $1/2$ and α.

Theorem 2.3. *Assume that the assumptions of Theorem 2.1 are satisfied for $\beta=1$. Then*

$$\frac{T_n}{a(n\pi^*(n))^2} \overset{\mathrm{d}}{\longrightarrow} 2L_2(1), \quad n\to\infty, \tag{2.15}$$

and

$$\frac{X_n}{w(n)} \xrightarrow{\text{d}} (2L_2(1))^{-1/2}, \quad n\to\infty. \tag{2.16}$$

Theorem 2.4. *Assume that the assumptions of Theorem 2.2 are satisfied for $\beta = 1$. Then*

$$\frac{T_n}{(n\pi^*(n))^{1/\alpha}} \xrightarrow{\text{d}} 2L_2(1), \quad n\to\infty \tag{2.17}$$

and

$$\frac{X_n}{\kappa(n)} \xrightarrow{\text{d}} (2L_2(1))^{-\alpha}, \quad n\to\infty.$$

2.5. **Comparison to earlier limit theorems.** It is more convenient to discuss limit results for T_n rather than X_n. Distributional limit theorems for X_n and T_n are proved in [26] for the case where ξ is $\mathbb{P}$-a.s. bounded (the corresponding environment may be called *weakly sparse*). Then, as expected, the distribution of ξ does not affect the asymptotic behavior of T_n in a significant way. The key parameter is $\alpha > 0$ for which $\mathbb{E}\rho^\alpha = 1$ (as in $(\rho 1)$), and T_n, properly normalized and centered, converges in distribution to an α-stable law (if $\alpha \geq 2$ the corresponding limit is Gaussian), see Proposition 3.9 [26]. For instance, if $\alpha \in (0,1)$, then 'T_n grows like $n^{1/\alpha}$'. The arguments rely on a change of measure which transfers the RWSRE into a random walk in a Markov environment. As a consequence of $\mathbb{P}$-a.s. boundedness of ξ, the Markov chain driving the environment has a finite state space which makes certain results of [27] applicable.

In order to go beyond bounded ξ one has to develop a different approach (one possibility exploited both in [26] and in the present article is to use a link with certain branching processes with immigration in a random environment). In the case of moderately sparse environment, that is, $\mathbb{E}\xi < \infty$ it is shown in [6] that the asymptotics T_n strongly depends on the interplay of parameters α and β and the behavior of a slowly varying function ℓ (see conditions $(\rho 1)$ and (ξ) for the definition). In all cases, the limit distribution of T_n, properly normalized and centered, is still stable. However, the normalization is $n^{1/\alpha}$ when the distribution of ρ dominates that of ξ, whereas it is $n^{1/\alpha}L(n)$ for a slowly varying L when the distribution of ξ dominates that of ρ. Summarizing, the results of both [26] and [6] bear a strong resemblance with those of [23] which is concerned with the nonsparse case $\xi = 1$ $\mathbb{P}$-a.s.

On the technical level the difference between the cases of moderate and strong sparsity is carefully explained at the beginning of Section 4. The strong sparsity strongly manifests itself in Theorem 2.1. In it, unlike the earlier limit theorems discussed above the normalization for T_n is n^2 as if (X_n) was a SRW. However, the connection with a SRW does not extend to the limit distribution which is rather exotic and seems to be new in the context of RWRE and in general. In Theorem 2.2 the limit distribution is still nonstable. However, the limit result obtained here looks more similar to those in the moderately sparse case. Loosely speaking, the normalization in the cited theorem can be interpreted as the time-changed version of $n^{1/\alpha}$. The case $\beta = 1$ treated in Theorems 2.3 and 2.4 can be thought of as *almost moderate*. The closeness to the moderate sparsity only reflects in a stable limit distribution, whereas the normalization for T_n is different from those appearing in [6].

3. An associated branching process

3.1. **The construction.** The relation between certain random walks and branching processes goes back to Harris [20]. Later on, it was successfully applied, in an extended form, in [23] to obtain distributional limit theorems for random walks in an iid random environment. Since then branching processes have become a useful tool in the analysis of one-dimensional RWRE. The presentation below follows closely that in [23] or [6].

Fix $n \in \mathbb{N}$ and consider the random variables

$$U_i^{(n)} = \#\{k < T_n : X_k = i, X_{k+1} = i-1\}, \qquad i \leq n.$$

Since $X_{T_n} = n$ and $X_0 = 0$ we have, for $n \in \mathbb{N}$,

$$\begin{aligned} T_n &= \# \text{ of steps during } [0, T_n) \\ &= \# \text{ of steps to the right during } [0, T_n) + \# \text{ of steps to the left during } [0, T_n) \\ &= n + 2 \cdot \# \text{ of steps to the left during } [0, T_n) \end{aligned}$$

which gives

$$T_n = n + 2 \sum_{i=-\infty}^{n} U_i^{(n)}, \quad n \in \mathbb{N}. \tag{3.1}$$

Recall from Section 2.1 that, under the setting of the present paper, X is transient to the right, that is, $\lim_{n\to\infty} X_n = +\infty$ $\mathbb{P}$-a.s. This entails

$$\sum_{i<0} U_i^{(n)} \leq \text{ total time spent by } X \text{ in } (-\infty, 0) < \infty \quad \mathbb{P}-\text{a.s.} \tag{3.2}$$

In particular,

$$T_n = n + 2\sum_{i=0}^{n} U_i^{(n)} + O_{\mathbb{P}}(1), \quad n \in \mathbb{N}, \tag{3.3}$$

where $O_{\mathbb{P}}(1)$ is a term which is bounded in probability. As a consequence, distributional limit theorems for T_n will follow from those for $n + 2\sum_{i=0}^{n} U_i^{(n)}$. The latter variables possess an elegant stochastic structure since, as argued below, for fixed environment ω, $U_n^{(n)}, U_{n-1}^{(n)}, U_{n-2}^{(n)}, \ldots U_0^{(n)}$ form a sequence of the first n generations of a inhomogeneous branching process with unit immigration.

In what follows, for $p \in (0, 1)$, $\mathrm{Geom}(p)$ is a shorthand for a geometric distribution with success probability p, that is,

$$\mathrm{Geom}(p)\{\ell\} = p(1-p)^{\ell}, \quad \ell \in \mathbb{N}_0.$$

Note that $U_n^{(n)} = 0$ and that $U_{n-1}^{(n)}$ is equal to the number of excursions to the left of $n-1$ before the first visit to n. Due to the transitivity of X, $U_{n-1}^{(n)}$ is distributed according to $\mathrm{Geom}(\omega_{n-1})$. Further, observe that, for $i = 1, \ldots, n-2$, $U_{n-i-1}^{(n)}$ can be represented as follows:

$$U_{n-i-1}^{(n)} = \sum_{k=1}^{U_{n-i}^{(n)}} V_k^{(n-i-1)} + V_0^{(n-i-1)}, \tag{3.4}$$

where, for $k \in \mathbb{N}$, $V_k^{(n-i-1)}$ denotes the number of excursions to the left of $n-i-1$ during the kth excursion to the left of $n-i$ and $V_0^{(n-i-1)}$ is the number of excursions to the left of $n-i-1$ before the first excursion to the left of $n-i$. Notice, since X is transitive and enjoys the Markov property with respect to the quenched probability, for each fixed ω, $V_1^{(n-i-1)}$, $V_2^{(n-i-1)}, \ldots$ are independent random variables with distribution $\mathrm{Geom}(\omega_{n-i-1})$ which are also independent of $U_{n-i}^{(n)}$. This shows that, under $\mathbb{P}_\omega$, $U_n^{(n)}, U_{n-1}^{(n)}, U_{n-2}^{(n)}, \ldots, U_0^{(n)}$ are the consecutive generation sizes in a inhomogeneous branching process with unit immigration in which the particles and the immigrant in the $(i-1)$th generation ($i = 1, \ldots, n-1$) reproduce according to $\mathrm{Geom}(\omega_{n-i})$ distribution.

To ease the notation, we introduce another branching process $Z = (Z_k)_{k\geq 0}$ which evolution can be described as follows. We start with $Z_0 = 0$ particles. At the generation $n = 1$, the first immigrant enters the system and gives birth to $Z_1 = G_0^{(1)}$ new particles, where $G_0^{(1)}$ has distribution Geom(ω_1). At the generation $n = 2$, another immigrant enters the system and all $Z_1 + 1$ particles reproduce independently according to distribution Geom(ω_2). The offspring of the first generation particles (including the immigrant) form the second generation. The system evolves according to these rules, with one new immigrant entering the system at each generation. In general, for each $n \in \mathbb{N}$, Z_n admits the following representation

$$Z_n = \sum_{k=1}^{Z_{n-1}} G_k^{(n)} + G_0^{(n)}, \tag{3.5}$$

where $G_0^{(n)}$ is the number of offspring of the nth immigrant and $G_k^{(n)}$ is the number of offspring of the kth particle in the $(n-1)$st generation (we set $G_k^{(n)} = 0$ if the kth particle in the $(n-1)$st generation does not exist). Thus, the process Z does not count the immigrants. Plainly, under $\mathbb{P}_\omega$, for each $n \in \mathbb{N}$, $G_0^{(n)}$, $G_1^{(n)}, \ldots$ are independent random variables with distribution Geom(ω_n) which are independent of Z_n. Whenever the environment is sparse, while the process Z reproduces according to distribution Geom(λ_{k+1}) at time S_k for $k \in \mathbb{N}_0$, most of the time, it evolves as a critical Galton–Watson process with unit immigration and the offspring distribution Geom(1/2), to be denoted by $Z^{\rm crit} = (Z_n^{\rm crit})_{n\in\mathbb{N}_0}$. In particular, for $n \in \mathbb{N}$, given $(\xi_j, \rho_j)_{1\le j\le n}$,

$$\sum_{i=0}^{S_n} U_i^{(S_n)} \overset{\rm d}{=} \sum_{k=0}^{S_n} Z_k. \tag{3.6}$$

For later needs we note that $Z^{\rm crit}$ is a particular instance of the process Z which corresponds to $\omega_n = 1/2$. In particular, $Z^{\rm crit}$ satisfies (3.5) in which $(G_k^{(n)})_{k\in\mathbb{N}_0, n\in\mathbb{N}}$ are independent random variables with distribution Geom(1/2).

3.2. **The notation.** For $k, n \in \mathbb{N}$, denote by $Z(k, n)$ the number of progeny residing in the nth generation of the kth immigrant. In particular, $Z(k, k)$ is the number of offspring of this immigrant and

$$Z_n = \sum_{k=1}^{n} Z(k, n), \quad n \in \mathbb{N}.$$

Moreover, for each $k \in \mathbb{N}$, $(Z(k, n))_{n\ge k}$ forms a branching process in a random environment (without immigration). Since at each generation the random reproduction law is the same for all particles, the processes $(Z(1, n))_{n\ge1}$, $(Z(2, n))_{n\ge2}$, $(Z(3, n))_{n\ge3}$ ... are dependent with respect to the annealed probability but are independent with respect to the quenched probability. For $n \in \mathbb{N}$ and $1 \le i \le n$, let $Y(i, n)$ denote the number of progeny in the generations $i, i + 1, \ldots, n$ of the ith immigrant, that is,

$$Y(i, n) = \sum_{k=i}^{n} Z(i, k).$$

Similarly, for $i \in \mathbb{N}$, we denote by Y_i the total progeny of the ith immigrant, that is,

$$Y_i = Y(i, \infty) = \sum_{k\ge i} Z(i, k).$$

We also define W_n to be the total population size in the first n generations, that is,

$$W_n = \sum_{j=1}^{n} Z_j, \quad n \in \mathbb{N}.$$

In view of the structure of the environment it is natural to divide the population into blocks which include generations $1, \dots, S_1$; $S_1 + 1, \dots, S_2$ and so on. To set out the necessary notation, we write

$$\mathbb{Z}_n = Z_{S_n}, \quad n \in \mathbb{N}$$

for the number of particles in the generation S_n, and

$$\mathbb{W}_n = W_{S_n} - W_{S_{n-1}} = \sum_{j=S_{n-1}+1}^{S_n} Z_j, \quad n \in \mathbb{N}$$

for the total population in the generations $S_{n-1} + 1, \dots, S_n$.

Put $W_n^{\rm crit} = \sum_{k=1}^n Z_k^{\rm crit}$ for $n \in \mathbb{N}$, so that $W_n^{\rm crit}$ is the total progeny in the first n generations of $Z^{\rm crit}$, see the end of Section 3.1 for the definition of $Z^{\rm crit}$. It is known that

$$n^{-2} W_n^{\rm crit} \xrightarrow{\rm d} \vartheta, \quad n \to \infty, \tag{3.7}$$

where ϑ is a random variable with the Laplace transform given in (2.4), see Theorem 5 in [28], and that

$$\lim_{n\to\infty} \mathbb{E}(n^{-2} W_n^{\rm crit})^s = \mathbb{E}\vartheta^s, \qquad s > 0, \tag{3.8}$$

see Lemma 6.5 in [6]. These properties of $Z^{\rm crit}$ will play an essential role in our proofs.

4. The strategy of the proofs

To prove a distributional limit theorem for T_n it is natural to use a decomposition

$$T_n = T_{S_{\nu(n)-1}} + (T_n - T_{S_{\nu(n)-1}}), \quad n \in \mathbb{N}, \tag{4.1}$$

where

$$\nu(t) = \inf\{n \in \mathbb{N} \,:\, S_n > t\}, \quad t \ge 0.$$

In principle, the asymptotic behavior of $T_{S_{\nu(n)-1}}$ may be regulated by that of S_n, W_{S_n} or both, see formulae (3.3) and (3.6). In the paper [6] which treats the case of moderate sparsity $\mathbb{E}\xi < \infty$ while the contribution of the second summand is negligible, the asymptotics of the first summand $T_{S_{\nu(n)-1}}$ is driven by W_{S_n} alone. The latter is explained by the fact that the contribution of S_n is only seen in the form of a law of large numbers and, as such, degenerate in the limit. The case of strong sparsity $\mathbb{E}\xi = \infty$ we are interested in here is more involved. Indeed, now the asymptotics of $T_{S_{\nu(n)-1}}$ is affected by (S_n, W_{S_n}), for, under (ξ), S_n, properly normalized, converges in distribution to a nondegenerate random variable. Further, in Theorem 2.1 the contributions of the summands in (4.1) are comparable. Therefore, one has to investigate their joint asymptotic behavior which leads to technical complications. On the other hand, the asymptotics of $T_n - T_{S_{\nu(n)-1}}$ alone is relatively easy to deal with, for the principal component of this random variable is given by the first-passage time of a reflected SRW stopped at an independent time. The other main results, Theorems 2.2, 2.3 and 2.4, are simpler than Theorem 2.1 because the first summand in (4.1) dominates the second.

The text below is borrowed from Section 4 in [6], with minor alterations and additions. While dealing with W_{S_n} our main arguments follow the strategy invented by Kesten et al. [23]. Namely, for large n, we decompose W_{S_n} as a sum of random variables which are iid under the annealed probability $\mathbb{P}$. For this purpose we define extinction times

$$\tau_0 := 0, \quad \tau_k := \min\{j > \tau_{k-1} : \mathbb{Z}_j = 0\}, \quad k \in \mathbb{N}. \tag{4.2}$$

Let us emphasize that the extinctions of Z in the generations other than S_1, $S_2, \dots$ are ignored. Set

$$\overline{\mathbb{W}}_{\tau_n} := W_{S_{\tau_n}} - W_{S_{\tau_{n-1}}}, \quad n \in \mathbb{N}$$

and note that $(\overline{\mathbb{W}}_{\tau_n}, \tau_n - \tau_{n-1})_{n\in\mathbb{N}}$ are iid random vectors under $\mathbb{P}$. Since the random variables in question are non-negative we have, for $n \in \mathbb{N}$,

$$\sum_{k=1}^{\tau_n^*} \overline{\mathbb{W}}_{\tau_k} \le \sum_{k=1}^{S_n} Z_k \le \sum_{k=1}^{\tau_n^*+1} \overline{\mathbb{W}}_{\tau_k} \quad \mathbb{P}-\text{a.s.}, \tag{4.3}$$

where τ_n^* is the number of extinctions of Z in the generations $S_0, \ldots, S_n$, that is,

$$\tau_n^* := \max\{k \ge 0 : \tau_k \le n\}, \quad n \in \mathbb{N}.$$

Lemma 4.1 given next states that the extinctions occur rather often.

Lemma 4.1. *Assume that* $\mathbb{E}\log\rho \in [-\infty, 0)$ *and* $\mathbb{E}\log\xi < \infty$. *Then* $\mathbb{E}\tau_1 < \infty$. *If additionally* $\mathbb{E}\rho^\varepsilon < \infty$ *and* $\mathbb{E}\xi^\varepsilon < \infty$ *for some* $\varepsilon > 0$, *then* $\mathbb{E}\exp(\gamma\tau_1) < \infty$ *for some* $\gamma > 0$.

The proof of this lemma can be found in the Appendix of [6]. Under the assumptions of our main results Lemma 4.1 ensures that $\mathtt{m} = \mathbb{E}\tau_1 < \infty$. The strong law of large numbers for renewal processes $(\tau_n^*)_{n\in\mathbb{N}_0}$ makes it plausible that, for large n,

$$W_{S_n} \approx \sum_{k=1}^{[\mathtt{m}^{-1}n]} \overline{\mathbb{W}}_{\tau_k}.$$

The right-hand side, properly centered and normalized, converges in distribution if, and only if, the distribution of $\overline{\mathbb{W}}_{\tau_1}$ belongs to the domain of attraction of a stable law. According to Lemma 5.6, the latter is indeed the case under the assumptions of our theorems.

An important technical ingredient of our proofs is the distribution tail behavior of the vector $(S_{\tau_1}, \overline{\mathbb{W}}_{\tau_1})$. To investigate it we have to discuss the structure of $\overline{\mathbb{W}}_{\tau_1}$ in more details. To this end, for $i \in \mathbb{N}$, we divide particles residing in the generations $S_{i-1}+1, \ldots, S_i$ into groups:

- $\mathcal{P}_{1,i}$ – the progeny residing in the generations $S_{i-1}+1, \ldots, S_i - 1$ of the immigrants arriving in the generations $S_{i-1}, \ldots, S_i - 2$, the number of these being
$$\mathbb{W}_i^0 := \sum_{j=S_{i-1}+1}^{S_i-1} \sum_{k=j}^{S_i-1} Z(j,k);$$
- $\mathcal{P}_{2,i}$ – the progeny residing in the generations $S_{i-1}+1, \ldots, S_i - 1$ of the immigrants arriving in the generations $0, 1, \ldots, S_{i-1} - 1$, the number of these being
$$\mathbb{W}_i^{\downarrow} := \sum_{j=1}^{S_{i-1}} \sum_{k=S_{i-1}+1}^{S_i-1} Z(j,k);$$
- $\mathcal{P}_{3,i}$ – particles of the generation S_i, the number of these being $\mathbb{Z}_i$.

The aforementioned partition of the population which is depicted on Figure 4.1 induces the following decompositions which hold $\mathbb{P}$-a.s.

$$\mathbb{W}_i = \mathbb{W}_i^0 + \mathbb{W}_i^{\downarrow} + \mathbb{Z}_i, \quad i \in \mathbb{N}$$

and

$$\overline{\mathbb{W}}_{\tau_1} = \sum_{i=1}^{\tau_1} \mathbb{W}_i^0 + \sum_{i=1}^{\tau_1} \mathbb{W}_i^{\downarrow} + \sum_{i=1}^{\tau_1} \mathbb{Z}_i. \tag{4.4}$$

Finally, we explain how we are going to treat the second summand in (4.1). We represent $T_n - T_{S_{\nu(n)-1}}$ as the sum of two components: the times spent by $(X_k)_{k=T_{S_{\nu(n)-1}}+1,\ldots,T_n}$ in $(-\infty, S_{\nu(n)-1})$ and $[S_{\nu(n)-1}, n]$, respectively. We shall prove in Lemmas 6.3 and 6.7 below that under the assumptions of our main theorems, the first component is asymptotically negligible. Before presenting our reasoning for the second component we find it convenient to recall a few classical notions and formulate a technical lemma.

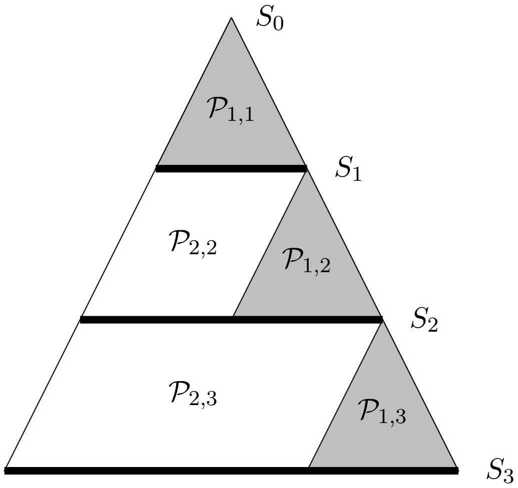

FIGURE 4.1. The generations 0 through S_3 of the BPRE Z and the partition of the corresponding population into parts $\mathcal{P}_{i,j}$, $i, j = 1, 2, 3$. The bold horizontal lines represent particles in the generations S_1, S_2 and S_3, that is, those comprising the groups $\mathcal{P}_{3,i}$, $i = 1, 2, 3$. By definition, $\mathcal{P}_{2,1} = \oslash$.

Let D denotes the Skorokhod space of right-continuous functions defined on $[0, \infty)$ with finite limits from the left at positive points. The two commonly used topologies that the Skorokhod space D is equipped with are J_1- and M_1-topologies. We refer to [2, 19] and [41] for comprehensive accounts of the J_1- and the M_1-topologies, respectively. In the sequel $\overset{\mathrm{J}_1}{\Rightarrow}$ and $\overset{\mathrm{M}_1}{\Rightarrow}$ will mean weak convergence on D when endowed with the J_1-topology and the M_1-topology, respectively. Our main results are one-dimensional distributional limit theorems. However, we find it useful to appeal, at some intermediate steps, to functional limit theorems on D. Working in this more general setting simplifies considerably proofs of limit theorems involving compositions. Theorem 13.2.2 in [41] stated as Lemma 4.2 below provides a necessary technical background.

Lemma 4.2. *Let $k \in \mathbb{N}$. The composition mapping $((x_1, \dots, x_k), \psi) \mapsto (x_1 \circ \psi, \dots, x_k \circ \psi)$ is J_1- continuous at vectors $(x_1, \dots, x_k) \in D^k$ with nonnegative coordinates and nonnegative continuous and strictly increasing ψ.*

Let $(X'_k)_{k\in\mathbb{N}_0}$ be a starting at zero simple random walk with reflection to the right at the origin, that is, $X'_0 = 0$, $\mathbb{P}\{X'_{k+1} = i \pm 1 | X'_k = i\} = 1/2$ for $k, i \in \mathbb{N}$ and $\mathbb{P}\{X'_{k+1} = 1 | X'_k = 0\} = 1$ for $k \in \mathbb{N}_0$. We shall assume that $(X'_k)_{k\in\mathbb{N}_0}$ is independent of $(\xi_j, \rho_j)_{j\in\mathbb{N}}$ and Z. Set

$$T'_n := \inf\{k \in \mathbb{N}_0 : X'_k = n\}, \quad n \in \mathbb{N}_0. \tag{4.5}$$

With this notation at hand we observe that

$$\text{given } \{S_{\nu(n)-1} = j\} \text{ the time spent by } (X_k)_{k=T_{S_{\nu(n)-1}}+1,\dots,T_n} \text{ in } [S_{\nu(n)-1}, n] \tag{4.6}$$
$$\text{has the same distribution as } T'_{n-j}.$$

It is well-known that

$$n^{-1/2} X'_{[n\cdot]} \overset{\mathrm{J}_1}{\Rightarrow} B(\cdot), \quad n \to \infty,$$

where $B := (B(t))_{t\geq 0}$ is a reflected Brownian motion. By a standard inversion argument, this yields

$$n^{-2} T'_{[n\cdot]} \overset{\mathrm{J}_1}{\Rightarrow} M(\cdot), \quad n \to \infty, \tag{4.7}$$

where $M(t) := \inf\{s > 0 : B(s) = t\}$ for $t \geq 0$. By Proposition 3.7 on p. 71 in [34]

$$\mathbb{E}\exp(-sM(t)) = \frac{1}{\cosh(t\sqrt{2s})}, \quad s \geq 0.$$

Recalling (2.4) we conclude that $M(1) \stackrel{\mathrm{d}}{=} 2\vartheta$. These facts explain the appearance of ϑ in Theorem 2.1.

5. The distribution tail behavior of S_{τ_1} and $\mathbb{W}_{\tau_1}$

To prove our main results we have to know the asymptotics of $\mathbb{P}\{S_{\tau_1} > t\}$, $\mathbb{P}\{\mathbb{W}_{\tau_1} > t\}$ and $\mathbb{P}\big\{S_{\tau_1} > g(t)x_1, \overline{\mathbb{W}}_{\tau_1} > f(t)x_2\big\}$ as $t \to \infty$ for suitable functions f and g.

5.1. **The marginal behavior.**

Lemma 5.1. *Assume that condition* (ξ) *holds for* $\beta \in (0,1)$ *and that* $\mathbb{E}\log\rho \in [-\infty, 0)$. *Then*

$$\mathbb{P}\big\{S_{\tau_1} > t\big\} \ \sim \ (\mathbb{E}\tau_1)\mathbb{P}\{\xi > t\} \ \sim \ (\mathbb{E}\tau_1)t^{-\beta}\ell(t), \quad t \to \infty \tag{5.1}$$

and

$$\mathbb{P}\big\{S_{\tau_1} > t\big\} \ \sim \ \mathbb{P}\{\max_{1\leq k\leq \tau_1} \xi_k > t\}, \quad t \to \infty. \tag{5.2}$$

Proof. The assumptions guarantee $\mathbb{E}\log\xi < \infty$ which in turn secures $\mathbb{E}\tau_1 < \infty$ by Lemma 4.1.

The random variable τ_1 *does not depend on the future of the sequence* $(\xi_i)_{i\in\mathbb{N}}$, that is, for each $n \in \mathbb{N}$, the collections of random variables

$$(\xi_1, \ldots, \xi_n, \mathbf{1}_{\{\tau_1 \leq n\}}) \quad \text{and} \quad (\xi_{n+1}, \xi_{n+2}, \ldots)$$

are independent. With this at hand a specialization of Theorem 1 in [25] yields

$$\mathbb{E}(S_{\tau_1} \wedge t) \ \sim \ (\mathbb{E}\tau_1)\mathbb{E}(\xi \wedge t), \quad t \to \infty.$$

By Karamata's theorem (Proposition 1.5.8 in [3])

$$\mathbb{E}(\xi \wedge t) \ \sim \ (1-\beta)^{-1}t^{1-\beta}\ell(t), \quad t \to \infty.$$

An application of the monotone density theorem (Theorem 1.7.2 in [3]) completes the proof of (5.1).

Turning to the proof of (5.2), write

$$\begin{aligned}\mathbb{P}\{\max_{1\leq k\leq \tau_1} \xi_k > t\} &= \sum_{k\geq 1}\mathbb{P}\{\max_{1\leq i\leq k-1} \xi_i \leq t, \xi_k > t, \tau_1 \geq k\} \\ &= \mathbb{P}\{\xi > t\}\sum_{k\geq 1}\mathbb{P}\{\max_{1\leq i\leq k-1} \xi_i \leq t, \tau_1 \geq k\},\end{aligned}$$

where the last equality is a consequence of the fact that τ_1 does not depend on the future of $(\xi_i)_{i\in\mathbb{N}}$. By the dominated convergence theorem

$$\lim_{t\to\infty}\frac{\mathbb{P}\{\max_{1\leq k\leq \tau_1} \xi_k > t\}}{\mathbb{P}\{\xi > t\}} = \mathbb{E}\tau_1.$$

□

It is worth mentioning that Theorem 1 in [25] cited in the previous proof treats standard random walks with two-sided jumps of infinite mean stopped at an arbitrary random variable of finite mean which does not depend on the future of the sequence of jumps. In particular, the regular variation of the distribution tail of a jump is not assumed.

Below we present a collection of auxiliary results borrowed from Section 5 in [6] which will be used in the sequel.

Lemma 5.2 (Lemma 5.1 in [6]). *Assume that* (2.3) *holds with some* $\beta > 0$. *Then*

$$\mathbb{P}\{\mathbb{W}_1^0 > t\} \sim (\mathbb{E}\vartheta^{\beta/2})t^{-\beta/2}\ell(t^{1/2}), \quad t \to \infty,$$

where ϑ *is a random variable with the Laplace transform given in* (2.4).

Lemma 5.3 (Lemma 5.2 in [6]). *Assume that* $\mathbb{E}\log\rho \in [-\infty, 0)$ *and that, for some* $s \leq 2$, $\mathbb{E}(\rho\xi)^s$ *and* $\mathbb{E}\xi^s$ *are finite. Then* $\mathbb{E}\mathbb{Z}_1^s < \infty$ *and there exists a positive constant* C *such that, for all* $n \in \mathbb{N}$,

$$\mathbb{E}\mathbb{Z}_n^s \leq \begin{cases} C & \text{if } \gamma < 1, \\ Cn & \text{if } \gamma = 1, \\ C\gamma^n & \text{if } \gamma > 1, \end{cases}$$

where $\gamma = \mathbb{E}\rho^s$. *If additionally* $\mathbb{E}\xi^{2s} < \infty$, *then*

$$\mathbb{E}\mathbb{W}_1^s < \infty.$$

Lemma 5.4 (Lemma 5.4 in [6]). *Assume that, for some* $s \leq 2$, $\mathbb{E}\rho^s < 1$, $\mathbb{E}(\rho\xi)^s$ *and* $\mathbb{E}\xi^s$ *are finite. Then, for all* $s_0 \in (0, s)$,

$$\mathbb{E}\left(\sum_{i=1}^{\tau_1}\mathbb{Z}_i\right)^{s_0} < \infty.$$

If additionally $\mathbb{E}\xi^{3s/2} < \infty$, *then*

$$\mathbb{E}\left(\sum_{i=1}^{\tau_1}\mathbb{W}_i^{\downarrow}\right)^{s_0} < \infty. \tag{5.3}$$

Lemma 5.5 (Lemma 5.5 in [6]). *Assume that* (ρ1) *holds for some* $\alpha \in (0, 2]$, $\mathbb{E}\xi^{3\alpha/2} < \infty$ *and* $\mathbb{E}(\rho\xi)^\alpha < \infty$. *Then*

$$\mathbb{P}\left\{\sum_{k=1}^{\tau_1}\left(\mathbb{Z}_k + \mathbb{W}_k^{\downarrow}\right) > t\right\} \sim (\mathbb{E}\tau_1)\mathcal{C}_Z(\alpha)t^{-\alpha}, \quad t \to \infty$$

for a positive constant $\mathcal{C}_Z(\alpha)$ *which can be represented as follows:*

$$\mathcal{C}_Z(\alpha) = \lim_{A\to\infty}\mathbb{E}\mathbb{Z}_{\sigma_A}^\alpha \mathbf{1}_{\{\sigma_A < \tau_1\}} \cdot \lim_{x\to\infty} x^\alpha \mathbb{P}\left\{\sum_{k\geq 0}\rho_1\ldots\rho_k\xi_{k+1} > x\right\}.$$

Here, $\sigma_A = \inf\{i \in \mathbb{N} : Z_j > A \text{ for some } j \leq S_i\}$. *Both limits exist and are finite.*

The assertion regarding the form of $\mathcal{C}_Z(\alpha)$ can be derived from the proof of Lemma 5.5 in [6]. Note that an explicit expression for $\mathcal{C}_Z(\alpha)$ is not known.

Lemma 5.6 (Proposition 5.7 in [6]). *The following asymptotic relations hold.*

(C1) *If* (ρ1) *holds for some* $\alpha \in (0, 2]$, *either* $\mathbb{E}\xi^{2\alpha} < \infty$ *or* (2.3) *holds with* $\beta = 2\alpha$, $\lim_{t\to\infty}\ell(t) = 0$, *and* $\mathbb{E}(\rho\xi)^\alpha < \infty$, *then*

$$\mathbb{P}\{\overline{\mathbb{W}}_{\tau_1} > t\} \sim (\mathbb{E}\tau_1)\mathcal{C}_Z(\alpha)t^{-\alpha}, \quad t \to \infty,$$

where $\mathcal{C}_Z(\alpha)$ *is the same constant as in Lemma 5.5.*

(C2) *If* (ρ1) *holds for some* $\alpha \in (0, 2)$, (2.3) *holds with* $\beta = 2\alpha$ *and* $\lim_{t\to\infty}\ell(t) = \mathcal{C}_\ell \in (0, \infty)$, $\mathbb{E}\rho^{\alpha+\varepsilon} < \infty$ *and* $\mathbb{E}\rho^\alpha\xi^{\alpha+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *then*

$$\mathbb{P}\{\overline{\mathbb{W}}_{\tau_1} > t\} \sim (\mathbb{E}\tau_1)\left((\mathbb{E}\vartheta^\alpha)\mathcal{C}_\ell + \mathcal{C}_Z(\alpha)\right)t^{-\alpha}, \quad t \to \infty.$$

(C3) *If* (ρ1) *holds for some* $\alpha \in (0, 2]$, (2.3) *holds with* $\beta = 2\alpha$ *and* $\lim_{t\to\infty}\ell(t) = \infty$, *and* $\mathbb{E}(\rho\xi)^\alpha < \infty$, *then*

$$\mathbb{P}\{\overline{\mathbb{W}}_{\tau_1} > t\} \sim (\mathbb{E}\tau_1)(\mathbb{E}\vartheta^\alpha)t^{-\alpha}\ell(t^{1/2}), \quad t \to \infty.$$

(C4) *If* ($\rho2$) *holds,* (2.3) *holds for some* $\beta \in (0,4)$ *such that* $\beta/2 \in \mathcal{I}$ *and* $\mathbb{E}(\rho\xi)^{\beta/2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$*, then*

$$\mathbb{P}\{\overline{\mathbb{W}}_{\tau_1} > t\} \sim (\mathbb{E}\tau_1)(\mathbb{E}\vartheta^{\beta/2})t^{-\beta/2}\ell(t^{1/2}), \quad t \to \infty.$$

5.2. **The joint behavior.** The asymptotic behavior of $t\mathbb{P}\big\{S_{\tau_1} > g(t)x_1, \overline{\mathbb{W}}_{\tau_1} > f(t)x_2\big\}$ as $t \to \infty$ is determined by the mutual interplay of the distributions of ξ and ρ. While Proposition 5.7 treats the situation in which the distribution of ξ dominates, Proposition 5.8 is concerned with the case in which the contributions of the distributions of ξ and ρ are comparable.

Proposition 5.7. *Assume that the assumptions of Theorem 2.1 are satisfied for* $\beta \in (0,1]$*, with the exception that condition* ($\xi\rho1$) *is not required. Then, for* $x_1, x_2 > 0$*,*

$$\mathbb{P}\big\{S_{\tau_1} > tx_1, \overline{\mathbb{W}}_{\tau_1} > t^2x_2\big\} \sim (\mathbb{E}\tau_1)\mathbb{E}\big[\min\big(x_1^{-\beta}, x_2^{-\beta/2}\vartheta^{\beta/2}\big)\big]\ell(t)t^{-\beta}, \quad t \to \infty,$$

where a random variable ϑ *has the Laplace transform given by* (2.4)*.*

Proposition 5.8. *Assume that the assumptions of Theorem 2.2 are satisfied for* $\beta \in (0,1]$*, with the exception that condition* ($\xi\rho2$) *is not required. Then*

$$\lim_{t\to\infty} t\mathbb{P}\big\{S_{\tau_1} > a(t), \overline{\mathbb{W}}_{\tau_1} > t^{1/\alpha}\big\} = 0.$$

Our proofs of both propositions rely on decomposition (4.4) and 'the principle of one big jump' which is commonly used when analyzing random variables with regularly varying distribution tails. In view of (5.2) the random variable S_{τ_1} takes a large value if and only if at least one of $\xi_1, \xi_2, \ldots, \xi_{\tau_1}$ is large. We shall choose a stopping time $T = T(t)$ such that $\xi_T \approx \max_{1\le k\le\tau_1}\xi_k \approx S_{\tau_1}$ on the event $\big\{\max_{1\le k\le\tau_1}\xi_k > t\big\}$ and then show that $\mathbb{W}_T^0$ dominates all the other terms in decomposition (4.4). According to (3.7) the variable $\mathbb{W}_T^0$ should be of magnitude t^2 on the event $\big\{\max_{1\le k\le\tau_1}\xi_k > t\big\}$ (see Lemma 5.11 for more details). Summarizing, it is plausible that

$$\mathbb{P}\big\{S_{\tau_1} > tx_1, \overline{\mathbb{W}}_{\tau_1} > t^2x_2\big\} \approx \mathbb{P}\big\{\xi_T > tx_1, \mathbb{W}_T^0 > t^2x_2\big\}, \quad t \to \infty. \tag{5.4}$$

The rigorous proofs of Propositions 5.7 and 5.8 are similar to the proof of Proposition 6.1 in [6]. However, since we need a joint, rather than marginal, asymptotic behavior, the details are more involved. We start with a lemma that provides the asymptotic behavior of the right-hand side in (5.4).

Lemma 5.9. *Let* ς *be an integer-valued random variable independent of* $(W_n^{\rm crit})_{n\in\mathbb{N}_0}$ *and such that*

$$\mathbb{P}\{\varsigma > t\} \sim t^{-\beta}\ell(t), \quad t \to \infty$$

for some $\beta > 0$ *and some* ℓ *slowly varying at* ∞*. Then, for* $x_1, x_2 > 0$*,*

$$\mathbb{P}\big\{\varsigma > tx_1, W_\varsigma^{\rm crit} > t^2x_2\big\} \sim \mathbb{E}\big[\min\big(x_1^{-\beta}, x_2^{-\beta/2}\vartheta^{\beta/2}\big)\big]\ell(t)t^{-\beta}, \quad t \to \infty,$$

where ϑ *is a random variable with the Laplace transform given in* (2.4)*.*

Proof. Put

$$v_x = \inf\{k \in \mathbb{N} : W_k^{\rm crit} > x\}, \quad x > 1.$$

Since $W_n^{\rm crit}$ is monotone, it diverges to $+\infty$ a.s. This ensures that v_x is finite a.s. For fixed $x_1, x_2 > 0$ and sufficiently large t,

$$\mathbb{P}\big\{\varsigma > tx_1, W_\varsigma^{\rm crit} > t^2x_2\big\} = \mathbb{P}\big\{\varsigma > \max(tx_1, v_{t^2x_2})\big\} = \mathbb{E}R\big(\max(tx_1, v_{t^2x_2})\big),$$

where $R(y) = \mathbb{P}\{\varsigma > y\}$ for $y > 0$. An application of a standard inversion technique to (3.7) yields

$$t^{-1/2}v_t \xrightarrow{\rm d} \vartheta^{-1/2}, \quad t \to \infty.$$

Hence,

$$t^{-1}\max(tx_1, v_{t^2x_2}) \xrightarrow{\rm d} \max(x_1, x_2^{1/2}\vartheta^{-1/2}), \quad t\to\infty$$

and subsequently

$$\frac{R\big(\max(tx_1, v_{t^2x_2})\big)}{R(t)} \xrightarrow{\rm d} \big[\max(x_1, x_2^{1/2}\vartheta^{-1/2})\big]^{-\beta} = \min\big(x_1^{-\beta}, x_2^{-\beta/2}\vartheta^{\beta/2}\big), \quad t\to\infty$$

having utilized the regular variation of R. Write

$$\frac{R\big(\max(tx_1, v_{t^2x_2})\big)}{R(t)} \le \frac{R(tx_1)}{R(t)} + \frac{R\big(v_{t^2x_2}\big)}{R\big(tx_2^{1/2}\big)}\frac{R\big(tx_2^{1/2}\big)}{R(t)}.$$

It is shown in the proof of Proposition 6.1 in [6] that the family $\Big(\frac{R(v_{t^2x_2})}{R(tx_2^{1/2})}\Big)_{t\ge t_0}$ is uniformly integrable for large enough $t_0>0$. This in combination with Potter's bound for regularly varying functions (Theorem 1.5.6(iii) in [3]) enables us to conclude that the family $\Big(\frac{R(\max(tx_1, v_{t^2x_2}))}{R(t)}\Big)_{t\ge t_1}$ is uniformly integrable for large enough $t_1>0$. Therefore,

$$\frac{\mathbb{P}\{\varsigma > tx_1, W_\varsigma^{\rm crit} > t^2x_2\}}{\mathbb{P}\{\varsigma>t\}} = \frac{\mathbb{E}R(\max(tx_1, v_{t^2x_2}))}{R(t)} \to \mathbb{E}\big[\min\big(x_1^{-\beta}, x_2^{-\beta/2}\vartheta^{\beta/2}\big)\big], \quad t\to\infty$$

which completes the proof. □

Some parts of the proofs of Propositions 5.7 and 5.8 can be treated along similar lines. As a preparation, we prove an auxiliary result.

Lemma 5.10. *Assume that either the assumptions of Theorem 2.1, with the case (A) and the condition ($\xi\rho1$) being excluded, or Theorem 2.2, with the condition ($\xi\rho2$) being excluded, are satisfied for $\beta\in(0,1]$. Let $\delta\in(0,\alpha)$ and b_1, b_2 and b_3 be positive functions diverging to $+\infty$. Then, as $t\to\infty$, uniformly in $k\in\mathbb{N}$,*

$$\mathbb{P}\Big\{\xi_k > b_1(t), k\le\tau_1, \sum_{i=1}^{k-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) > b_2(t)\Big\} = O\big(\mathbb{P}\{\xi>b_1(t)\}b_2(t)^{-\alpha}\big);$$

$$\mathbb{P}\big\{\xi_k > b_1(t), \mathbb{W}_k^{\downarrow} > b_2(t)\big\} = O\big(\mathbb{E}[\xi^\delta \mathbf{1}_{\{\xi>b_1(t)\}}]b_2(t)^{-\delta}\big)$$

and, uniformly in $k=1,2,\dots,[b_3(t)]$,

$$\mathbb{P}\Big\{\xi_k > b_1(t),\ \mathbb{Z}_k + \sum_{j=1}^{\mathbb{Z}_k} Y^*_{j,S_k} > b_2(t)\Big\} = O\big(\mathbb{P}\{\xi>b_1(t)\}^{\varepsilon/(\alpha+\varepsilon)}b_2(t)^{-\alpha}b_3(t)\big)$$

with the same ε as defined in Theorems 2.1 and 2.2. Here, for $j\in\mathbb{N}$, $j\le\mathbb{Z}_k$, Y^*_{j,S_k} denotes the total progeny of the jth particle in the generation S_k.

Proof. The first relation is justified as follows: uniformly in $k\in\mathbb{N}$,

$$\mathbb{P}\Big\{\xi_k > b_1(t), k\le\tau_1, \sum_{i=1}^{k-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) > b_2(t)\Big\}$$

$$= \mathbb{P}\{\xi_k > b_1(t)\}\mathbb{P}\Big\{k\le\tau_1,\ \sum_{i=1}^{k-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) > b_2(t)\Big\}$$

$$\le \mathbb{P}\{\xi>b_1(t)\}\mathbb{P}\Big\{\sum_{i=1}^{\tau_1}\big(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}\big) > b_2(t)\Big\} = O\big(\mathbb{P}\{\xi>b_1(t)\}b_2(t)^{-\alpha}\big), \quad t\to\infty,$$

where the second inequality follows from the fact that τ_1 does not depend on the future of $(\xi_i)_{i\in\mathbb{N}}$, and the last equality is a consequence of Lemma 5.5.

While treating the second relation we use a representation

$$\mathbb{W}_1^{\downarrow}=0,\quad \mathbb{W}_k^{\downarrow}=\sum_{i=1}^{\mathbb{Z}_{k-1}}D_i^{(k)},\quad k\geq 2,\ \mathbb{P}-\text{a.s.},$$

where $D_i^{(k)}$ is the number of progeny in the generations $S_{k-1}+1,\dots,S_k-1$ of the ith particle in the generation S_{k-1}. For fixed $k\geq 2$, under $\mathbb{P}_\omega$, $D_1^{(k)}$, $D_2^{(k)},\dots$ are iid and independent of $\mathbb{Z}_{k-1}$, and one can check that $\mathbb{E}_\omega[D_i^{(k)}]=\xi_k-1$. With this at hand we write, for any $\delta\in(0,\alpha)$ and any $k\in\mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}\big\{\xi_k>b_1(t),\mathbb{W}_k^{\downarrow}>b_2(t)\big\}\leq b_2(t)^{-\delta}\mathbb{E}\Bigg[\mathbf{1}_{\{\xi_k>b_1(t)\}}\Bigg(\sum_{i=1}^{\mathbb{Z}_{k-1}}D_i^{(k)}\Bigg)^{\delta}\Bigg]\\
&=b_2(t)^{-\delta}\mathbb{E}\Bigg[\mathbf{1}_{\{\xi_k>b_1(t)\}}\mathbb{E}_\omega\Bigg[\Bigg(\sum_{i=1}^{\mathbb{Z}_{k-1}}D_i^{(k)}\Bigg)^{\delta}\Bigg|\mathbb{Z}_{k-1}\Bigg]\Bigg]\\
&\leq b_2(t)^{-\delta}\mathbb{E}\Bigg[\mathbf{1}_{\{\xi_k>b_1(t)\}}\mathbb{E}_\omega\Bigg[\sum_{i=1}^{\mathbb{Z}_{k-1}}D_i^{(k)}\Bigg|\mathbb{Z}_{k-1}\Bigg]^{\delta}\Bigg]\\
&\leq b_2(t)^{-\delta}\mathbb{E}[\xi^{\delta}\mathbf{1}_{\{\xi>b_1(t)\}}]\mathbb{E}[\mathbb{Z}_{k-1}^{\delta}],
\end{aligned}$$

where the first line is obtained with the help of Markov's inequality, and the penultimate line follows by an application of the conditional Jensen's inequality. Since $\mathbb{E}\rho^{\delta}<1$, an appeal to Lemma 5.3 yields $\sup_{k\geq 1}\mathbb{E}[\mathbb{Z}_k^{\delta}]<\infty$.

Turning to the analysis of the third relation we first note that, for fixed $k\in\mathbb{N}$, Y_{1,S_k}^*, $Y_{2,S_k}^*,\dots$ are $\mathbb{P}_\omega$-independent of copies of Y_1 which are $\mathbb{P}$-independent of ξ_k. Therefore, according to Lemma 7.2 in [6], there exists a (nonrandom) constant $A>0$ such that, for $x>0$,

$$\mathbb{P}\Bigg\{\sum_{j=1}^{\mathbb{Z}_k}Y_{j,S_k}^*>x\Bigg|\mathbb{Z}_k,\xi_k\Bigg\}\leq A\mathbb{Z}_k^{\alpha}x^{-\alpha}\quad\mathbb{P}-\text{a.s.}\tag{5.5}$$

Also, it can be checked (see the proof of Lemma 5.2 in [6] for details) that, for $k\in\mathbb{N}$,

$$\mathbb{E}_\omega\mathbb{Z}_k=\rho_k\mathbb{E}_\omega\mathbb{Z}_{k-1}+\rho_k\xi_k\leq(1+\mathbb{E}_\omega\mathbb{Z}_{k-1})\rho_k\xi_k,\quad\mathbb{P}-\text{a.s.}\tag{5.6}$$

Here, the last inequality follows from $\xi_k\geq 1$ $\mathbb{P}$-a.s. Write, for $k\in\mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}\Bigg\{\xi_k>b_1(t),\ \mathbb{Z}_k+\sum_{j=1}^{\mathbb{Z}_k}Y_{j,S_k}^*>b_2(t)\Bigg\}\\
&\leq\mathbb{P}\Big\{\xi_k>b_1(t),\ \mathbb{Z}_k>2^{-1}b_2(t)\Big\}+\mathbb{P}\Bigg\{\xi_k>b_1(t),\ \sum_{j=1}^{\mathbb{Z}_k}Y_{j,S_k}^*>2^{-1}b_2(t)\Bigg\}\\
&\leq\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}\mathbb{P}_\omega\Big\{\mathbb{Z}_k>2^{-1}b_2(t)\Big\}+\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}\mathbb{P}\Bigg\{\sum_{j=1}^{\mathbb{Z}_k}Y_{j,S_k}^*>2^{-1}b_2(t)\Bigg|\mathbb{Z}_k,\xi_k\Bigg\}\\
&\leq 2^{\alpha}b_2(t)^{-\alpha}\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}\mathbb{E}_\omega\mathbb{Z}_k^{\alpha}+2^{\alpha}Ab_2(t)^{-\alpha}\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}\mathbb{Z}_k^{\alpha}\\
&=2^{\alpha}(1+A)b_2(t)^{-\alpha}\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}\mathbb{E}_\omega\mathbb{Z}_k^{\alpha}\leq 2^{\alpha}(1+A)b_2(t)^{-\alpha}\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}(\mathbb{E}_\omega\mathbb{Z}_k)^{\alpha}
\end{aligned}$$

having utilized Markov's inequality for the third line, inequality (5.5) for the fourth and the conditional Jensen's inequality (observe that $\alpha\in(0,1/2]$) for the fifth. Further, for

$k=1,2,\dots,[b_3(t)]$,

$$\begin{aligned}&\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}(\mathbb{E}_\omega\mathbb{Z}_k)^\alpha\\&\leq\mathbb{E}(1+\mathbb{E}_\omega\mathbb{Z}_{k-1})^\alpha\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}(\rho_k\xi_k)^\alpha\\&\leq\Big(1+\mathbb{E}(\rho\xi)^\alpha\Big(\sum_{j=0}^{k-1}\mathbb{E}\rho^\alpha\Big)\Big)\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}(\rho_k\xi_k)^\alpha\leq(1+\mathbb{E}(\rho\xi)^\alpha)k\mathbb{E}\mathbf{1}_{\{\xi_k>b_1(t)\}}(\rho_k\xi_k)^\alpha\\&\leq(1+\mathbb{E}(\rho\xi)^\alpha)b_3(t)\mathbb{E}\mathbf{1}_{\{\xi>b_1(t)\}}(\rho\xi)^\alpha\\&\leq(1+\mathbb{E}(\rho\xi)^\alpha)\Big(\mathbb{E}(\rho\xi)^{\alpha+\varepsilon}\Big)^{\alpha/(\alpha+\varepsilon)}b_3(t)\mathbb{P}\{\xi>b_1(t)\}^{\varepsilon/(\alpha+\varepsilon)},\end{aligned}$$

where the second line follows from (5.6), and the last line is obtained with the help of Hölder's inequality. Combining pieces together completes the proof of the third relation. □

Fix $x_1>0$ and define the stopping time

$$T=T(t)=\inf\{i\in\mathbb{N}:\xi_i>(t-t^{3/4})x_1\},$$

where, as usual, $\inf\varnothing=\infty$. Put

$$\mathbb{W}^0=\sum_{i=1}^{\tau_1}\mathbb{W}_i^0$$

and note that, for $i\in\mathbb{N}$, given ξ_i,

$$\mathbb{W}_i^0\overset{\mathrm{d}}{=}W_{\xi_i-1}^{\mathrm{crit}},\tag{5.7}$$

where $W_0^{\mathrm{crit}}=0$, W_n^{crit} for $n\in\mathbb{N}$ is the total progeny in the first n generations of Z^{crit} and ξ_i is assumed independent of $(W_n^{\mathrm{crit}})_{n\in\mathbb{N}_0}$. As a consequence, the random variables $\mathbb{W}_1^0$, $\mathbb{W}_2^0,\dots$ are identically distributed. Also, it is clear that they are independent.

Lemma 5.11. *Under the assumptions of Proposition 5.7 there exists a constant C such that, for $x_1,x_2>0$, as $t\to\infty$,*

$$\begin{aligned}\mathbb{P}\big\{S_{\tau_1}>tx_1,\mathbb{W}^0>t^2x_2\big\}=\mathbb{P}\big\{&S_{\tau_1}>tx_1,T\leq\tau_1<C\log t,\mathbb{W}_T^0>(t-t^{3/4})^2x_2\\&\text{and }\xi_i\leq t^{2/3}x_1,\mathbb{W}_i^0\leq t^{5/3}x_2\text{ for all }i\neq T\big\}+o(t^{-\beta}).\end{aligned}$$

Proof. The assumptions ensure that $\mathbb{E}\rho^\varepsilon<\infty$ and $\mathbb{E}\xi^\varepsilon<\infty$ for some $\varepsilon>0$. Hence, $\mathbb{E}e^{r\tau_1}<\infty$ for some $r>0$ by Lemma 4.1.

We start by proving a similar statement for the first coordinate alone: as $t\to\infty$,

$$\mathbb{P}\{S_{\tau_1}>tx_1\}=\mathbb{P}\big\{S_{\tau_1}>tx_1,T\leq\tau_1<C\log t\text{ and }\xi_i\leq t^{2/3}x_1\text{ for all }i\neq T\big\}+o(t^{-\beta})\tag{5.8}$$

for any $C>2\beta/r$. By Markov's inequality

$$\mathbb{P}\{\tau_1\geq C\log t\}\leq t^{-Cr}\mathbb{E}e^{r\tau_1}=o(t^{-2\beta}),\quad t\to\infty.\tag{5.9}$$

Further, since

$$\begin{aligned}\mathbb{P}\big\{S_{\tau_1}>tx_1,\tau_1<C\log t\ \text{ and }\ \xi_i>t^{2/3}x_1,\xi_j>t^{2/3}x_1\ \text{ for some }i<j\leq\tau_1\big\}\\\leq2^{-1}C^2(\log t)^2\mathbb{P}\big\{\xi>t^{2/3}x_1\big\}^2=o(t^{-\beta}),\end{aligned}$$

we conclude that, as $t\to\infty$,

$$\begin{aligned}\mathbb{P}\{S_{\tau_1}>tx_1\}=\mathbb{P}\big\{S_{\tau_1}>tx_1,\tau_1<C\log t,\xi_j>(t-t^{3/4})x_1\text{ for some unique }j\leq\tau_1\\\text{and }\xi_i\leq t^{2/3}x_1\text{ for all }i\leq\tau_1,i\neq j\big\}+o(t^{-\beta})\end{aligned}$$

because the sum S_{τ_1} must exceed tx_1. Therefore, on the event $\{S_{\tau_1}>tx_1\}$ with t large enough we have $T=j\leq\tau_1$ and thereupon $T\leq\tau_1$ which in turn yields (5.8).

Analogously we can prove that, as $t \to \infty$,

$$\mathbb{P}\big\{S_{\tau_1} > tx_1, \mathbb{W}^0 > t^2 x_2\big\} = \mathbb{P}\big\{S_{\tau_1} > tx_1, \mathbb{W}^0 > t^2 x_2, T \le \tau_1 < C\log t \text{ and } \xi_i \le t^{2/3}x_1 \text{ for all } i \ne T\big\} + o(t^{-\beta}).$$

Choosing $s > 3\beta$ and appealing to (5.7) and (3.8) yields

$$\begin{aligned}
&\mathbb{P}\big\{\tau_1 < C\log t, \xi_i \le t^{2/3}x_1, \mathbb{W}_i^0 > t^{5/3}x_2 \text{ for some } i \le \tau_1\big\} \\
&\le \sum_{i=1}^{[C\log t]} \mathbb{P}\big\{\xi_i \le t^{2/3}x_1, \mathbb{W}_i^0 > t^{5/3}x_2\big\} \le C\log t\,\mathbb{P}\{W^{\rm crit}_{[t^{2/3}x_1]} > t^{5/3}x_2\} \\
&\le Cx_2^{-s}\log t \frac{\mathbb{E}(W^{\rm crit}_{[t^{2/3}x_1]})^s}{t^{4s/3}} \cdot \frac{t^{4s/3}}{t^{5s/3}} = o(t^{-\beta}), \quad t \to \infty
\end{aligned}$$

having utilized Markov's inequality for the last line. Since each $\mathbb{W}_i^0$ for $i \ne T$ does not exceed $t^{5/3}x_2$, the variable $\mathbb{W}_T^0 = \sum_{i=1}^{T-1}\mathbb{W}_i^0 + \mathbb{W}_T^0$ can only be larger than t^2x_2 for large t provided that the summand $\mathbb{W}_T^0$ is larger than $(t - t^{3/4})^2 x_2$. This completes the proof of the lemma. □

Proof of Proposition 5.7. Our proof consists of two steps. First we show that, for $x_1, x_2 > 0$,

$$\mathbb{P}\{S_{\tau_1} > tx_1, \mathbb{W}^0 > t^2x_2\} \sim (\mathbb{E}\tau_1)\mathbb{E}\big[\min\big(x_1^{-\beta}, x_2^{-\beta/2}\vartheta^{\beta/2}\big)\big]\ell(t)t^{-\beta}, \quad t \to \infty \tag{5.10}$$

and then that

$$\mathbb{P}\{S_{\tau_1} > tx_1, \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^2x_2\} = o\big(\ell(t)t^{-\beta}\big), \quad t \to \infty. \tag{5.11}$$

Proof of (5.10). Due to Lemma 5.11 we are left with investigating the asymptotic behavior of

$$P(t) = \mathbb{P}\Big\{S_{\tau_1} > tx_1, T \le \tau_1 < C\log t, \mathbb{W}_T^0 > (t - t^{3/4})^2 x_2 \text{ and } \xi_i \le t^{2/3}x_1, \mathbb{W}_i^0 \le t^{5/3}x_2 \text{ for all } i \ne T\Big\},$$

where C is a constant.

We need a more general version of the observation made in the proof of Lemma 5.1: for each $n \in \mathbb{N}$, the families $((\xi_k, \mathbb{W}_k^0)_{k\le n}, \mathbf{1}_{\{\tau_1 \le n\}})$ and $(\xi_k, \mathbb{W}_k^0)_{k>n}$ are independent, that is, the random variable τ_1 does not depend on the future of the sequence $(\xi_i, \mathbb{W}_i^0)_{i\in\mathbb{N}}$. This in combination with distributional equality (5.7) and Lemma 5.9 yields

$$\begin{aligned}
P(t) &\le \sum_{j\ge1}\mathbb{P}\Big\{\xi_j > (t - t^{3/4})x_1, \mathbb{W}_j^0 > (t - t^{3/4})^2x_2, \tau_1 \ge j\Big\} \\
&= \sum_{j\ge1}\mathbb{P}\Big\{\xi_j > (t - t^{3/4})x_1, \mathbb{W}_j^0 > (t - t^{3/4})^2x_2\Big\}\mathbb{P}\{\tau_1 \ge j\} \\
&= \mathbb{P}\Big\{\xi_1 > (t - t^{3/4})x_1, W^{\rm crit}_{\xi_1 - 1} > (t - t^{3/4})^2x_2\Big\}\sum_{j\ge1}\mathbb{P}\{\tau_1 \ge j\} \\
&\sim (\mathbb{E}\tau_1)\mathbb{E}\big[\min\big(x_1^{-\beta}, x_2^{-\beta/2}\vartheta^{\beta/2}\big)\big]t^{-\beta}\ell(t), \quad t \to \infty.
\end{aligned}$$

To obtain a lower bound for $P(t)$ we shall use the following inequality

$$\mathbb{P}\{\xi_1 \le t^{2/3}x_1, \mathbb{W}_1^0 \le t^{5/3}x_2\} \ge 1 - \mathbb{P}\{\xi_1 > t^{2/3}x_1\} - \mathbb{P}\{\mathbb{W}_1^0 > t^{5/3}x_2\} \ge 1 - t^{-\beta/3} \tag{5.12}$$

which holds by virtue of (2.3) and Lemma 5.2 for t large enough. Recalling (5.7) and appealing again to the fact that τ_1 does not depend on the future of the sequence $\{(\xi_k, \mathbb{W}_k^0)\}_{k\in\mathbb{N}}$

we obtain

$$\begin{aligned}
P(t) &\geq \sum_{j\geq 1} \mathbb{P}\big\{j \leq \tau_1 < C\log t, \xi_j > tx_1, \mathbb{W}_j^0 > t^2 x_2 \\
&\qquad\qquad\qquad \text{and } \xi_i \leq t^{2/3}x_1, \mathbb{W}_i^0 \leq t^{5/3}x_2 \text{ for all } i \neq j\big\} \\
&\geq \sum_{j\geq 1} \mathbb{P}\big\{j \leq \tau_1 < C\log t, \xi_i \leq t^{2/3}x_1, \mathbb{W}_i^0 \leq t^{5/3}x_2 \text{ for all } i < j\big\} \\
&\qquad \mathbb{P}\big\{\xi_j > tx_1, \mathbb{W}_j^0 > t^2x_2\big\}\mathbb{P}\big\{\xi_i \leq t^{2/3}x_1, \mathbb{W}_i^0 \leq t^{5/3}x_2 \text{ for all } j < i < C\log t\big\} \\
&\geq \mathbb{P}\big\{\xi_1 > tx_1, W_{\xi_1 - 1}^{\rm crit} > t^2x_2\big\}(1-t^{-\beta/3})^{C\log t} \\
&\quad \cdot \sum_{j\geq 1} \mathbb{P}\big\{j \leq \tau_1 < C\log t, \xi_i \leq t^{2/3}x_1, \mathbb{W}_i^0 \leq t^{5/3}x_2 \text{ for all } i < j\big\}.
\end{aligned}$$

In view of Lemma 5.9 it remains to note that

$$\liminf_{t\to\infty} \sum_{j\geq 1} \mathbb{P}\big\{j \leq \tau_1 < C\log t, \xi_i \leq t^{2/3}x_1, \mathbb{W}_i^0 \leq t^{5/3}x_2 \text{ for all } i < j\big\} \geq \mathbb{E}\tau_1$$

by Fatou's lemma.

Proof of (5.11). In the case (A) of Theorem 2.1 relation (5.11) is just a consequence of Lemma 5.4 and Markov's inequality:

$$\begin{aligned}
&\mathbb{P}\big\{S_{\tau_1} > tx_1, \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^2x_2\big\} \\
&\leq \mathbb{P}\bigg\{\sum_{i=1}^{\tau_1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) > t^2x_2\bigg\} \\
&\leq x_2^{-(\beta+\gamma)/2}\mathbb{E}\bigg(\sum_{i=1}^{\tau_1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow})\bigg)^{(\beta+\gamma)/2} \cdot t^{-(\beta+\gamma)} \\
&\leq x_2^{-(\beta+\gamma)/2}\bigg(\mathbb{E}\bigg(\sum_{i=1}^{\tau_1}\mathbb{Z}_i\bigg)^{(\beta+\gamma)/2} + \mathbb{E}\bigg(\sum_{i=1}^{\tau_1}\mathbb{W}_i^{\downarrow}\bigg)^{(\beta+\gamma)/2}\bigg) \cdot t^{-(\beta+\gamma)},
\end{aligned}$$

where $\gamma > 0$ is small enough (in particular, $\beta + \gamma < 2$), and the last inequality is justified by subadditivity of $s \mapsto s^{(\beta+\gamma)/2}$ for $s \geq 0$.

Assume from now on that either the case (B1) or (B2) of Theorem 2.1 prevails. Arguing as in the proof of Lemma 5.11 one can check that it is sufficient to show that, for any $x_1, x_2 > 0$ and a constant $C > 0$,

$$\mathbb{P}\big\{\xi_T > tx_1, T \leq \tau_1 \leq C\log t,\ \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^2x_2\big\} \;=\; o\big(\ell(t)t^{-\beta}\big), \quad t \to \infty. \tag{5.13}$$

Observe that decomposition (4.4) implies that on the event $\{T \leq \tau_1\}$

$$\overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 = \sum_{i=1}^{T-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) + \mathbb{W}_T^{\downarrow} + \mathbb{Z}_T + \sum_{j=1}^{\mathbb{Z}_T} Y_{j,S_T}^* + \sum_{k=S_T+1}^{S_{\tau_1}} Y_k,$$

where, for $j \in \mathbb{N}$, $j \le \mathbb{Z}_T$, Y^*_{j,S_T} denotes the total progeny of the jth particle in the generation S_T. Thus, to ensure (5.13) it is sufficient to check that, as $t \to \infty$,

$$I_1(t) = \mathbb{P}\Big\{\xi_T > tx_1, T \le \tau_1 < C\log t,\ \sum_{i=1}^{T-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) > t^2x_2\Big\} = o\big(\ell(t)t^{-\beta}\big),$$
$$I_2(t) = \mathbb{P}\big\{\xi_T > tx_1, T \le \tau_1 < C\log t,\ \mathbb{W}_T^{\downarrow} > t^2x_2\big\} = o\big(\ell(t)t^{-\beta}\big),$$
$$I_3(t) = \mathbb{P}\Big\{\xi_T > tx_1, T \le \tau_1 < C\log t,\ \mathbb{Z}_T + \sum_{j=1}^{\mathbb{Z}_T} Y^*_{j,S_T} > t^2x_2\Big\} = o\big(\ell(t)t^{-\beta}\big),$$
$$I_4(t) = \mathbb{P}\Big\{\xi_T > tx_1, T \le \tau_1 < C\log t,\ \sum_{k=S_T+1}^{S_{\tau_1}} Y_k > t^2x_2\Big\} = o\big(\ell(t)t^{-\beta}\big).$$

To treat $I_1(t)$, $I_2(t)$ and $I_3(t)$ we use Lemma 5.10 with $b_1(t) = tx_1$, $b_2(t) = t^2x_2$ and $b_3(t) = C\log t$. Recalling that $\alpha = \beta/2$ we obtain

$$I_1(t) \le \sum_{k=1}^{[C\log t]} \mathbb{P}\Big\{\xi_k > tx_1,\ \max_{1\le i\le k-1} \xi_i \le (t - t^{3/4})x_1, k \le \tau_1, \sum_{i=1}^{k-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) > t^2x_2\Big\}$$
$$= O\big(\mathbb{P}\{\xi > tx_1\}t^{-\beta}\log t\big) = o\big(\ell(t)t^{-\beta}\big), \quad t \to \infty.$$

Further, for any $\delta \in (0, \beta/2)$,

$$I_2(t) \le \sum_{k=2}^{[C\log t]} \mathbb{P}\big\{\xi_k > tx_1, \mathbb{W}_k^{\downarrow} > t^2x_2\big\} = O\big(\mathbb{E}\big[\xi^\delta \mathbf{1}_{\{\xi > tx_1\}}\big]t^{-2\delta}\log t\big), \quad t \to \infty.$$

According to Karamata's theorem (Theorem 1.6.5 in [3]) the function $t \mapsto \mathbb{E}\big[\xi^\delta \mathbf{1}_{\{\xi>tx_1\}}\big] \times t^{-2\delta}\log t$ is regularly varying at ∞ of index $-\beta-\delta$, whence $I_2(t) = o(t^{-\beta}\ell(t))$ as $t \to \infty$. Passing to $I_3(t)$ we infer

$$I_3(t) \le \sum_{k=1}^{[C\log t]} \mathbb{P}\Big\{\xi_k > tx_1,\ \mathbb{Z}_k + \sum_{j=1}^{\mathbb{Z}_k} Y^*_{j,S_k} > t^2x_2\Big\} = O\big(\mathbb{P}\{\xi > tx_1\}^{\varepsilon/(\alpha+\varepsilon)}t^{-\beta}(\log t)^2\big)$$
$$= o\big(t^{-\beta}\ell(t)\big), \quad t \to \infty.$$

Finally, the relation for $I_4(t)$ holds true just because ξ_T and $\sum_{k=S_T+1}^{S_{\tau_1}} Y_k$ are independent. □

Proof of Proposition 5.8. Recalling the asymptotic relation obtained in Lemma 5.2 and arguing as in the proof of Lemma 5.1 we conclude that

$$\mathbb{P}\Big\{\mathbb{W}^0 = \sum_{k=1}^{\tau_1} \mathbb{W}_k^0 > t\Big\} \ \sim\ (\mathbb{E}\tau_1)\mathbb{P}\{\mathbb{W}_1^0 > t\}, \quad t \to \infty$$

which implies

$$\mathbb{P}\left\{\mathbb{W}^0 > t^{1/\alpha}\right\} \ \sim\ (\mathbb{E}\tau_1)(\mathbb{E}\vartheta^{\beta/2})\ell\big(t^{1/(2\alpha)}\big)t^{-\beta/2\alpha} = o(t^{-1}), \quad t \to \infty.$$

Thus, it is sufficient to prove

$$\mathbb{P}\big\{S_{\tau_1} > a(t), \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^{1/\alpha}\big\} = o(t^{-1}), \quad t \to \infty.$$

In view of $\mathbb{P}\big\{\max_{1\le k\le\tau_1} \xi_k > t\big\} \sim \mathbb{P}\{S_{\tau_1} > t\}$ as $t \to \infty$ (see (5.2)),

$$\mathbb{P}\{S_{\tau_1} > t, \max_{1\le k\le\tau_1} \xi_k \le t\} = \mathbb{P}\{S_{\tau_1} > t\} - \mathbb{P}\{\max_{1\le k\le\tau_1} \xi_k > t\} = o(\mathbb{P}\{S_{\tau_1} > t\}),$$

whence

$$\mathbb{P}\big\{S_{\tau_1} > a(t),\ \max_{1\le k\le \tau_1} \xi_k \le a(t), \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^{1/\alpha}\big\} = o(t^{-1}),\quad t\to\infty.$$

As a consequence, we are left with showing that

$$\mathbb{P}\big\{\max_{1\le k\le \tau_1} \xi_k > a(t), \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^{1/\alpha}\big\} = o(t^{-1}),\quad t\to\infty.$$

By Lemma 4.1, $\mathbb{E}\exp(r\tau_1)<\infty$ for some $r>0$. This implies that there exists $C>0$ such that $\sum_{k>[C\log t]}\mathbb{P}\{\tau_1\ge k\} = o(t^{-1})$ as $t\to\infty$ and thereupon

$$\mathbb{P}\big\{\max_{1\le k\le \tau_1} \xi_k > a(t), \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^{1/\alpha}\big\}$$
$$\le \sum_{k=1}^{[C\log t]} \mathbb{P}\big\{\xi_k > a(t),\ k\le \tau_1, \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^{1/\alpha}\big\} + o(t^{-1}),\quad t\to\infty.$$

To ensure that the first summand on the right-hand side is $o(t^{-1})$ it is more than sufficient if we can check that, for some $\gamma>0$,

$$\mathbb{P}\{\xi_k > a(t),\ k\le\tau_1,\ \overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 > t^{1/\alpha}\} = o(t^{-1-\gamma}),\quad t\to\infty$$

uniformly in $k=1,\dots,[C\log t]$. The latter is accomplished by making use of a decomposition similar to the one used in the proof of Proposition 5.7, namely: on the event $\{k\le\tau_1\}$,

$$\overline{\mathbb{W}}_{\tau_1} - \mathbb{W}^0 = \sum_{i=1}^{k-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) + \mathbb{W}_k^{\downarrow} + \mathbb{Z}_k + \sum_{j=1}^{\mathbb{Z}_k} Y^*_{j,S_k} + \sum_{j=S_k+1}^{S_{\tau_1}} Y_j.$$

Summarizing, our task boils down to proving that, uniformly in $k=1,\dots,[C\log t]$, as $t\to\infty$,

$$J_1(k,t) = \mathbb{P}\Big\{\xi_k > a(t), k\le\tau_1,\ \sum_{i=1}^{k-1}(\mathbb{Z}_i + \mathbb{W}_i^{\downarrow}) > t^{1/\alpha}\Big\} = o\big(t^{-1-\gamma}\big),$$
$$J_2(k,t) = \mathbb{P}\big\{\xi_k > a(t), k\le\tau_1,\ \mathbb{W}_k^{\downarrow} > t^{1/\alpha}\big\} = o\big(t^{-1-\gamma}\big),$$
$$J_3(k,t) = \mathbb{P}\Big\{\xi_k > a(t), k\le\tau_1,\ \mathbb{Z}_k + \sum_{j=1}^{\mathbb{Z}_k} Y^*_{j,S_k} > t^{1/\alpha}\Big\} = o\big(t^{-1-\gamma}\big),$$
$$J_4(k,t) = \mathbb{P}\Big\{\xi_k > a(t), k\le\tau_1,\ \sum_{j=S_k+1}^{S_{\tau_1}} Y_j > t^{1/\alpha}\Big\} = o\big(t^{-1-\gamma}\big).$$

To prove the limit relations for $J_1(k,t)$, $J_2(k,t)$ and $J_3(k,t)$ we apply Lemma 5.10 with $b_1(t)=a(t)$, $b_2(t)=t^{1/\alpha}$ and $b_3(t)=C\log t$. This enables us to conclude that $J_1(k,t) = O\big(\mathbb{P}\{\xi>a(t)\}t^{-1}\big) = O(t^{-2})$ uniformly in $k\in\mathbb{N}$ as $t\to\infty$. Also, for any $\delta\in(0,\alpha)$,

$$J_2(k,t) \le \mathbb{P}\big\{\xi_k > a(t), \mathbb{W}_k^{\downarrow} > t^{1/\alpha}\big\} = O\big(t^{-\delta/\alpha}\mathbb{E}\big[\xi^\delta \mathbf{1}_{\{\xi>a(t)\}}\big]\big),\quad t\to\infty$$

uniformly in $k\in\mathbb{N}$. Invoking Karamata's theorem (Theorem 1.6.5 in [3]) we infer that the function $t\mapsto t^{-\delta/\alpha}\mathbb{E}\big[\xi^\delta\mathbf{1}_{\{\xi>a(t)\}}\big]$ is regularly varying at ∞ of index $-1-\delta(\alpha^{-1}-\beta^{-1})<-1$, hence $J_2(k,t)=o(t^{-1-\gamma})$ uniformly in $k\in\mathbb{N}$ as $t\to\infty$. Further, as $t\to\infty$,

$$J_3(k,t) \le \mathbb{P}\Big\{\xi_k > a(t),\ \mathbb{Z}_k + \sum_{j=1}^{\mathbb{Z}_k} Y^*_{j,S_k} > t^{1/\alpha}\Big\} = O(t^{-1-\varepsilon/(\alpha+\varepsilon)}\log t) = o\big(t^{-1-\gamma}\big)$$

uniformly in $k=1,2,\dots,[C\log t]$. The asymptotic estimate for $J_4(k,t)$ is justified by the independence of ξ_k and $\sum_{j=S_k+1}^{S_{\tau_1}} Y_j$. □

6. The proofs

Recall the notation
$$\nu(t)=\inf\{n\in\mathbb{N}\,:\,S_n>t\},\quad t\geq 0.$$
Put
$$U(t)=\mathbb{E}\nu(t)=\sum_{k\geq 0}\mathbb{P}\{S_k\leq t\},\quad t\geq 0, \tag{6.1}$$
so that U is the renewal function. It is well-known (see, for instance, formula (2.1) in [10]) that (2.3) with $\beta\in(0,1]$ entails
$$\lim_{t\to\infty}\frac{m(t)}{t}U(t)=(\Gamma(2-\beta)\Gamma(1+\beta))^{-1}, \tag{6.2}$$
where Γ is the Euler gamma function and the function m is defined in (2.13).

We start with several technical results. The relevance of the random variables Y_n defined in Lemma 6.1 is justified by formula (6.11).

Lemma 6.1. *Put*
$$Y_n=\sum_{j=0}^{S_{\nu(n)-1}}\left(U_j^{(n)}-U_j^{(S_{\nu(n)-1})}\right),\quad n\in\mathbb{N}.$$
Then
$$\begin{aligned}\mathbb{E}_\omega Y_n=&(n-S_{\nu(n)-1})\rho_1\rho_2\cdot\ldots\cdot\rho_{\nu(n)}\\ &+(n-S_{\nu(n)-1})\rho_{\nu(n)}(\xi_1\rho_2\cdot\ldots\cdot\rho_{\nu(n)-1}+\ldots+\xi_{\nu(n)-1}).\end{aligned}$$

Proof. Recurrence relation (3.4) entails, for $k\in\mathbb{N}$ and $j=0,1,\ldots,k-1$,
$$\mathbb{E}_\omega U_j^{(k)}=\sum_{i=j}^{k-1}\prod_{r=j}^{i}q_r\quad\text{a.s.},$$
where $q_r=(1-\omega_r)/\omega_r$ for $r\in\mathbb{N}_0$ and thereupon
$$\begin{aligned}\mathbb{E}_\omega\sum_{j=0}^{S_{\nu(n)-1}}\left(U_j^{(n)}-U_j^{(S_{\nu(n)-1})}\right)&=\sum_{j=0}^{S_{\nu(n)-1}}\left(\sum_{i=j}^{n-1}\prod_{r=j}^{i}q_r-\sum_{i=j}^{S_{\nu(n)-1}-1}\prod_{r=j}^{i}q_r\right)\\ &=\sum_{j=0}^{S_{\nu(n)-1}}\sum_{i=S_{\nu(n)-1}}^{n-1}\prod_{r=j}^{i}q_r\\ &=\sum_{i=S_{\nu(n)-1}}^{n-1}\prod_{r=0}^{i}q_r+\sum_{m=1}^{\nu(n)-1}\sum_{j=S_{m-1}+1}^{S_m}\sum_{i=S_{\nu(n)-1}}^{n-1}\prod_{r=j}^{i}q_r\\ &=(n-S_{\nu(n)-1})\rho_1\rho_2\cdot\ldots\cdot\rho_{\nu(n)}\\ &+(n-S_{\nu(n)-1})\rho_{\nu(n)}(\xi_1\rho_2\cdot\ldots\cdot\rho_{\nu(n)-1}+\ldots+\xi_{\nu(n)-1}).\end{aligned}$$
□

In Lemma 6.2 we show that the contribution of $\rho_{\nu(n)}$ is negligible in an appropriate sense. Recall that, for $i=1,2$, the functions c_i were defined in the conditions ($\xi\rho$ i) in Section 2.3 when $\beta\in(0,1)$ and in (2.14) when $\beta=1$.

Lemma 6.2. (i) *Assume that (ξ) and ($\xi\rho$1) hold for $\beta\in(0,1]$ and that $\mathbb{E}\rho^\gamma<\infty$ for some $\gamma>\beta/2$. Then $\rho_{\nu(n)}/c_1(n)\overset{\mathbb{P}}{\longrightarrow}0$ as $n\to\infty$.*

(ii) *Assume that (ξ) and ($\xi\rho$2) hold for $\beta\in(0,1]$ and some $\alpha\leq\beta/2$ and $\mathbb{E}\rho^\gamma<\infty$ for some $\gamma>\alpha$. Then $\rho_{\nu(n)}/c_2(n)\overset{\mathbb{P}}{\longrightarrow}0$ as $n\to\infty$.*

Proof. We first check that ($\xi\rho 1$) in combination with (ξ) entails

$$\lim_{t\to\infty}\frac{\mathbb{P}\{\xi>t^{1/2},\rho>\varepsilon c_1(t)\}}{\mathbb{P}\{\xi>t\}}=0 \tag{6.3}$$

for all $\varepsilon>0$. It suffices to prove this for $\varepsilon\in(0,2)$. Fix any such an ε and pick t_0 so large to ensure that $c_1(\varepsilon t/2)/c_1(t)\le\varepsilon$ for $t\ge t_0$. This is possible because c_1 is regularly varying at ∞ of index 1. Then, for $t\ge t_0$,

$$\begin{aligned}\frac{\mathbb{P}\{\xi>t^{1/2},\,\rho>\varepsilon c_1(t)\}}{\mathbb{P}\{\xi>t\}}&\le\frac{\mathbb{P}\{\xi>t^{1/2},\,\rho>c_1(\varepsilon t/2)\}}{\mathbb{P}\{\xi>t\}}\\&\le\frac{\mathbb{P}\{\xi>(\varepsilon t/2)^{1/2},\,\rho>c_1(\varepsilon t/2)\}}{\mathbb{P}\{\xi>\varepsilon t/2\}}\frac{\mathbb{P}\{\xi>\varepsilon t/2\}}{\mathbb{P}\{\xi>t\}}.\end{aligned}$$

The right-hand side tends to zero as $t\to\infty$ in view of ($\xi\rho 1$) and the regular variation of $s\mapsto\mathbb{P}\{\xi>s\}$. This completes the proof of (6.3).

As a consequence of (6.2) we have

$$\mathbb{P}\{\xi>t\}U(t)=O(1),\quad t\to\infty. \tag{6.4}$$

For any $\delta>0$,

$$\begin{aligned}\mathbb{P}\{\rho_{\nu(n)}>\delta c_1(n)\}&=\int_{[0,\,n-n^{1/2}]}\mathbb{P}\{\xi>n-y,\,\rho>\delta c_1(n)\}\mathrm{d}U(y)\\&\quad+\int_{(n-n^{1/2},\,n]}\mathbb{P}\{\xi>n-y,\,\rho>\delta c_1(n)\}\mathrm{d}U(y)=:I_1(n)+I_2(n).\end{aligned}$$

Further,

$$I_1(n)\le\frac{\mathbb{P}\{\xi>n^{1/2},\,\rho>\delta c_1(n)\}}{\mathbb{P}\{\xi>n\}}\big(\mathbb{P}\{\xi>n\}U(n)\big)\;\to\;0,\quad n\to\infty$$

by (6.3) and (6.4). As for $I_2(n)$ we have

$$\begin{aligned}I_2(n)&\le\mathbb{P}\{\rho>\delta c_1(n)\}(U(n)-U(n-n^{1/2}))\le\mathbb{P}\{\rho>\delta c_1(n)\}U(n^{1/2})\\&\le(\mathbb{E}\rho^\gamma)(\delta c_1(n))^{-\gamma}U(n^{1/2})\end{aligned}$$

by subadditivity of U and Markov's inequality. According to (6.2) U is regularly varying at ∞ of index β. Also, c_1 is regularly varying of index 1. Therefore, the right-hand side of the last centered formula converges to zero as $n\to\infty$. This completes the proof of part (i). The proof of part (ii) is analogous. Therefore, we only discuss principal steps.

Similarly to (6.3) we have

$$\lim_{t\to\infty}\frac{\mathbb{P}\{\xi>t^{\alpha/\beta},\rho>\varepsilon c_2(t)\}}{\mathbb{P}\{\xi>t\}}=0 \tag{6.5}$$

for all $\varepsilon>0$. Let $\delta>0$. Using a decomposition

$$\begin{aligned}\mathbb{P}\{\rho_{\nu(n)}>\delta c_2(n)\}&=\int_{[0,\,n-n^{\alpha/\beta}]}\mathbb{P}\{\xi>n-y,\,\rho>\delta c_2(n)\}\mathrm{d}U(y)\\&\quad+\int_{(n-n^{\alpha/\beta},\,n]}\mathbb{P}\{\xi>n-y,\,\rho>\delta c_2(n)\}\mathrm{d}U(y)=:J_1(n)+J_2(n)\end{aligned}$$

we further obtain

$$J_1(n)\le\frac{\mathbb{P}\{\xi>n^{\alpha/\beta},\,\rho>\delta c_2(n)\}}{\mathbb{P}\{\xi>n\}}\big(\mathbb{P}\{\xi>n\}U(n)\big)\;\to\;0,\quad n\to\infty$$

by (6.5) and (6.4) and

$$\begin{aligned} J_2(n) &\le \mathbb{P}\{\rho > \delta c_2(n)\}(U(n) - U(n - n^{\alpha/\beta})) \le \mathbb{P}\{\rho > \delta c_2(n)\}U(n^{\alpha/\beta}) \\ &\le (\mathbb{E}\rho^\gamma)(\delta c_2(n))^{-\gamma}U(n^{\alpha/\beta}). \end{aligned}$$

The right-hand side of the last centered formula converges to zero as $n \to \infty$ because $U(t^{\alpha/\beta})$ and $c_2(t)^\gamma$ are regularly varying at ∞ of indices α and $(\beta-\alpha)\alpha^{-1}\gamma$, respectively, and $\alpha < (\beta - \alpha)\alpha^{-1}\gamma$. The latter is secured by $\gamma > \alpha$ and $\beta \ge 2\alpha$. □

6.1. The case $\beta \in (0, 1)$.

Lemma 6.3. *Assume that $\mathbb{E}\log\rho \in [-\infty, 0)$ and that (ξ) holds for $\beta \in (0,1)$.*

(i) *If ($\xi\rho1$) holds, and $\mathbb{E}\rho^\gamma < \infty$ for some $\gamma > \beta/2$, then*

$$n^{-2}Y_n \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty. \tag{6.6}$$

(ii) *If ($\xi\rho2$) holds for some $\alpha \le \beta/2$, and $\mathbb{E}\rho^\gamma < \infty$ for some $\gamma > \alpha$, then*

$$\mathbb{P}\{\xi > n\}^{1/\alpha}Y_n \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty. \tag{6.7}$$

Proof. For part (i) it suffices to prove that

$$n^{-2}\mathbb{E}_\omega Y_n \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty \tag{6.8}$$

because while $\mathbb{P}_\omega\{Y_n > \varepsilon n^2\} \overset{\mathbb{P}}{\longrightarrow} 0$ for all $\varepsilon > 0$ as $n \to \infty$ then follows by Markov's inequality, $\mathbb{P}\{Y_n > \varepsilon n^2\} = \mathbb{E}\mathbb{P}_\omega\{Y_n > \varepsilon n^2\} \to 0$ for all $\varepsilon > 0$ as $n \to \infty$ is justified by the Lebesgue dominated convergence theorem.

According to Lemma 6.1,

$$\mathbb{E}_\omega Y_n = (n - S_{\nu(n)-1})\rho_1 \cdot \ldots \cdot \rho_{\nu(n)} + (n - S_{\nu(n)-1})\rho_{\nu(n)}(\xi_1\rho_2 \cdot \ldots \cdot \rho_{\nu(n)-1} + \ldots + \xi_{\nu(n)-1}).$$

In view of $\mathbb{E}\log\rho \in [-\infty, 0)$ we have $\lim_{n\to\infty}\rho_1 \cdot \ldots \cdot \rho_n = 0$ a.s., whence $\lim_{n\to\infty}\rho_1 \cdot \ldots \cdot \rho_{\nu(n)-1} = \lim_{n\to\infty}\rho_1 \cdot \ldots \cdot \rho_{\nu(n)} = 0$ a.s. because $\lim_{n\to\infty}\nu(n) = \infty$ a.s. This in combination with $n - S_{\nu(n)-1} \le n$ a.s. proves that

$$(n - S_{\nu(n)-1})\rho_1\rho_2 \cdot \ldots \cdot \rho_{\nu(n)} = o(n), \quad n \to \infty \text{ a.s.}$$

By Lemma 6.2, $n^{-1}\rho_{\nu(n)} \overset{\mathbb{P}}{\longrightarrow} 0$ as $n \to \infty$. Therefore, (6.8) follows if we can show that $\xi_1\rho_2 \cdot \ldots \cdot \rho_{\nu(n)-1} + \ldots + \xi_{\nu(n)-1}$ is bounded in probability, that is, for any $a \in (0,1)$ there exists $b = b(a) > 0$ such that

$$\mathbb{P}\{\xi_1\rho_2 \cdot \ldots \cdot \rho_{\nu(n)-1} + \ldots + \xi_{\nu(n)-1} > b\} \le a. \tag{6.9}$$

Write, for any $x > 0$,

$$\begin{aligned} &\mathbb{P}\{\xi_1\rho_2 \cdot \ldots \cdot \rho_{\nu(n)-1} + \ldots + \xi_{\nu(n)-1} > x\} \\ =\ &\sum_{k\ge2}\mathbb{P}\{\xi_1\rho_2 \cdot \ldots \cdot \rho_{k-1} + \ldots + \xi_{k-1} > x, S_{k-1} \le n, S_{k-1} + \xi_k > n\} \\ =\ &\sum_{k\ge2}\mathbb{P}\{\xi_1 + \xi_2\rho_1 + \ldots + \xi_{k-1}\rho_1 \cdot \ldots \cdot \rho_{k-2} > x, S_{k-1} \le n, S_{k-1} + \xi_k > n\} \\ =\ &\mathbb{P}\{\xi_1 + \xi_2\rho_1 + \ldots + \xi_{\nu(n)-1}\rho_1 \cdot \ldots \cdot \rho_{\nu(n)-2} > x\} \\ \le\ &\mathbb{P}\{\xi_1 + \xi_2\rho_1 + \xi_3\rho_1\rho_2 + \ldots > x\}. \end{aligned}$$

This proves (6.9) (hence, (6.8)) because $\mathbb{E}\log\xi < \infty$ which is a consequence of (ξ) together with $\mathbb{E}\log\rho \in [-\infty, 0)$ ensures that the series $\xi_1 + \xi_2\rho_1 + \xi_3\rho_1\rho_2 + \ldots$ converges a.s.

Part (ii) follows from (6.9) and the observation

$$\frac{(n - S_{\nu(n)-1})\rho_{\nu(n)}}{(\mathbb{P}\{\xi > n\}^{-1/\alpha}} \le \frac{\rho_{\nu(n)}}{n^{-1}\mathbb{P}\{\xi > n\}^{-1/\alpha}} = \frac{\rho_{\nu(n)}}{c_2(n)} \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty,$$

where the limit relation is guaranteed by Lemma 6.2. □

Lemma 6.4. *Assume that $\beta \in (0,1)$. The measure μ defined in (2.5) satisfies*

$$\int_{|\mathbf{x}|\neq 0} (|\mathbf{x}| \wedge 1)\mu(\mathrm{d}\mathbf{x}) < \infty. \tag{6.10}$$

In particular, μ is the Lévy measure of the two-dimensional Lévy process $\mathbf{L}$ defined in (2.6).

Proof. One part of (6.10) is trivial:

$$\int_{|\mathbf{x}|>1} \mu(\mathrm{d}\mathbf{x}) = \mu\{(u,v)\in\mathbb{K} : u^2+v^2>1\} \le \mu\{(u,v)\in\mathbb{K} : u>2^{-1/2} \text{ or } v>2^{-1/2}\} < \infty.$$

To prove the other part of (6.10), set, for $n \in \mathbb{N}_0$, $A_n := \{(u,v)\in\mathbb{K} : u > 2^{-n} \text{ or } v > 2^{-n}\}$. Now observe that $(0,1]^2 = \bigcup_{n\ge 1}(A_n \backslash A_{n-1})$ and that $\mu(A_n) \le \mathcal{C}_\mu 2^{n\beta/2} + 2^{n\beta}$ for $n \in \mathbb{N}_0$. Using these in combination with the inequality $\sqrt{x_1^2+x_2^2} \le \sqrt{2}(x_1 \vee x_2)$ which holds for nonnegative x_1 and x_2 we obtain

$$\begin{aligned} 2^{-1/2}\int_{0<|\mathbf{x}|\le 1} |\mathbf{x}|\mu(\mathrm{d}\mathbf{x}) &\le 2^{-1/2}\int_{(0,\,1]^2} |\mathbf{x}|\mu(\mathrm{d}\mathbf{x}) \le \sum_{n\ge 1}\int_{A_n\backslash A_{n-1}} (x_1\vee x_2)\mu(\mathrm{d}\mathbf{x}) \\ &\le \sum_{n\ge1} 2^{-(n-1)}\mu(A_n) \le \sum_{n\ge 1} 2^{-(n-1)}(\mathcal{C}_\mu 2^{n\beta/2}+2^{n\beta}) < \infty. \end{aligned}$$

To justify the last inequality we recall that $\beta \in (0,1)$. □

We are ready to prove the main results.

Proof of Theorem 2.1. The transition from (2.8) to (2.9) is straightforward. Hence, we only prove (2.8). While either of the conditions imposed on the distribution of ρ ensures that $\mathbb{E}\log\rho \in [-\infty, 0)$, condition (ξ) guarantees $\mathbb{E}\log\xi < \infty$. This means that (2.1) holds. Starting with (3.1) we obtain a decomposition: for $n \in \mathbb{N}$,

$$\begin{aligned} T_n &= S_{\nu(n)-1} + 2\sum_{i=0}^{S_{\nu(n)-1}} U_i^{(S_{\nu(n)-1})} + (n - S_{\nu(n)-1}) + 2\sum_{i=S_{\nu(n)-1}+1}^{n} U_i^{(n)} \\ &+ 2\sum_{i=0}^{S_{\nu(n)-1}} \big(U_i^{(n)} - U_i^{(S_{\nu(n)-1})}\big) + 2\sum_{i<0} U_i^{(n)}. \end{aligned} \tag{6.11}$$

Since the random walk X is transient to the right (recall (2.1)) the last summand is bounded in probability as $n \to \infty$.

If condition $(\rho1)$ holds with $\alpha = \beta/2$, then part (i) of Lemma 6.3 applies with $\gamma > \beta/2$ as defined in $(\rho1)$. If condition $(\rho2)$ holds with $\beta/2 \in \mathcal{I}$, then part (i) of Lemma 6.3 applies with any $\gamma > \beta/2$ such that $\gamma \in \mathcal{I}$. In any event, we conclude that relation (6.6) holds. Thus, (2.8) is a consequence of

$$n^{-2}\Big(S_{\nu(n)-1} + 2\sum_{i=0}^{S_{\nu(n)-1}} U_i^{(S_{\nu(n)-1})} + (n - S_{\nu(n)-1}) + 2\sum_{i=S_{\nu(n)-1}+1}^{n} U_i^{(n)}\Big) \overset{\mathrm{d}}{\longrightarrow} 2\chi, \quad n\to\infty. \tag{6.12}$$

In view of $S_{\nu(n)-1} \le n$ we have

$$n^{-2} S_{\nu(n)-1} \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty. \tag{6.13}$$

Recall the definition of T'_n given in (4.5). We claim that, for $n \in \mathbb{N}$,

$$\begin{aligned}\varrho_n &:= 2\sum_{i=0}^{S_{\nu(n)-1}} U_i^{(S_{\nu(n)-1})} + (n - S_{\nu(n)-1}) + 2\sum_{i=S_{\nu(n)-1}+1}^{n} U_i^{(n)} \\ &\overset{\mathrm{d}}{=} 2\sum_{k=1}^{S_{\nu(n)-1}} Z_k + T'_{n-S_{\nu(n)-1}} := \varpi_n\end{aligned} \tag{6.14}$$

which shows it is enough to prove

$$n^{-2}\Big(2\sum_{k=1}^{S_{\nu(n)-1}} Z_k + T'_{n-S_{\nu(n)-1}}\Big) \overset{\mathrm{d}}{\longrightarrow} 2\chi, \quad n\to\infty. \tag{6.15}$$

To check (6.14), we write, for $n \in \mathbb{N}$ and $x \geq 0$,

$$\begin{aligned}\mathbb{P}\{\varrho_n \leq x\} &= \sum_{k\geq 1}\mathbb{E}\Big[\mathbb{P}\Big\{2\sum_{i=0}^{S_{k-1}} U_i^{(S_{k-1})} + (n - S_{k-1}) + 2\sum_{i=S_{k-1}+1}^{n} U_i^{(n)} \leq x, \\ &\qquad S_{k-1}\leq n, S_k > n \big| (\xi_j,\rho_j)_{1\leq j\leq k-1}\Big\}\Big] \\ &= \sum_{k\geq 1}\mathbb{E}\Big[\mathbb{P}\Big\{2\sum_{i=1}^{S_{k-1}} Z_i + T'_{n-S_{k-1}} \leq x,\ S_{k-1}\leq n, S_k > n\big|(\xi_j,\rho_j)_{1\leq j\leq k-1}\Big\}\Big] \\ &= \mathbb{P}\{\varpi_n \leq x\},\end{aligned}$$

where for the second equality we have used formula (3.6), the conditional independence of $\sum_{i=0}^{S_{k-1}} U_i^{(S_{k-1})}$ and $\sum_{i=S_{k-1}+1}^{n} U_i^{(n)}$, given $(\xi_j,\rho_j)_{1\leq j\leq k-1}$, and the fact that, given $(\xi_j,\rho_j)_{1\leq j\leq k-1}$, the random variable $n - S_{k-1} + 2\sum_{i=S_{k-1}+1}^{n} U_i^{(n)}$ has the same distribution as $T'_{n-S_{k-1}}$, see (4.6). Passing to the proof of (6.15) we note that an appeal to (4.3) yields

$$\sum_{k=1}^{\tau^*_{\nu(n)-1}} \overline{\mathbb{W}}_{\tau_k} \leq \sum_{k=1}^{S_{\nu(n)-1}} Z_k \leq \sum_{k=1}^{\tau^*_{\nu(n)-1}+1} \overline{\mathbb{W}}_{\tau_k} \quad \mathbb{P}-\text{a.s.} \tag{6.16}$$

Formula (6.15) holds provided that

$$n^{-2}\Big(2\sum_{k=1}^{\tau^*_{\nu(n)-1}} \overline{\mathbb{W}}_{\tau_k} + T'_{n-S_{\nu(n)-1}}\Big) \overset{\mathrm{d}}{\longrightarrow} 2\chi, \quad n\to\infty \tag{6.17}$$

and

$$n^{-2}\Big(2\sum_{k=1}^{\tau^*_{\nu(n)-1}+1} \overline{\mathbb{W}}_{\tau_k} + T'_{n-S_{\nu(n)-1}}\Big) \overset{\mathrm{d}}{\longrightarrow} 2\chi, \quad n\to\infty$$

We shall only check (6.17). The proof of the other limit relation is analogous.

Recall that a is a positive function satisfying $\lim_{t\to\infty} t\mathbb{P}\{\xi_1 > a(t)\} = 1$. By Lemma 5.1, $\mathbb{E}\tau_1 < \infty$ and

$$\lim_{t\to\infty} t\mathbb{P}\{S_{\tau_1} > a(t)x_1\} = (\mathbb{E}\tau_1)x_1^{-\beta}, \quad x_1 > 0.$$

Further, parts (C2), (C3) and (C4) of Lemma 5.6 ensure

$$\lim_{t\to\infty} t\mathbb{P}\{\overline{\mathbb{W}}_{\tau_1} > a(t)^2 x_2\} = (\mathbb{E}\tau_1)\mathcal{C}_\mu x_2^{-\beta/2}, \quad x_2 > 0.$$

These limit relations in combination with Proposition 5.7 demonstrate that

$$\lim_{t\to\infty} t\mathbb{P}\{S_{\tau_1} > a(t)x_1 \text{ or } \overline{\mathbb{W}}_{\tau_1} > a(t)^2 x_2\} = (\mathbb{E}\tau_1)\mu\{(u,v)\in\mathbb{K} : u > x_1 \text{ or } v > x_2\}$$

for all $x_1, x_2 > 0$, where μ is a measure defined in (2.5). By Lemma 6.1 in [32], the latter implies that

$$n\mathbb{P}\Big\{\Big(\frac{S_{\tau_1}}{a(n)}, \frac{\overline{\mathbb{W}}_{\tau_1}}{a(n)^2}\Big) \in \cdot\Big\} \overset{\text{v}}{\longrightarrow} (\mathbb{E}\tau_1)\mu(\cdot), \quad n\to\infty,$$

where $\overset{\text{v}}{\longrightarrow}$ denotes vague convergence in the set of locally finite (Radon) measures on $\mathbb{K}$. By Theorem 4 in [33],

$$\Big(\frac{\sum_{k=1}^{[n\cdot]}(S_{\tau_k} - S_{\tau_{k-1}})}{a(n)}, \frac{\sum_{k=1}^{[n\cdot]}\overline{\mathbb{W}}_{\tau_k}}{a(n)^2}\Big) \Rightarrow \mathbf{L}(h(\cdot)), \quad n\to\infty \tag{6.18}$$

in the J_1-topology on D^2, where $h(t) = (\mathbb{E}\tau_1)t$ for $t \geq 0$.

In view of $\mathbb{E}\tau_1 < \infty$, $(\tau_n^*)_{n\in\mathbb{N}}$ is the renewal process which corresponds to the finite mean standard random walk $(\tau_k)_{k\in\mathbb{N}_0}$. According to the weak law of large numbers for renewal processes $n^{-1}\tau_n^* \overset{\mathbb{P}}{\longrightarrow} (\mathbb{E}\tau_1)^{-1}$ as $n\to\infty$. It is well-known that this can be strengthened to

$$\sup_{t\in[0,T]} |n^{-1}\tau^*_{[nt]} - (\mathbb{E}\tau_1)^{-1}t| \overset{\mathbb{P}}{\longrightarrow} 0, \quad n\to\infty \tag{6.19}$$

for all $T > 0$ or equivalently

$$n^{-1}\tau^*_{[n\cdot]} \overset{\text{J}_1}{\Rightarrow} g(\cdot), \quad n\to\infty, \tag{6.20}$$

where $g(t) = (\mathbb{E}\tau_1)^{-1}t$ for $t \geq 0$. Since the limit is deterministic, (6.18) and (6.20) can be combined into the joint convergence

$$\Big(\Big(\frac{\sum_{k=1}^{[n\cdot]}(S_{\tau_k} - S_{\tau_{k-1}})}{a(n)}, \frac{\sum_{k=1}^{[n\cdot]}\overline{\mathbb{W}}_{\tau_k}}{a(n)^2}\Big), \frac{\tau^*_{[n\cdot]}}{n}\Big) \Rightarrow \big(\mathbf{L}(h(\cdot)), g(\cdot)\big), \quad n\to\infty \tag{6.21}$$

in the product J_1-topology on $D^2 \times D$. Furthermore, since the convergence in (6.19) is uniform, then passing to versions of $\Big(\frac{\sum_{k=1}^{[n\cdot]}(S_{\tau_k} - S_{\tau_{k-1}})}{a(n)}, \frac{\sum_{k=1}^{[n\cdot]}\overline{\mathbb{W}}_{\tau_k}}{a(n)^2}\Big)$ and $\frac{\tau^*_{[n\cdot]}}{n}$ which converge $\mathbb{P}$-a.s. we infer that we can use the same homeomorphisms $\lambda_n(t)$ which appear in the definition of the J_1-convergence for both terms. This shows that (6.21) holds in the J_1-topology on D^3. An application of Lemma 4.2 together with the continuous mapping theorem yields

$$\Big(\frac{\sum_{k=1}^{\tau^*_{[n\cdot]}}(S_{\tau_k} - S_{\tau_{k-1}})}{a(n)}, \frac{\sum_{k=1}^{\tau^*_{[n\cdot]}}\overline{\mathbb{W}}_{\tau_k}}{a(n)^2}\Big) \Rightarrow \mathbf{L}(\cdot), \quad n\to\infty \tag{6.22}$$

and

$$\Big(\frac{\sum_{k=1}^{\tau^*_{[n\cdot]}+1}(S_{\tau_k} - S_{\tau_{k-1}})}{a(n)}, \frac{\sum_{k=1}^{\tau^*_{[n\cdot]}}\overline{\mathbb{W}}_{\tau_k}}{a(n)^2}\Big) \Rightarrow \mathbf{L}(\cdot), \quad n\to\infty \tag{6.23}$$

in the J_1-topology on D^2. Since the random walk $(S_n)_{n\in\mathbb{N}_0}$ is $\mathbb{P}$-a.s. nondecreasing and $\tau_{\tau_n^*} \leq n \leq \tau_{\tau_n^*+1}$ $\mathbb{P}$-a.s., relations (6.22) and (6.23) entail

$$\Big(\frac{\sum_{k=1}^{[n\cdot]}\xi_k}{a(n)}, \frac{\sum_{k=1}^{\tau^*_{[n\cdot]}}\overline{\mathbb{W}}_{\tau_k}}{a(n)^2}\Big) \Rightarrow \mathbf{L}(\cdot), \quad n\to\infty \tag{6.24}$$

in the J_1-topology on D^2.

An argument leading to Theorem 3.6 in[1] [37] enables us to conclude that the last limit relation entails
(6.25)
$$\Big(\frac{\sum_{k=1}^{\nu([n\cdot])-1}\xi_k}{n},\frac{\sum_{k=1}^{\tau^*_{\nu([n\cdot])-1}}\overline{\mathbb{W}}_{\tau_k}}{n^2}\Big)\ \Rightarrow\ \big(((L_1\circ L_1^{\leftarrow})(\cdot)-)^+,(((L_2\circ L_1^{\leftarrow})(\cdot)-))^+\big),\quad n\to\infty$$
in the J_1-topology on D^2, where we write $(X(t)^+)$ for $(X(t+))$.

By Lemma 4.2 (iv) in [37] the limit process in (6.25) admits no fixed discontinuities. In view of this we obtain
$$\Big(\frac{\sum_{k=1}^{\nu(n)-1}\xi_k}{n},\frac{\sum_{k=1}^{\tau^*_{\nu(n)-1}}\overline{\mathbb{W}}_{\tau_k}}{n^2}\Big)\ \stackrel{\rm d}{\longrightarrow}\ (L_1(L_1^{\leftarrow}(1)-),L_2(L_1^{\leftarrow}(1)-)),\quad n\to\infty$$
as a consequence of (6.25). An application of (4.7) yields
$$(6.26)\quad \Big(\frac{S_{\nu(n)-1}}{n},\frac{\sum_{k=1}^{\tau^*_{\nu(n)-1}}\overline{\mathbb{W}}_{\tau_k}}{n^2},\frac{T'_n}{n^2}\Big)\ \stackrel{\rm d}{\longrightarrow}\ \big(L_1(L_1^{\leftarrow}(1)-),L_2(L_1^{\leftarrow}(1)-),M(1)\big),\quad n\to\infty$$
having utilized the fact that T'_n is independent of the other components on the left-hand side. In view of $n-S_{\nu(n)-1}\stackrel{\mathbb{P}}{\longrightarrow}\infty$ as $n\to\infty$ this implies that, as $n\to\infty$,
(6.27)
$$\Big(\frac{n-S_{\nu(n)-1}}{n},\frac{\sum_{k=1}^{\tau^*_{\nu(n)-1}}\overline{\mathbb{W}}_{\tau_k}}{n^2},\frac{T'_{n-S_{\nu(n)-1}}}{(n-S_{\nu(n)-1})^2}\Big)\ \stackrel{\rm d}{\longrightarrow}\ \big(1-L_1(L_1^{\leftarrow}(1)-),L_2(L_1^{\leftarrow}(1)-),M(1)\big)$$
and thereupon
$$\frac{2\sum_{k=1}^{\tau^*_{\nu(n)-1}}\overline{\mathbb{W}}_{\tau_k}+T'_{n-S_{\nu(n)-1}}}{n^2}\ \stackrel{\rm d}{\longrightarrow}\ 2L_2(L_1^{\leftarrow}(1)-)+M(1)(1-L_1(L_1^{\leftarrow}(1)-))^2\stackrel{\rm d}{=}2\chi,\ \ n\to\infty.$$
Thus, relation (6.17) holds true. □

Proof of Theorem 2.2. Relation (2.11) is an immediate consequence of (2.10). Therefore, we only focus on (2.10). Its proof proceeds along the lines of the proof of Theorem 2.1 but is much simpler. In view of this, we only give a sketch.

We shall use decomposition (6.11). We already know from the proof of Theorem 2.1 that the last summand in (6.11) is bounded in probability. Further, note that under the assumptions of Theorem 2.2 all the conditions of part (ii) Lemma 6.3 are met. Thus,
$$(6.28)\qquad \mathbb{P}\{\xi>n\}^{1/\alpha}\sum_{j=0}^{S_{\nu(n)-1}}\Big(U_j^{(n)}-U_j^{(S_{\nu(n)-1})}\Big)\ \stackrel{\mathbb{P}}{\longrightarrow}\ 0,\quad n\to\infty.$$
Also,
$$\frac{S_{\nu(n)-1}}{\mathbb{P}\{\xi>n\}^{-1/\alpha}}\ \stackrel{\mathbb{P}}{\longrightarrow}\ 0,\quad n\to\infty$$
because $S_{\nu(n)-1}\le n$ and the denominator varies regularly of index $\beta/\alpha\ge 2$. In view of these limit relations, (2.10) is a consequence of
$$\mathbb{P}\{\xi>n\}^{1/\alpha}\Big(2\sum_{i=0}^{S_{\nu(n)-1}}U_i^{(S_{\nu(n)-1})}+(n-S_{\nu(n)-1})+2\sum_{i=S_{\nu(n)-1}+1}^{n}U_i^{(n)}\Big)\ \stackrel{\rm d}{\longrightarrow}\ 2\widehat{L}_2(\widehat{L}_1^{\leftarrow}(1))$$

[1]The independence assumption imposed in the cited result is only made to ensure a limit relation like (6.24). The passage from (6.24) to (6.25) is justified by the continuous mapping theorem and as such does not require the aforementioned independence.

as $n \to \infty$, which in its turn is implied by

$$\mathbb{P}\{\xi > n\}^{1/\alpha}\Big(2\sum_{k=1}^{\tau^*_{\nu(n)-1}} \overline{\mathbb{W}}_{\tau_k} + T'_{n-S_{\nu(n)-1}}\Big) \overset{\mathrm{d}}{\longrightarrow} 2\widehat{L}_2(\widehat{L}_1^{\leftarrow}(1)-) \overset{\mathrm{d}}{=} 2\widehat{L}_2(\widehat{L}_1^{\leftarrow}(1)), \quad n \to \infty$$

by the same reasoning as given in the proof of Theorem 2.1. Condition ($\rho1$) ensures that $\mathbb{E}\log\rho \in [-\infty, 0)$. By Lemma 5.1, $\mathbb{E}\tau_1 < \infty$ and

$$\lim_{t\to\infty} t\mathbb{P}\{S_{\tau_1} > a(t)x_1\} = (\mathbb{E}\tau_1)x_1^{-\beta}, \quad x_1 > 0.$$

According to part (C1) of Lemma 5.6,

$$\lim_{t\to\infty} t\mathbb{P}\{\overline{\mathbb{W}}_{\tau_1} > t^{1/\alpha}x_2\} = (\mathbb{E}\tau_1)\mathcal{C}_Z(\alpha)x_2^{-\alpha}, \quad x_2 > 0.$$

Observe that the cited result applies in the case $\alpha \in (0, \beta/2)$ in view of $\mathbb{E}\xi^{2\alpha} < \infty$ which is secured by (ξ). These limit relations in combination with Proposition 5.8 demonstrate that

$$\lim_{t\to\infty} t\mathbb{P}\{S_{\tau_1} > a(t)x_1 \text{ or } \overline{\mathbb{W}}_{\tau_1} > t^{1/\alpha}x_2\} = (\mathbb{E}\tau_1)(x_1^{-\beta} + \mathcal{C}_Z(\alpha)x_2^{-\alpha})$$

for all $x_1, x_2 > 0$. Arguing as in the proof of Theorem 2.1 we arrive at counterparts of (6.18), (6.24) and (6.27), respectively,

$$\Big(\frac{\sum_{k=1}^{[n\cdot]}(S_{\tau_k} - S_{\tau_{k-1}})}{a(n)}, \frac{\sum_{k=1}^{[n\cdot]}\overline{\mathbb{W}}_{\tau_k}}{n^{1/\alpha}}\Big) \Rightarrow (\widehat{L}_1(h(\cdot)), \widehat{L}_2(h(\cdot))), \quad n \to \infty$$

in the J_1-topology on D^2, where, as before, $h(t) = (\mathbb{E}\tau_1)t$ for $t \geq 0$;

$$\Big(\frac{\sum_{k=1}^{[n\cdot]}\xi_k}{a(n)}, \frac{\sum_{k=1}^{\tau^*_{[n\cdot]}}\overline{\mathbb{W}}_{\tau_k}}{n^{1/\alpha}}\Big) \Rightarrow (\widehat{L}_1(\cdot), \widehat{L}_2(\cdot)), \quad n \to \infty \tag{6.29}$$

in the J_1-topology on D^2; as $n \to \infty$,

$$\Big(\frac{n - S_{\nu(n)-1}}{n}, \frac{\sum_{k=1}^{\tau^*_{\nu(n)-1}}\overline{\mathbb{W}}_{\tau_k}}{\mathbb{P}\{\xi > n\}^{-1/\alpha}}, \frac{T'_{n-S_{\nu(n)-1}}}{(n - S_{\nu(n)-1})^2}\Big) \overset{\mathrm{d}}{\longrightarrow} \big(1 - \widehat{L}_1(\widehat{L}_1^{\leftarrow}(1)-), \widehat{L}_2(\widehat{L}_1^{\leftarrow}(1)-), M(1)\big).$$

Since $\mathbb{P}\{\xi > n\}^{-1/\alpha} \sim n^{\beta/\alpha}\ell(n)^{-1/\alpha}$ we infer $\lim_{n\to\infty} n^{-2}\mathbb{P}\{\xi > n\}^{-1/\alpha} = \infty$ and thereupon

$$\mathbb{P}\{\xi > n\}^{1/\alpha}\Big(2\sum_{k=1}^{\tau^*_{\nu(n)-1}} \overline{\mathbb{W}}_{\tau_k} + T'_{n-S_{\nu(n)-1}}\Big) \overset{\mathrm{d}}{\longrightarrow} 2\widehat{L}_2(\widehat{L}_1^{\leftarrow}(1)-) \overset{\mathrm{d}}{=} 2\widehat{L}_2(\widehat{L}_1^{\leftarrow}(1)), \quad n \to \infty,$$

where the last distributional equality is implied by the independence. □

6.2. **The case $\beta = 1$.** We start by proving several auxiliary results. The functions a, π and π^* appearing below are defined in Section 2.4.

Lemma 6.5. *Assume that (ξ) holds for $\beta = 1$. Then, for every $T > 0$,*

$$\sup_{u\in[0,T]}\Big|\frac{\nu(tu)}{t\pi^*(t)} - u\Big| \overset{\mathbb{P}}{\longrightarrow} 0, \quad t \to \infty.$$

In particular,

$$\frac{S_{\nu(t)}}{t} \overset{\mathbb{P}}{\longrightarrow} 1, \quad t \to \infty. \tag{6.30}$$

Proof. It is known (see, for instance, Example 2 on p. 1034 in [18]) that if $t \mapsto \mathbb{P}\{\xi > t\}$ is regularly varying at ∞ of index -1, then

$$\frac{S_{[t\cdot]} - t(\cdot)\pi(t)}{a(t)} \overset{\mathrm{J}_1}{\Rightarrow} Y_1(\cdot), \quad t \to \infty$$

in the J_1-topology on D, where $(Y_1(u))_{u\geq 0}$ is a 1-stable spectrally positive Lévy process. This yields

$$\frac{S_{[t\cdot]}}{t\pi(t)} \overset{\mathrm{J}_1}{\Rightarrow} f(\cdot), \quad t \to \infty, \tag{6.31}$$

where $f(u) = u$ for $u \geq 0$. Using continuity of the inversion (see [40]) we deduce

$$\frac{\nu(t(\cdot)\pi(t))}{t} = \inf\left\{s \geq 0 : \frac{S_{[ts]}}{t\pi(t)} > (\cdot)\right\} \overset{\mathrm{M}_1}{\Rightarrow} f(\cdot), \quad t \to \infty.$$

Since the limit is continuous, the convergence is actually locally uniform, that is, for every $T > 0$,

$$\sup_{u\in[0,T]} \left|\frac{\nu(tu\pi(t))}{t} - u\right| \overset{\mathbb{P}}{\longrightarrow} 0, \quad t \to \infty.$$

Replacing t with $t\pi^*(t)$ and using $\lim_{t\to\infty} \pi^*(t)\pi(t\pi^*(t)) = 1$ we obtain the first claim of the lemma.

Relation (6.30) follows from (6.31) with $t\pi^*(t)$ replacing t in combination with the first claim and Lemma 4.2 with $k = 1$. □

Lemma 6.6. *Assume that (ξ) holds for $\beta = 1$. Then*

$$\frac{n - S_{\nu(n)-1}}{a(n\pi^*(n))} \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty.$$

Proof. Fix $\delta > 0$ and write

$$\mathbb{P}\{n - S_{\nu(n)-1} \geq \delta a(n\pi^*(n))\} \leq \mathbb{P}\left\{\frac{m(n - S_{\nu(n)-1})}{m(n)} \geq \frac{m(\delta a(n\pi^*(n)))}{m(n)}\right\}.$$

By Theorem 6 in [10],

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{m(n - S_{\nu(n)-1})}{m(n)} \leq x\right\} = x, \quad x \in [0, 1].$$

Thus, it is enough to show that

$$\liminf_{n\to\infty} \frac{m(\delta a(n\pi^*(n)))}{m(n)} \geq 1.$$

Since m is slowly varying, this limit relation is equivalent to

$$\liminf_{n\to\infty} \frac{m(a(n\pi^*(n)))}{m(n)} = \liminf_{n\to\infty} \frac{\pi(n\pi^*(n))}{m(n)} = \liminf_{n\to\infty} \frac{1}{\pi^*(n)m(n)} \geq 1.$$

The last inequality follows by an application of Fatou's lemma together with Lemma 6.5:

$$1 = \mathbb{E} \lim_{n\to\infty} \frac{\nu(n)}{n\pi^*(n)} \leq \liminf_{n\to\infty} \frac{U(n)}{n\pi^*(n)} = \liminf_{n\to\infty} \frac{U(n)m(n)}{n} \frac{1}{m(n)\pi^*(n)} = \liminf_{n\to\infty} \frac{1}{m(n)\pi^*(n)},$$

where the first limit is understood as the limit in probability, U denotes the renewal function (see (6.1)), and the last equality follows from (6.2). □

Lemma 6.7 given next is a counterpart of Lemma 6.3.

Lemma 6.7. *Assume that $\mathbb{E} \log \rho \in [-\infty, 0)$ and that (ξ) holds for $\beta = 1$.*

(i) *If (ξρ1) holds, and $\mathbb{E}\rho^\gamma < \infty$ for some $\gamma > 1/2$, then*

$$a(n\pi^*(n))^{-2} Y_n \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty. \tag{6.32}$$

(ii) *If ($\xi\rho2$) holds for some $\alpha \le 1/2$ and $\mathbb{E}\rho^\gamma < \infty$ for some $\gamma > \alpha$, then*

$$(n\pi^*(n))^{-1/\alpha} Y_n \xrightarrow{\mathbb{P}} 0, \quad n \to \infty. \tag{6.33}$$

The proof is omitted, for it follows along the same lines as the proof of Lemma 6.3.

Proof of Theorem 2.3. Recall that $\beta = 1$. If condition ($\rho1$) holds with $\alpha = \beta/2 = 1/2$, then Lemma 6.7(i) applies with $\gamma > 1/2$ as defined in ($\rho1$). If condition ($\rho2$) holds with $1/2 \in \mathcal{I}$, then Lemma 6.7(i) applies with any $\gamma > 1/2$ such that $\gamma \in \mathcal{I}$. Thus, in any event, (6.32) holds.

Using once again decomposition (6.11) we conclude that (2.15) is a consequence of

$$\frac{\left(S_{\nu(n)-1} + 2\sum_{i=0}^{S_{\nu(n)-1}} U_i^{(S_{\nu(n)-1})} + (n - S_{\nu(n)-1}) + 2\sum_{i=S_{\nu(n)-1}+1}^{n} U_i^{(n)}\right)}{a(n\pi^*(n))^2} \xrightarrow{\mathrm{d}} 2L_2(1) \tag{6.34}$$

as $n \to \infty$. In view of $S_{\nu(n)-1} \le n$ $\mathbb{P}$-a.s.,

$$\frac{S_{\nu(n)-1}}{a(n\pi^*(n))^2} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty \tag{6.35}$$

because the denominator is regularly varying at ∞ of index 2.

From the proof of Theorem 2.1 we know that distributional equality (6.14) holds. Hence, it is enough to show that

$$\frac{\sum_{k=1}^{S_{\nu(n)-1}} Z_k}{a(n\pi^*(n))^2} \xrightarrow{\mathrm{d}} L_2(1), \quad n \to \infty \tag{6.36}$$

and

$$\frac{T'_{n-S_{\nu(n)-1}}}{a(n\pi^*(n))^2} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty. \tag{6.37}$$

We first prove (6.37). Using (4.7) in combination with $n - S_{\nu(n)-1} \xrightarrow{\mathbb{P}} +\infty$ as $n \to \infty$ and the independence of $(T'_k)_{k\in\mathbb{N}_0}$ and $n - S_{\nu(n)-1}$ we infer

$$\frac{T'_{n-S_{\nu(n)-1}}}{(n - S_{\nu(n)-1})^2} \xrightarrow{\mathrm{d}} 2\vartheta, \quad n \to \infty.$$

With this at hand, (6.37) follows from Lemma 6.6.

In order to prove (6.36) note that in formula (6.22) we still have convergence of the second components, that is,

$$\frac{\sum_{k=1}^{\tau^*_{[n\cdot]}} \overline{\mathbb{W}}_{\tau_k}}{a(n)^2} \overset{\mathrm{J_1}}{\Rightarrow} L_2(\cdot), \quad n \to \infty$$

or, equivalently,

$$\frac{\sum_{k=1}^{\tau^*_{[n\pi^*(n)(\cdot)]}} \overline{\mathbb{W}}_{\tau_k}}{a(n\pi^*(n))^2} \overset{\mathrm{J_1}}{\Rightarrow} L_2(\cdot), \quad n \to \infty.$$

By Lemma 6.5,

$$\frac{\nu(n\cdot) - 1}{n\pi^*(n)} \overset{\mathrm{J_1}}{\Rightarrow} f(\cdot), \quad n \to \infty,$$

where $f(t) = t$ for $t \ge 0$. Using once again Lemma 4.2 with $k = 1$ we infer

$$\frac{\sum_{k=1}^{\tau^*_{\nu(n)-1}} \overline{\mathbb{W}}_{\tau_k}}{a(n\pi^*(n))^2} \xrightarrow{\mathrm{d}} L_2(1), \quad n \to \infty.$$

The same limit relation holds with $\nu(n)$ replacing $\nu(n) - 1$. In view of (6.16) we arrive at (6.36). □

Proof of Theorem 2.4 . As before, we only focus on the formula involving T_n, that is, (2.17). The proof of (2.17) is similar to but much simpler than the proof of Theorem 2.2. In view of this, we only give a sketch.

According to the proof of Theorem 2.2 the last summand in (6.11) is bounded in probability. Under the assumptions of Theorem 2.4 the conditions of Lemma 6.7(ii) are satisfied, whence

$$(n\pi^*(n))^{-1/\alpha} \sum_{j=0}^{S_{\nu(n)-1}} \Big(U_j^{(n)} - U_j^{(S_{\nu(n)-1})}\Big) \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty \tag{6.38}$$

Further,

$$\frac{S_{\nu(n)-1}}{(n\pi^*(n))^{1/\alpha}} \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty$$

because $S_{\nu(n)-1} \le n$ $\mathbb{P}$-a.s. and the denominator is regularly varying at ∞ of index $1/\alpha \ge 2$. In view of (6.29)

$$\frac{\sum_{k=1}^{\tau^*_{[n\pi^*(n)(\cdot)]}} \overline{\mathbb{W}}_{\tau_k}}{(n\pi^*(n))^{1/\alpha}} \overset{\mathrm{J}_1}{\Rightarrow} L_2(\cdot), \quad n \to \infty.$$

As in the proof of Theorem 2.3, an appeal to J_1-continuity of the composition and Lemma 6.5 enables us to conclude that

$$\frac{\sum_{k=1}^{\tau^*_{\nu(n)-1}} \overline{\mathbb{W}}_{\tau_k}}{(n\pi^*(n))^{1/\alpha}} \overset{\mathrm{J}_1}{\Rightarrow} L_2(\cdot), \quad n \to \infty,$$

and that its counterpart holds with $\nu(n)$ replacing $\nu(n) - 1$. Finally, we claim that

$$\frac{T'_{n-S_{\nu(n)-1}}}{(n\pi^*(n))^{1/\alpha}} \overset{\mathbb{P}}{\longrightarrow} 0, \quad n \to \infty.$$

Indeed, this is a consequence of (4.7), Lemma 6.6 and the limit relation

$$\lim_{n\to\infty} \frac{a(n\pi^*(n))^2}{(n\pi^*(n))^{1/\alpha}} = 0$$

that we are now going to prove. When $\alpha < 1/2$, the latter holds, for the function a is then regularly varying at ∞ of index $1 < 1/(2\alpha)$. When $\alpha = 1/2$, we have $\lim_{t\to\infty} \ell(t) = 0$ by assumption. This entails $a(t) = o(t)$ as $t \to \infty$, and the limit relation follows. □

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