MODERATE PARTS IN REGENERATIVE COMPOSITIONS: THE CASE OF REGULAR VARIATION

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ABSTRACT. A regenerative random composition of integer n is constructed by allocating n standard exponential points over a countable number of intervals, comprising the complement of the closed range of a subordinator S. Assuming that the Lévy measure of S is infinite and regularly varying at zero of index $-\alpha$, $\alpha \in (0,1)$, we find an explicit threshold $r = r(n)$, such that the number $K_{n,r(n)}$ of blocks of size $r(n)$ converges in distribution without any normalization to a mixed Poisson distribution. The sequence $(r(n))$ turns out to be regularly varying with index $\alpha/(\alpha+1)$ and the mixing distribution is that of the exponential functional of S. We also discuss asymptotic behavior of $K_{n,w(n)}$ in cases when $w(n)$ diverges but grows slower than $r(n)$. Our findings complement previously known strong laws of large numbers for $K_{n,r}$ in case of a fixed $r \in \mathbb{N}$. As a key tool we employ new Abelian theorems for Laplace–Stiletjes transforms of regularly varying functions with the indexes of regular variation diverging to infinity.

1. INTRODUCTION AND THE MAIN RESULT

Let $S = (S(t))_{t\geq 0}$ be a drift-free subordinator with no killing and a Lévy measure ν on $(0,\infty)$. The classical Itô decomposition reads as

$$S(t) = \sum_{k:\tau_k \leq t} j_k, \quad t \geq 0,$$

with probability one, where $\sum \delta_{(\tau_k, j_k)}$ is a Poisson point process with values in $[0,\infty) \times (0,\infty)$ and intensity measure $\mathbf{LEB} \times \nu$. Here δ_x is a Dirac point measure at x and $\mathbf{LEB}$ is the standard Lebesgue measure on $[0,\infty)$. A random closed subset of $[0,\infty)$ defined by $\mathscr{R} := \mathrm{cl}\{S(t) : t \geq 0\}$ is called the range of S and the open complement $\mathscr{R}^c := [0,\infty) \setminus \mathscr{R}$ can be expressed as the union $\mathscr{R}^c = \cup_k (S(\tau_k-), S(\tau_k)) = \cup_k (S(\tau_k-), S(\tau_k-) + j_k)$ of countably many open intervals, where $\{\tau_k\}$ is the set of jump epochs of S. We call the disjoint intervals comprising $\mathscr{R}^c$ *boxes*. Further, let $(E_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a random variable E with the standard exponential distribution, and $(E_k)_{k\in\mathbb{N}}$ is independent of S. The points $(E_k)_{k\in\mathbb{N}}$ are called *balls*. Since the Lebesgue measure of $\mathscr{R}$ is zero with probability one, see Proposition 1.8 in [2], each ball E_j with probability one falls into one of the boxes $(S(\tau_k-), S(\tau_k))$. A family $(\mathscr{C}_n)_{n\in\mathbb{N}}$, where $\mathscr{C}_n$ is the vector of nonzero occupancy numbers of the intervals $(S(\tau_k-), S(\tau_k))$, written in their natural order, defines a coherent sequence of random compositions in the following sense. For every $n \in \mathbb{N}$, the sum of coordinates of $\mathscr{C}_n$ is equal to n and one can pass from the composition of n given by $\mathscr{C}_n$ to the composition of $n-1$ defined by $\mathscr{C}_{n-1}$ be removing the point E_n from a box it occupies.

The family $(\mathscr{C}_n)_{n\in\mathbb{N}}$ possesses a distinguishing property called *regeneration* inherited from the regenerative property of the set $\mathscr{R}$ combined with the memoryless property of the exponential distribution. Consider composition $\mathscr{C}_n$ of integer n and suppose that the first summand is equal to $m < n$, then deleting this part yields a composition on $n-m$ which has the same distribution as $\mathscr{C}_{n-m}$. In view of this property, the sequence $(\mathscr{C}_n)_{n\in\mathbb{N}}$ is called *regenerative composition structure*, as introduced in [10].

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For $n,k,r \in \mathbb{N}$, set

$$\mathscr{Z}_{n,k} := \#\{1 \le j \le n : E_j \in (S(\tau_k-), S(\tau_k))\} \quad \text{and} \quad K_{n,r} := \sum_{k\ge 1} \mathbb{1}_{\{\mathscr{Z}_{n,k}=r\}}.$$

Thus, $K_{n,r}$ is the number of boxes occupied by exactly r balls and $K_n := \sum_{r=1}^{n} K_{n,r}$ is the total number of occupied boxes. The asymptotic analysis, as $n \to \infty$, of K_n and $K_{n,r}$ for regenerative compositions has received a considerable attention in the past decades. The model exhibits a wide range of possible asymptotic regimes depending on the tail behavior of the governing Lévy measure ν. It is common to distinguish three situations:

(i) the case of finite ν in which the corresponding construction is called *the Bernoulli sieve*;
(ii) the case where ν is infinite and the function $y \mapsto \nu([y,\infty))$ is slowly varying at zero;
(iii) the case where ν is infinite and the function $y \mapsto \nu([y,\infty))$ is regularly varying at zero with index $-\alpha$, $\alpha \in (0,1)$.

In case (i), in which the subordinator S is a compound Poisson process, further subdivision stems from the tail behavior (both at zero and infinity) of the distribution of the generic jump of S. For example, if this distribution has finite mean and is nonlattice, then there exists a nondegenerate random vector $(K_1, K_2, \dots)$ such that

$$(K_{n,1}, K_{n,2}, \dots) \xrightarrow[n\to\infty]{d} (K_1, K_2, \dots), \tag{1}$$

see Theorem 3.3 in [9]. Under the additional assumption that the distribution of the generic jump of S belongs to a domain of attraction of a stable law with the index of stability lying in $(1,2]$, the total number of occupied boxes K_n, properly centred and normalized, converges in distribution to a stable law, see [5] and [8]. Furthermore, a functional limit theorem for the process $[0,1] \ni z \mapsto \sum_{r \le n^z} K_{n,r}$, is also available, see Theorem 2.2 in [1]. In particular, the aforementioned results demonstrate that the main contribution to K_n in case (i) is given by $K_{n,r}$'s with r lying between n^a and n^b, $0 < a < b \le 1$, whereas small counts $K_{n,r}$, with r fixed, are negligible. A completely different picture occurs in case (iii) in which K_n has the same magnitude as $K_{n,r}$, $r = 1, 2, \dots$ in the following sense. There exists a regularly varying normalization a_n and a sequence of positive deterministic constants $(c_r)_{r\in\mathbb{N}}$ summing up to one, such that

$$\frac{K_n}{a_n} \xrightarrow[n\to\infty]{a.s.} \mathscr{I}_\alpha \quad \text{and} \quad \frac{K_{n,r}}{a_n} \xrightarrow[n\to\infty]{a.s.} c_r \mathscr{I}_\alpha, \quad r \in \mathbb{N}, \tag{2}$$

where $\mathscr{I}_\alpha$ is the exponential functional of the subordinator S, that is,

$$\mathscr{I}_\alpha := \int_0^\infty e^{-\alpha S(\tau)} \mathrm{d}\tau, \tag{3}$$

see Theorem 4.1 in [11]. Intermediate regimes in case (ii), in which $K_{n,r}$ diverges but grows slowly than K_n, see [6], [7] and [12], make the picture even more diverse.

The main purpose of this short note is a further investigation of regenerative compositions in case (iii). A natural question arising while comparing (1) and (2) is the following. What is a threshold $r = r(n)$ such that $K_{n,r(n)}$ converges without any normalization, as $n \to \infty$, to a finite nondegenerate limit and what is the limit?

A complete answer to the above question is provided by our first main result. Throughout the paper the notation $f(t) \sim g(t)$ is used to denote asymptotic equivalence, that is, $\lim_{t\to\infty} f(t)/g(t) = 1$.

Theorem 1.1. *Assume that S is a drift-free subordinator with no killing and a Lévy measure ν on $(0,\infty)$ satisfying*

$$\nu([y,\infty)) = y^{-\alpha} \ell(1/y), \quad y \downarrow 0, \tag{4}$$

for some $\alpha \in (0,1)$ and a slowly varying at infinity function ℓ. Let $r = r(t)$ by any positive function such that

$$\lim_{t\to\infty} \frac{\alpha t^{\alpha}}{(r(t))^{\alpha+1}} \ell\left(\frac{t}{r(t)}\right) = 1, \tag{5}$$

and $r_i = r_i(t)$, $i = 1,\dots,m$ be integer-valued functions such that $r_i(t) \sim u_i r(t)$ for some $0 < u_1 < u_2 < \cdots < u_m < \infty$, as $t \to \infty$. Then

$$\left(K_{n,r_1(n)}, K_{n,r_2(n)}, \dots, K_{n,r_m(n)}\right) \xrightarrow[n\to\infty]{d} (\mathscr{P}_1, \mathscr{P}_2, \dots, \mathscr{P}_m),$$

where, given the subordinator S, $(\mathscr{P}_i)_{i=1,\dots,m}$ are mutually independent Poisson random variables with $\mathbb{E}[\mathscr{P}_i|S] = u_i^{-\alpha-1}\mathscr{I}_\alpha$ and $\mathscr{I}_\alpha$ is defined by (3).

Remark 1.2. The existence and uniqueness (up to asymptotic equivalence) of a function r satisfying (5) follows by a standard argument involving de Bruijn conjugates. Put $\ell_1(t) := \alpha^{1/(\alpha+1)}\ell^{1/(\alpha+1)}(t)$. Then ℓ_1 is slowly varying at infinity and thus possesses a de Bruijn conjugate, say ℓ_1^*, see Theorem 1.5.13 in [3], such that $\lim_{t\to\infty} \ell_1^*(t)\ell_1(t\ell_1^*(t)) = 1$. Then $r(t) = t^{\alpha/(\alpha+1)}/\ell_1^*(t^{1/(\alpha+1)})$ satisfies (5). In particular, r is regularly varying at infinity with index $\alpha/(\alpha+1) \in (0, 1/2)$.

In order to formulate our next results, let us introduce the following classes of functions:

- $\mathscr{W}^r$, a class of positive functions defined in a neighborhood of infinity such that $w \in \mathscr{W}^r$ iff $w(t) = o(r(t))$ and $w(t) \to \infty$ as $t \to \infty$, where r is defined by (5);
- $\mathscr{W}^r_{RV}$, a subclass of $\mathscr{W}^r$ comprised of functions which are regularly varying at infinity, necessarily with index of regular variation in $[0, \alpha/(\alpha+1)]$;
- $\mathscr{Q}$, a class of positive functions defined in a neighborhood of infinity such that $q \in \mathscr{Q}$ iff $q(t) = o(t)$ and $q(t) \to \infty$ as $t \to \infty$.

Obviously, $\mathscr{W}^r_{RV} \subset \mathscr{W}^r \subset \mathscr{Q}$. We also denote by $\mathscr{N}$ the class of $\mathbb{N}$-valued functions.

An interesting question naturally occurring upon examination of Theorem 1.1 is the following. What happens with $K_{n,w(n)}$ for the functions $w \in \mathscr{W}^r \cap \mathscr{N}$, that is, for $w(n)$ growing slowly than $r(n)$? The answer to this question under an additional assumption of regular variation is provided by our second main result.

Theorem 1.3. *Let $w \in \mathscr{W}^r_{RV} \cap \mathscr{N}$. Under the same assumptions on the subordinator S as in Theorem 1.1 we have*

$$\frac{(w(n))^{\alpha+1}K_{n,w(n)}}{n^{\alpha}\ell(n/w(n))} \xrightarrow[n\to\infty]{P} \alpha\mathscr{I}_\alpha,$$

with $\mathscr{I}_\alpha$ given by (3).

From the previous two results it is natural to expect that if $q \in \mathscr{Q} \cap \mathscr{N}$ is such that $r(t) = o(q(t))$, then

$$K_{n,q(n)} \xrightarrow[n\to\infty]{P} 0, \tag{6}$$

and this is indeed the case, as we shall show later in the proofs. However, it is possible to formulate a non-trivial limit theorem for such "rapidly increasing" sequence by considering a closely related functional. For $i \in \mathbb{N}$, put

$$K_{n,\geq i} := \sum_{k\geq 1} \mathbb{1}_{\{\mathscr{Z}_{n,k} \geq i\}},$$

so $K_{n,\geq i}$ is the number of boxes occupied by *at least i* balls. Our last main result provides a limit theorem for $K_{n,\geq q(n)}$ for arbitrary $q \in \mathscr{Q} \cap \mathscr{N}$ not necessarily satisfying $r(t) = o(q(t))$ nor regularly varying. In a sense the next result is the easiest one, because the sequence $(K_{n,\geq i})_{n\in\mathbb{N}}$ is monotone for every fixed $i \in \mathbb{N}$. This also partially explains why almost no assumptions on q are needed.

Theorem 1.4. *Under the same assumptions on the subordinator S as in Theorem 1.1 and for $q \in \mathscr{Q} \cap \mathscr{N}$, the following holds:*

$$\frac{(q(n))^{\alpha} K_{n,\geq q(n)}}{n^{\alpha}\ell(n/q(n))} \overset{P}{\underset{n\to\infty}{\longrightarrow}} \mathscr{I}_{\alpha}.$$

We close the introduction by specializing our main results to stable subordinators, which are typical representatives of the family of subordinators with regularly varying infinite Lévy measures.

Example 1.5. Let S be an α-stable subordinator, that is,

$$\nu([y,\infty)) = \frac{y^{-\alpha}}{\Gamma(1-\alpha)}, \quad y>0,$$

for some $\alpha \in (0,1)$. Thus, (4) holds with $\ell(y) \equiv 1/\Gamma(1-\alpha)$, and (5) holds with

$$r(t) = \left(\frac{\alpha}{\Gamma(1-\alpha)}\right)^{1/(\alpha+1)} t^{\alpha/(\alpha+1)}, \quad t>0.$$

In particular, taking $m=1$ and $u_1 = (\Gamma(1-\alpha)/\alpha)^{1/(\alpha+1)}$ in Theorem 1.1, we obtain

$$K_{n,\lfloor n^{\alpha/(\alpha+1)}\rfloor} \overset{d}{\underset{n\to\infty}{\longrightarrow}} \mathscr{P},$$

where $\mathscr{P}$ has a mixed Poisson distribution with (conditional) mean $\mathbb{E}[\mathscr{P}|S] = \frac{\alpha}{\Gamma(1-\alpha)}\mathscr{I}_{\alpha}$. If $w=w(t)$ is a positive integer-valued function which is regularly varying at infinity and such that $w(t)\to\infty$ and $w(t) = o(t^{\alpha/(\alpha+1)})$, as $t\to\infty$, then

$$\frac{(w(n))^{\alpha+1} K_{n,w(n)}}{n^{\alpha}} \overset{P}{\underset{n\to\infty}{\longrightarrow}} \frac{\alpha}{\Gamma(1-\alpha)}\mathscr{I}_{\alpha}.$$

Finally, for arbitrary $q \in \mathscr{Q} \cap \mathscr{N}$, by Theorem 1.4

$$\frac{(q(n))^{\alpha} K_{n,\geq q(n)}}{n^{\alpha}} \overset{P}{\underset{n\to\infty}{\longrightarrow}} \frac{1}{\Gamma(1-\alpha)}\mathscr{I}_{\alpha}.$$

2. Preparatory results

The following two lemmas lie in the core of our proofs. Both results provide a kind of Abelian theorems for integrals involving regularly varying functions with the index of regular variation diverging to infinity, and might be of interest on their own. The essence of the proofs is an application of the saddle-point method.

Lemma 2.1. *Let ℓ be a locally bounded positive function slowly varying at infinity and $q \in \mathscr{Q}$. Then*

$$\int_0^{\infty} y^{q(t)-1} e^{-y} \ell(t/y) \mathrm{d}y \ \sim\ \Gamma(q(t))\ell(t/q(t)), \quad t\to\infty. \tag{7}$$

Note that for $q(t)\equiv q>0$ this is just the direct half of the classical Karamata theorem. Also note that in general $\ell(t/q(t))$ in the right-hand side cannot be replaced by $\ell(t)$.

Proof. Let a, A be positive constants such that $0<a<1<A<\infty$. Write

$$\int_0^{\infty} y^{q(t)-1} e^{-y}\ell(t/y)\mathrm{d}y = \left(\int_0^{aq(t)} + \int_{aq(t)}^{Aq(t)} + \int_{Aq(t)}^{\infty}\right) y^{q(t)-1}e^{-y}\ell(t/y)\mathrm{d}y =: I_1(t)+I_2(t)+I_3(t).$$

We start by showing that

$$\lim_{t\to\infty} \frac{I_1(t)}{\Gamma(q(t))\ell(t/q(t))} = 0. \tag{8}$$

For $\alpha > 0$, the function $y \mapsto y^\alpha e^{-y}$ is increasing on $[0, \alpha]$. Since $aq(t) \le q(t) - 1$ for large enough $t > 0$, we obtain

$$\int_0^{aq(t)} y^{q(t)-1} e^{-y} \ell(t/y) \mathrm{d}y \le (aq(t))^{q(t)-1} e^{-aq(t)} \int_0^{aq(t)} \ell(t/y) \mathrm{d}y \sim \ (aq(t))^{q(t)} e^{-aq(t)} \ell(t/(aq(t))), \quad t \to \infty, \tag{9}$$

where the last passage follows from Proposition 1.5.10 in [3] upon substitution $z = t/y$. Thus, by the slow variation of ℓ and the Stirling formula for the gamma-function,

$$\lim_{t\to\infty} \frac{I_1(t)}{\Gamma(q(t))\ell(t/q(t))} \le \lim_{t\to\infty} \frac{(aq(t))^{q(t)} e^{-aq(t)}}{\Gamma(q(t))} = \lim_{t\to\infty} \frac{(aq(t))^{q(t)} e^{-aq(t)}}{\sqrt{2\pi} q(t)^{q(t)-1/2} e^{-q(t)}} = \lim_{t\to\infty} \sqrt{q(t)/(2\pi)} (ae^{-a}e)^{q(t)}.$$

The last limit is equal to zero because $ae^{-a}e < 1$ for $0 < a < 1$ and (8) follows. Let us check that

$$\lim_{t\to\infty} \frac{I_3(t)}{\Gamma(q(t))\ell(t/q(t))} = 0. \tag{10}$$

Upon substitution $y = q(t)z$ we obtain

$$I_3(t) \ = \ (q(t))^{q(t)} \int_A^\infty z^{q(t)-1} e^{-q(t)z} \ell(t/(zq(t))) \mathrm{d}z \ \le \ t^{-1} (q(t))^{q(t)+1} \sup_{y \le t/(Aq(t))} (y\ell(y)) \int_A^\infty z^{q(t)} e^{-q(t)z} \mathrm{d}z.$$

By Theorem 1.5.3 in [3], the right-hand side of the last display is asymptotically equal to

$$A^{-1} (q(t))^{q(t)} \ell(t/q(t)) \int_A^\infty z^{q(t)} e^{-q(t)z} \mathrm{d}z.$$

Therefore, (10) is a consequence of

$$\lim_{t\to\infty} \frac{(q(t))^{q(t)} \int_A^\infty z^{q(t)} e^{-q(t)z} \mathrm{d}z}{\Gamma(q(t))} = 0. \tag{11}$$

To check the latter we write

$$\begin{aligned}
\int_A^\infty z^{q(t)} e^{-q(t)z} \mathrm{d}z &= \int_0^\infty (z+A)^{q(t)} e^{-q(t)(A+z)} \mathrm{d}z \\
&= (Ae^{-A})^{q(t)} \int_0^\infty (1 + z/A)^{q(t)} e^{-zq(t)} \mathrm{d}z \\
&\le (Ae^{-A})^{q(t)} \int_0^\infty e^{-z(1-A^{-1})q(t)} \mathrm{d}z \\
&= (Ae^{-A})^{q(t)} \frac{A}{(A-1)q(t)}.
\end{aligned} \tag{12}$$

Thus, the Stirling formula for the gamma-function yields (11) because $Ae^{-A}e < 1$ for $A > 1$. By applying (8) and (10) with $\ell \equiv 1$ we obtain

$$\int_{aq(t)}^{Aq(t)} y^{q(t)-1} e^{-y} \mathrm{d}y \ \sim \ \Gamma(q(t)), \quad t \to \infty.$$

This shows that (7) is a consequence of

$$\lim_{t\to\infty} \frac{I_2(t)}{\ell(t/q(t)) \int_{aq(t)}^{Aq(t)} y^{q(t)-1} e^{-y} \mathrm{d}y} = 1.$$

But this relation follows trivially from the uniform convergence theorem for slowly varying functions, see Theorem 1.5.2 in [3]. Indeed,

$$\frac{\inf_{z\in[a,A]}\ell(t/(zq(t)))}{\ell(t/q(t))} \leq \frac{I_2(t)}{\ell(t/q(t))\int_{aq(t)}^{Aq(t)} y^{q(t)-1}e^{-y}\mathrm{d}y} \leq \frac{\sup_{z\in[a,A]}\ell(t/(zq(t)))}{\ell(t/q(t))},$$

and both lower and upper bounds converge to one, as $t\to\infty$, by the aforementioned uniformity. □

Corollary 2.2. *Let U be a positive locally bounded function such that $U(x) = x^{\beta}\ell(x)$ for some ℓ slowly varying at infinity and $\beta\in\mathbb{R}$. If $q\in\mathcal{Q}$, then*

$$\int_0^{\infty} y^{q(t)-1}e^{-y}U(t/y)\mathrm{d}y \ \sim\ \Gamma(q(t))U(t/q(t)),\quad t\to\infty. \tag{13}$$

Proof. Follows from Lemma 2.1 by plugging $U(t/y) = (t/y)^{\beta}\ell(t/y)$ and using the asymptotic relation $\Gamma(q(t)-\beta)\sim(q(t))^{-\beta}\Gamma(q(t))$, since $q(t)\to\infty$, as $t\to\infty$. □

The next lemma is a counterpart of the Abelian implication of the Karamata theorem for Laplace–Stieltjes transforms.

Lemma 2.3. *Let $q\in\mathcal{Q}$ and $U:(0,\infty)\mapsto(0,\infty)$ be a right-continuous nonincreasing function and $U(x) = x^{-\gamma}\ell(1/x)$ for some ℓ slowly varying at infinity and $\gamma>0$. Then*

$$\int_{[0,\infty)} e^{-tx}\frac{(tx)^{q(t)}}{\Gamma(q(t)+1)}\mathrm{d}(-U(x)) \ \sim\ \gamma U(q(t)/t)/q(t) \ =\ \frac{\gamma t^{\gamma}\ell(t/q(t))}{q^{1+\gamma}(t)},\quad t\to\infty. \tag{14}$$

Proof. The equality in (14) is obvious. Making change of variable $x = q(t)y/t$ and integrating by parts, we obtain

$$\begin{aligned}\int_{[0,\infty)} e^{-tx}(tx)^{q(t)}\mathrm{d}(-U(x)) &= (q(t))^{q(t)}\int_{[0,\infty)}(ye^{-y})^{q(t)}\mathrm{d}_y(-U(q(t)y/t))\\ &= (q(t))^{q(t)}\int_{[0,\infty)}U(q(t)y/t)\mathrm{d}_y\left((ye^{-y})^{q(t)}\right)\\ &= (q(t))^{q(t)}\left(\int_{[0,y_1)}+\int_{[y_1,y_2]}+\int_{(y_2,\infty)}\right)\cdots\\ &= (q(t))^{q(t)}(J_1(y_1,t)+J_2(y_1,y_2,t)+J_3(y_2,t)),\end{aligned}$$

where $y_1<1<y_2$ are the real roots of the equation $ye^{-y}=1/4$, see Fig. 1. Arguing exactly as in the proof of equations (8) and (10), it can be checked the integrals $J_1(y_1,t)$ and $J_3(y_2,t)$ are negligible, that is,

$$\int_{[0,\infty)}U(q(t)y/t)\mathrm{d}_y\left((ye^{-y})^{q(t)}\right) \ \sim\ \int_{[y_1,y_2]}U(q(t)y/t)\mathrm{d}_y\left((ye^{-y})^{q(t)}\right),\quad t\to\infty. \tag{15}$$

Let us check that

$$\frac{(q(t))^{q(t)}}{\Gamma(1+q(t))}\int_{[y_1,y_2]}U(q(t)y/t)\mathrm{d}_y\left((ye^{-y})^{q(t)}\right) \ \sim\ \gamma U(q(t)/t)/q(t),\quad t\to\infty. \tag{16}$$

To this end, let $W_1:[1/4,\,e^{-1}]\mapsto[y_1,\,1]$ and $W_2:[1/4,\,e^{-1}]\mapsto[1,\,y_2]$ be the inverses of the function $y\mapsto ye^{-y}$ on its monotonicity intervals $[y_1,\,1]$ and $[1,\,y_2]$, respectively, see Fig. 1. Then, using the changes of variables

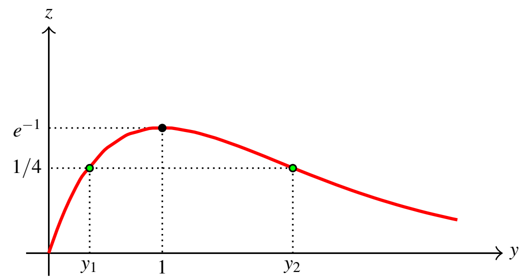

FIGURE 1. The plot of the function $y \mapsto ye^{-y}$ (solid red). The scale on the z-axis is three times larger than on the y-axis. The solid black dot is the maximum e^{-1} attained at $y = 1$, the green dots are the intersections with the horizontal line $z = 1/4$. Two inverse functions W_1 and W_2 are obtained by inverting monotone pieces of $y \mapsto ye^{-y}$ on $[y_1, 1]$ and $[1, y_2]$, respectively.

$y = W_{1,2}(z^{1/q(t)})$, we obtain

$$\begin{aligned}
&\int_{[y_1,y_2]} U\left(\frac{q(t)y}{t}\right) \mathrm{d}_y \left((ye^{-y})^{q(t)}\right) \\
&\quad = \left(\int_{[y_1,1]} + \int_{[1,y_2]}\right) U\left(\frac{q(t)y}{t}\right) \mathrm{d}_y \left((ye^{-y})^{q(t)}\right) \\
&\quad = \int_{[4^{-q(t)},e^{-q(t)}]} \left(U\left(\frac{q(t)}{t} W_1(z^{1/q(t)})\right) - U\left(\frac{q(t)}{t} W_2(z^{1/q(t)})\right)\right) \mathrm{d}z.
\end{aligned}$$

In view of the inequalities

$$y_1 \le W_1(z^{1/q(t)}) \le 1 \quad \text{and} \quad 1 \le W_2(z^{1/q(t)}) \le y_2,$$

which hold for all $z \in [4^{-q(t)}, e^{-q(t)}]$, the uniform convergence theorem for regularly varying functions, see Theorem 1.5.2 in [3], is applicable. Thus, for every fixed $\varepsilon \in (0,1)$ there exists $t_0 > 0$ such that for all $t > t_0$ and all $z \in [4^{-q(t)}, e^{-q(t)}]$, it holds

$$\begin{aligned}
(1-\varepsilon)U(q(t)/t) &\left((W_1(z^{1/q(t)}))^{-\gamma} - (W_2(z^{1/q(t)}))^{-\gamma}\right) \le \\
&U\left(\frac{q(t)}{t} W_1(z^{1/q(t)})\right) - U\left(\frac{q(t)}{t} W_2(z^{1/q(t)})\right) \le \\
&\qquad (1+\varepsilon)U(q(t)/t) \left((W_1(z^{1/q(t)}))^{-\gamma} - (W_2(z^{1/q(t)}))^{-\gamma}\right).
\end{aligned}$$

These inequalities demonstrate that (16) follows from

$$\lim_{t\to\infty} \frac{(q(t))^{q(t)+1}}{\Gamma(1+q(t))} \int_{[4^{-q(t)},e^{-q(t)}]} \left((W_1(z^{1/q(t)}))^{-\gamma} - (W_2(z^{1/q(t)}))^{-\gamma}\right) \mathrm{d}z \;=\; \gamma. \tag{17}$$

This latter can be checked by reversing the arguments. The changes of variables $y = W_{1,2}(z^{1/q(t)})$ yield

$$\begin{aligned}
&\frac{(q(t))^{q(t)+1}}{\Gamma(1+q(t))}\int_{[4^{-q(t)},e^{-q(t)}]}\Big((W_1(z^{1/q(t)}))^{-\gamma}-(W_2(z^{1/q(t)}))^{-\gamma}\Big)\,\mathrm{d}z\\
&\quad=\frac{(q(t))^{q(t)+1}}{\Gamma(1+q(t))}\int_{[y_1,y_2]}y^{-\gamma}\mathrm{d}_y\Big((ye^{-y})^{q(t)}\Big)\\
&\quad=\frac{(q(t))^{q(t)+1}}{\Gamma(1+q(t))}\frac{(q(t))^{\gamma}}{t^{\gamma}}\int_{[y_1,y_2]}(q(t)y/t)^{-\gamma}\mathrm{d}_y\Big((ye^{-y})^{q(t)}\Big)\\
&\quad\sim\ \frac{(q(t))^{q(t)+1}}{\Gamma(1+q(t))}\frac{(q(t))^{\gamma}}{t^{\gamma}}\int_{[0,\infty)}(q(t)y/t)^{-\gamma}\mathrm{d}_y\Big((ye^{-y})^{q(t)}\Big),\quad t\to\infty,
\end{aligned}$$

where the last passage follows from (15) applied with $U(y) = y^{-\gamma}$. It remains to note that

$$\int_{[0,\infty)}y^{-\gamma}\mathrm{d}_y\Big((ye^{-y})^{q(t)}\Big)=\gamma\int_0^\infty y^{q(t)-\gamma-1}e^{-yq(t)}\mathrm{d}y=\gamma(q(t))^{\gamma-q(t)}\Gamma(q(t)-\gamma),$$

which readily implies (17). The proof is complete. □

Remark 2.4. Using the same method we can also prove the analogue of Lemma 2.3 for nondecreasing integrators U. More precisely, if $q\in\mathscr{Q}$ and $U:(0,\infty)\mapsto(0,\infty)$ is a right-continuous nondecreasing function such that $U(x) = x^{\gamma}\ell(1/x)$ for some ℓ slowly varying at infinity and $\gamma > 0$, then

$$\int_{[0,\infty)}e^{-tx}\frac{(tx)^{q(t)}}{\Gamma(q(t)+1)}\mathrm{d}U(x)\ \sim\ \gamma U(q(t)/t)/q(t)\ =\ \frac{\gamma t^{-\gamma}\ell(t/q(t))}{q^{1-\gamma}(t)},\quad t\to\infty. \tag{18}$$

3. Proofs of Theorems 1.1, 1.3 and 1.4: Poissonization.

As in many previous works on occupancy schemes and random compositions we use the Poissonization–dePoissonization technique. Let $\Pi = (\Pi(t))_{t\geq 0}$ be a standard Poisson process which is assumed independent of the subordinator S and of the sample $(E_k)_{k\in\mathbb{N}}$. The Poissonized model is obtained by allocating the balls $(E_k)_{k\in\mathbb{N}}$ at the epochs of the Poisson process Π: the ball E_1 is dropped at the moment of the first jump of Π (which is standard exponential), the ball E_2 is dropped at the moment of the second jump of Π (which is the sum of two independent standard exponentials) and so on. The Poissonized model has an advantage that conditional on subordinator S the number of balls in different boxes form mutually independent Poisson processes with intensity of the number of balls in a particular interval being equal to its length. Note that the total number of balls allocated during $[0,t]$ is equal to $\Pi(t)$. For $r\in\mathbb{N}$ and $t\geq 0$, denote by $K_r(t) := K_{\Pi(t),r}$ the number of boxes containing r balls at time t in the Poissonized model. Note that, for $r\in\mathbb{N}$,

$$K_r(t)=\sum_{k\geq 1}\mathbb{1}_{\{\mathscr{Z}_{\Pi(t),k}=r\}},\quad t\geq 0, \tag{19}$$

and also

$$K_{\geq r}(t)=\sum_{k\geq 1}\mathbb{1}_{\{\mathscr{Z}_{\Pi(t),k}\geq r\}},\quad t\geq 0. \tag{20}$$

We first prove version of Theorems 1.1, 1.3 and 1.4 for the Poissonized model.

Proposition 3.1. *Under the assumptions of Theorem 1.1, it holds*

$$\big(K_{r_1(t)}(t),K_{r_2(t)}(t),\dots,K_{r_m(t)}(t)\big)\ \xrightarrow[t\to\infty]{d}\ (\mathscr{P}_1,\mathscr{P}_2,\dots,\mathscr{P}_m).$$

Proof. Let us show that, for arbitrary fixed $|z_i| \le 1$, $i = 1, \dots, m$

$$\mathbb{E}\left[\prod_{i=1}^m z_i^{K_{r_i(t)}(t)} \Big| S\right] \xrightarrow[t\to\infty]{a.s.} \exp\left(-\mathscr{I}_\alpha \sum_{i=1}^m u_i^{-\alpha-1}(1-z_i)\right). \tag{21}$$

From this the desired statement follows by applying the dominated convergence theorem because the right-hand side is the conditional (given S) multivariate generating function of $(\mathscr{P}_1, \mathscr{P}_2, \dots, \mathscr{P}_m)$.

To prove (21) pick $t_0 > 0$ so large that $r_i(t) < r_j(t)$ for all $t > t_0$ and $1 \le i < j \le m$. By the conditional independence of $(\mathscr{Z}_{\Pi(t),k})_{k\ge1}$ we have

$$\mathbb{E}\left[\prod_{i=1}^m z_i^{K_{r_i(t)}(t)} \Big| S\right] = \prod_{k\ge1}\mathbb{E}\left[\prod_{i=1}^m z_i^{\mathbb{1}_{\{\mathscr{Z}_{\Pi(t),k}=r_i(t)\}}} \Big| S\right] = \prod_{k\ge1}\left(1 - \sum_{i=1}^m (1-z_i)\mathbb{P}\{\mathscr{Z}_{\Pi(t),k} = r_i(t)|S\}\right).$$

By an elementary fact, see Lemma 4.8 in [13], the right-hand side of the last display converges a.s. to the right-hand side of (21) provided

$$\sum_{k\ge1}\mathbb{P}\{\mathscr{Z}_{\Pi(t),k} = r_i(t)|S\} \xrightarrow[t\to\infty]{a.s.} u_i^{-\alpha-1}\mathscr{I}_\alpha, \tag{22}$$

for all $i = 1, \dots, m$. To prove (22), let us introduce a box-counting (random) function

$$\rho(x) := \#\{k \ge 1 : e^{-S(\tau_k-)} - e^{-S(\tau_k)} > x\}, \quad x > 0. \tag{23}$$

Note that ρ is nonincreasing and $\rho(x) = 0$ for $x \ge 1$. In what follows we shall frequently write integrals with $-\rho(x)$ in the integrator over infinite intervals $(0,\infty)$ keeping in mind that that the actual domain of integration is $(0,1)$.

Since the conditional (given S) distribution of $\mathscr{Z}_{\Pi(t),k}$ is Poisson with mean $t(e^{-S(\tau_k-)} - e^{-S(\tau_k)})$, we can write

$$\sum_{k\ge1}\mathbb{P}\{\mathscr{Z}_{\Pi(t),k} = r_i(t)|S\} = \int_{(0,\infty)} e^{-tx}\frac{(tx)^{r_i(t)}}{r_i(t)!}\mathrm{d}(-\rho(x)), \quad i = 1, \dots, m.$$

By Theorem 5.1 in [11]

$$\frac{\rho(x)}{x^{-\alpha}\ell(1/x)} \xrightarrow{a.s.} \mathscr{I}_\alpha, \quad x \downarrow 0. \tag{24}$$

Applying Lemma 2.3 with $U(x) = \rho(x)$ and $q(t) = r_i(t)$ we infer

$$\sum_{k\ge1}\mathbb{P}\{\mathscr{Z}_{\Pi(t),k} = r_i(t)|S\} \sim \frac{\alpha\mathscr{I}_\alpha t^\alpha}{(r_i(t))^{\alpha+1}}\ell(t/r_i(t)), \quad t\to\infty \quad \text{a.s.}, \tag{25}$$

for all $i = 1, \dots, m$. Since

$$\frac{\alpha t^\alpha \ell(t/r_i(t))}{(r_i(t))^{\alpha+1}} \sim \frac{\alpha t^\alpha \ell(t/r(t))}{(u_i r(t))^{\alpha+1}} \to u_i^{-\alpha-1}, \quad t\to\infty,$$

we obtain (22) for all $i = 1, \dots, m$. The proof is complete. □

Using similar arguments we can also deduce the following counterpart of Theorem 1.3. Note that we do not assume regular variation of w here.

Proposition 3.2. *Assume that* $w \in \mathscr{W}^r \cap \mathscr{N}$. *Then* (4) *implies*

$$\frac{(w(t))^{\alpha+1}K_{w(t)}(t)}{t^\alpha\ell(t/w(t))} \xrightarrow[t\to\infty]{P} \alpha\mathscr{I}_\alpha.$$

Proof. From representation (19) and Lemma 2.3 applied with $U(x)=\rho(x)$ and $q(t)=w(t)$ it follows that

$$\mathbb{E}[K_{w(t)}(t)|S]=\sum_{k\geq 1}\mathbb{P}\{\mathscr{Z}_{\Pi(t),k}=w(t)|S\}\ \sim\ \frac{\alpha\mathscr{I}_{\alpha}t^{\alpha}}{(w(t))^{\alpha+1}}\ell(t/w(t)),\quad t\to\infty\quad \text{a.s.,} \tag{26}$$

and by our assumptions on w the right-hand side is divergent to $+\infty$ a.s. Further, from (19) it follows that

$$\mathrm{Var}\,[K_{w(t)}(t)|S]=\sum_{k\geq 1}\mathrm{Var}\,[\mathbb{1}_{\{\mathscr{Z}_{\Pi(t),k}=w(t)\}}|S]\leq \mathbb{E}[K_{w(t)}(t)|S].$$

Thus, by Chebyshev's inequality, for every fixed $\varepsilon>0$,

$$\mathbb{P}\left\{\left|\frac{K_{w(t)}(t)}{\mathbb{E}[K_{w(t)}(t)|S]}-1\right|>\varepsilon\Big|S\right\}\overset{a.s.}{\underset{t\to\infty}{\longrightarrow}}0.$$

By the dominated convergence theorem

$$\frac{K_{w(t)}(t)}{\mathbb{E}[K_{w(t)}(t)|S]}\overset{P}{\underset{t\to\infty}{\longrightarrow}}1,$$

which, by Slutsky's lemma and (26), yields the desired claim. □

Finally, here is a counterpart of Theorem 1.4 for the Poissonized model.

Proposition 3.3. *Assume that $q\in\mathscr{Q}\cap\mathscr{N}$. Then* (4) *implies*

$$\frac{(q(t))^{\alpha}K_{\geq q(t)}(t)}{t^{\alpha}\ell(t/q(t))}\overset{P}{\underset{t\to\infty}{\longrightarrow}}\mathscr{I}_{\alpha}.$$

Proof. Representation (20) implies

$$\begin{aligned}\mathbb{E}[K_{\geq q(t)}(t)|S]&=\sum_{k\geq 1}\mathbb{P}\{\mathscr{Z}_{\Pi(t),k}\geq q(t)|S\}=\int_{(0,\infty)}\left(\sum_{j\geq q(t)}e^{-tx}\frac{(tx)^j}{j!}\right)\mathrm{d}(-\rho(x))\\&=t\int_{(0,\infty)}\rho(x)e^{-tx}\frac{(tx)^{q(t)-1}}{(q(t)-1)!}\mathrm{d}x=\int_{(0,\infty)}\rho(y/t)e^{-y}\frac{y^{q(t)-1}}{(q(t)-1)!}\mathrm{d}x,\end{aligned}$$

where the penultimate equality follows upon integration by parts from an easily checked relation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sum_{j=k}^{\infty}e^{-\lambda}\frac{\lambda^j}{j!}\right)=e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!},\quad k\in\mathbb{N}.$$

Thus, Corollary 2.2 applied with $U(x)=\rho(1/x)$ yields

$$\mathbb{E}[K_{\geq q(t)}(t)|S]\ \sim\ \rho(q(t)/t)\ \sim\ \mathscr{I}_{\alpha}(t/q(t))^{\alpha}\ell(t/q(t)),\quad t\to\infty\quad \text{a.s.,}$$

by (24). Since $t/q(t)\to\infty$, as $t\to\infty$, we have $\mathbb{E}[K_{\geq q(t)}(t)|S]\to\infty$ a.s. From the inequality $\mathrm{Var}\,[K_{\geq q(t)}(t)|S]\leq \mathbb{E}[K_{\geq q(t)}(t)|S]$, we obtain

$$\frac{K_{\geq q(t)}(t)}{\mathbb{E}[K_{\geq q(t)}(t)|S]}\overset{P}{\underset{t\to\infty}{\longrightarrow}}1,$$

by Chebyshev's inequality and dominated convergence. This completes the proof of Proposition 3.3. □

From (25) is it also clear that whenever $q\in\mathcal{Q}\cap\mathcal{N}$ is such that $r(t)=o(q(t))$, then

$$\mathbb{P}\{K_{q(t)}(t)\neq 0|S\}=\mathbb{P}\{K_{q(t)}(t)\geq 1|S\}\leq \mathbb{E}[K_{q(t)}(t)|S]\xrightarrow[t\to\infty]{a.s.}0.$$

Thus,

$$K_{q(t)}(t)\xrightarrow[t\to\infty]{P}0, \tag{27}$$

which is the analogue of (6) for the Poissonized model.

4. Proofs of Theorems 1.1, 1.3 and 1.4: dePoissonization.

This part of the proofs is called dePoissonization and its aim is to deduce Theorems 1.1, 1.3 and 1.4 from Propositions 3.1, 3.2 and 3.3, respectively. We start with an easier implication Proposition 3.3 $\Rightarrow$ Theorem 1.4. It is simpler because $(K_{n,\geq i})_{n\in\mathbb{N}}$ is nondecreasing.

4.1. **Proof of Theorem 1.4 using Proposition 3.3.** Fix $\varepsilon\in(0,1)$ and note that on the event $\{\Pi((1-\varepsilon)t)\leq \lfloor t\rfloor\leq \Pi((1+\varepsilon)t)\}$ we have

$$\frac{(q(\lfloor t\rfloor))^{\alpha}K_{\geq q(\lfloor t\rfloor)}((1-\varepsilon)t)}{\lfloor t\rfloor^{\alpha}\ell(\lfloor t\rfloor/q(\lfloor t\rfloor))}\leq \frac{(q(\lfloor t\rfloor))^{\alpha}K_{\lfloor t\rfloor,\geq q(\lfloor t\rfloor)}}{\lfloor t\rfloor^{\alpha}\ell(\lfloor t\rfloor/q(\lfloor t\rfloor))}\leq \frac{(q(\lfloor t\rfloor))^{\alpha}K_{\geq q(\lfloor t\rfloor)}((1+\varepsilon)t)}{\lfloor t\rfloor^{\alpha}\ell(\lfloor t\rfloor/q(\lfloor t\rfloor))}.$$

Put $q^{\pm\varepsilon}(t):=q(\lfloor t/(1\pm\varepsilon)\rfloor)$ and note that $q\in\mathcal{Q}\cap\mathcal{N}$ implies $q^{\pm\varepsilon}\in\mathcal{Q}\cap\mathcal{N}$. Further,

$$\begin{aligned}\frac{(q(\lfloor t\rfloor))^{\alpha}K_{\geq q(\lfloor t\rfloor)}((1\pm\varepsilon)t)}{\lfloor t\rfloor^{\alpha}\ell(\lfloor t\rfloor/q(\lfloor t\rfloor))}&=\frac{(q^{\pm\varepsilon}(t(1\pm\varepsilon)))^{\alpha}K_{q^{\pm\varepsilon}(t(1\pm\varepsilon))}((1\pm\varepsilon)t)}{\lfloor t\rfloor^{\alpha}\ell(\lfloor t\rfloor/q^{\pm\varepsilon}(\lfloor(1\pm\varepsilon)t\rfloor))}\\&\sim\frac{(q^{\pm\varepsilon}(t(1\pm\varepsilon)))^{\alpha}K_{q^{\pm\varepsilon}(t(1\pm\varepsilon))}((1\pm\varepsilon)t)}{\lfloor t\rfloor^{\alpha}\ell(\lfloor(1\pm\varepsilon)t\rfloor/q^{\pm\varepsilon}(\lfloor(1\pm\varepsilon)t\rfloor))},\quad t\to\infty\quad\text{a.s.}\end{aligned}$$

by the slow variation of ℓ. By Proposition 3.3 the right-hand side converges in probability to $(1\pm\varepsilon)^{\alpha}\mathcal{I}_{\alpha}$, as $t\to\infty$. Sending $\varepsilon\downarrow 0$ and noting that

$$\lim_{t\to\infty}\mathbb{P}\{\Pi((1-\varepsilon)t)\leq\lfloor t\rfloor\leq\Pi((1+\varepsilon)t)\}=1, \tag{28}$$

we arrive at the conclusion of Theorem 1.4.

For the remaining implications, namely Proposition 3.1 $\Rightarrow$ Theorem 1.1 and Proposition 3.2 $\Rightarrow$ Theorem 1.3, more sophisticated arguments are necessary due to lack of monotonicity of $(K_{n,i})_{n\in\mathbb{N}}$, $i\in\mathbb{N}$. In particular, the first order result for $(\Pi(t))$ given by (28) is not sufficient.

4.2. **Proofs of Theorems 1.1 and 1.3 using Propositions 3.1 and 3.2.** We shall prove both implications simultaneously. Throughout this subsection $v(t)$ denotes either $r_i(t)$, for some fixed $i=1,\dots,m$, in the settings of Theorem 1.1 or $w(t)$ in the settings of Theorem 1.3. In both cases v is regularly varying and integer-valued. Set

$$c(t)\ :=\ t^{\alpha}\ell(t/v(\lfloor t\rfloor))/(v(\lfloor t\rfloor))^{\alpha+1}\ \sim\ t^{\alpha}\ell(t/v(t))/(v(t))^{\alpha+1},\quad t\to\infty.$$

It is enough to prove

$$\frac{K_{v(\lfloor t\rfloor)}(t)-K_{\lfloor t\rfloor,v(\lfloor t\rfloor)}}{c(t)}=\frac{K_{\Pi(t),v(\lfloor t\rfloor)}-K_{\lfloor t\rfloor,v(\lfloor t\rfloor)}}{c(t)}\xrightarrow[t\to\infty]{P}0. \tag{29}$$

Indeed, under the assumptions of Theorem 1.1, $c(t)\to c$, as $t\to\infty$, for some $c>0$. Therefore, (29) is equivalent to $\mathbb{P}\{K_{v(\lfloor t\rfloor)}(t)\neq K_{\lfloor t\rfloor,v(\lfloor t\rfloor)}\}\to 0$, as $t\to\infty$. This proves that Proposition 3.1 and (29) yield Theorem 1.1. Similarly, Proposition 3.2 and (29) yield Theorem 1.3.

Fix $T>0$, $\delta>0$ and note that for large enough $t>0$ we have

$$\mathbb{P}\{|K_{\Pi(t),\nu(\lfloor t\rfloor)}-K_{\lfloor t\rfloor,\nu(\lfloor t\rfloor)}|\geq \delta c(t)\}\leq 1-\mathbb{P}\{\Pi(t-T\sqrt{t})\leq \lfloor t\rfloor\leq \Pi(t+T\sqrt{t})\}$$
$$+\mathbb{P}\{\Pi(t-T\sqrt{t})\leq \lfloor t\rfloor\leq \Pi(t+T\sqrt{t}),\,|K_{\Pi(t),\nu(\lfloor t\rfloor)}-K_{\lfloor t\rfloor,\nu(\lfloor t\rfloor)}|\geq \delta c(t)\}.$$

By the central limit theorem for Poisson processes we see that it is enough to check

$$\lim_{T\to\infty}\limsup_{t\to\infty}\mathbb{P}\{\Pi(t-T\sqrt{t})\leq \lfloor t\rfloor\leq \Pi(t+T\sqrt{t}),|K_{\Pi(t),\nu(\lfloor t\rfloor)}-K_{\lfloor t\rfloor,\nu(\lfloor t\rfloor)}|\geq \delta c(t)\}=0,$$

for every $\delta>0$, which in turn follows from

$$\limsup_{t\to\infty}\frac{\mathbb{E}\left(\sup_{s\in[t-T\sqrt{t},t+T\sqrt{t}]}|K_{\Pi(s),\nu(\lfloor t\rfloor)}-K_{\Pi(t),\nu(\lfloor t\rfloor)}|\right)}{c(t)}=0, \tag{30}$$

for arbitrary fixed $T>0$. In order to prove (30) we estimate the supremum as follows:

$$\sup_{s\in[t-T\sqrt{t},t+T\sqrt{t}]}|K_{\Pi(s),\nu(\lfloor t\rfloor)}-K_{\Pi(t),\nu(\lfloor t\rfloor)}|\leq\sum_{k\geq 1}\sup_{s\in[t-T\sqrt{t},t+T\sqrt{t}]}|\mathbb{1}_{\{\mathscr{Z}_{\Pi(s),k}=\nu(\lfloor t\rfloor)\}}-\mathbb{1}_{\{\mathscr{Z}_{\Pi(t),k}=\nu(\lfloor t\rfloor)\}}|.$$

Note that the k-th summand on the right-hand side can be non-zero only in the following two scenarios:

- the number of balls in the k-th box at time $t-T\sqrt{t}$ was strictly smaller than $\nu(\lfloor t\rfloor)$ but at time $t+T\sqrt{t}$ it became larger or equal than $\nu(\lfloor t\rfloor)$, that is, at *some* epoch during $[t-T\sqrt{t},t+T\sqrt{t}]$ the number of balls in the k-th box increased to $\nu(\lfloor t\rfloor)$;
- at least one ball has fallen during $[t-T\sqrt{t},t+T\sqrt{t}]$ in the k-th box which contained exactly $\nu(\lfloor t\rfloor)$ balls at time $t-T\sqrt{t}$.

Thus,

$$\sup_{s\in[t-T\sqrt{t},t+T\sqrt{t}]}|K_{\Pi(s),\nu(\lfloor t\rfloor)}-K_{\Pi(t),\nu(\lfloor t\rfloor)}|\leq$$
$$\sum_{k\geq 1}\left(\left(\sum_{j=0}^{\nu(\lfloor t\rfloor)-1}\mathbb{1}_{\{\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}=j,\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}\geq\nu(\lfloor t\rfloor)\}}\right)+\mathbb{1}_{\{\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}=\nu(\lfloor t\rfloor),\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}>\nu(\lfloor t\rfloor)\}}\right),$$

and thereupon

$$\mathbb{E}\left(\sup_{s\in[t-T\sqrt{t},t+T\sqrt{t}]}|K_{\Pi(s),\nu(\lfloor t\rfloor)}-K_{\Pi(t),\nu(\lfloor t\rfloor)}|\right)\leq$$
$$\leq\sum_{k\geq 1}\sum_{j=0}^{\nu(\lfloor t\rfloor)-1}\mathbb{P}\{\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}=j,\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}\geq\nu(\lfloor t\rfloor)\}$$
$$+\sum_{k\geq 1}\mathbb{P}\{\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}=\nu(\lfloor t\rfloor),\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}>\nu(\lfloor t\rfloor)\}$$
$$=\sum_{k\geq 1}\mathbb{P}\{\nu(\lfloor t\rfloor)-(\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}-\mathscr{Z}_{\Pi(t-T\sqrt{t}),k})\leq\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}\leq\nu(\lfloor t\rfloor)-1\}$$
$$+\sum_{k\geq 1}\mathbb{P}\{\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}=\nu(\lfloor t\rfloor),\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}-\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}\geq 1\}.$$

Recall that, conditional on S, $(\mathscr{Z}_{\Pi(s),k})_{s\geq 0}$ is a Poisson process and in particular has independent and stationary increments. Thus, exploiting definition (23) of the function ρ and Markov's inequality for the second summand, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[t-T\sqrt{t},t+T\sqrt{t}]}|K_{\Pi(s),\nu(\lfloor t\rfloor)}-K_{\Pi(t-T\sqrt{t}),\nu(\lfloor t\rfloor)}|\right)\\
&\leq \mathbb{E}\sum_{k\geq 1}\mathbb{P}\{\nu(\lfloor t\rfloor)-(\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}-\mathscr{Z}_{\Pi(t-T\sqrt{t}),k})\leq \mathscr{Z}_{\Pi(t-T\sqrt{t}),k}\leq \nu(\lfloor t\rfloor)-1|S\}\\
&+\mathbb{E}\sum_{k\geq 1}\mathbb{P}\{\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}=\nu(\lfloor t\rfloor)|S\}\mathbb{P}\{\mathscr{Z}_{\Pi(t+T\sqrt{t}),k}-\mathscr{Z}_{\Pi(t-T\sqrt{t}),k}\geq 1|S\}\\
&=\int_{(0,\infty)}\mathbb{P}\{\nu(\lfloor t\rfloor)-\mathrm{Poi}'(2T\sqrt{t}x)\leq \mathrm{Poi}((t-T\sqrt{t})x)\leq \nu(\lfloor t\rfloor)-1\}\mathrm{d}(-\mathbb{E}\rho(x))\\
&+\int_{(0,\infty)}e^{-x(t-T\sqrt{t})}\frac{x^{\nu(\lfloor t\rfloor)}(t-T\sqrt{t})^{\nu(\lfloor t\rfloor)}}{\nu(\lfloor t\rfloor)!}(2Tx\sqrt{t})\mathrm{d}(-\mathbb{E}\rho(x))\\
&=: P_1(t)+P_2(t),
\end{aligned}$$

where Poi and Poi′ are independent Poisson random variables. We first deal with $P_2(t)$. By Theorem 5.1 in [11]

$$\mathbb{E}\rho(x)\ \sim\ \mathrm{const}\cdot x^{-\alpha}\ell(1/x),\quad x\to\infty, \tag{31}$$

where here and below "const" denotes some positive constants which does not depend on x and/or t but might depend on all other parameters. Let $(s(t))_{t\geq 0}$ be such that $s(t)-T\sqrt{s(t)}=t$ for all $t\geq 0$. Lemma 2.3 applied with $U(x)=\mathbb{E}\rho(x)$, t replaced by $s(t)$ and $q(t)=\nu(\lfloor s(t)\rfloor)+1$, yields

$$\begin{aligned}
P_2(s(t))&=\frac{2T\sqrt{s(t)}(\nu(\lfloor s(t)\rfloor)+1)}{t}\int_{(0,\infty)}e^{-xt}\frac{(tx)^{\nu(\lfloor s(t)\rfloor)+1}}{(\nu(\lfloor s(t)\rfloor)+1)!}\mathrm{d}\,(-\mathbb{E}\rho(x))\\
&\sim\ \mathrm{const}\cdot\frac{2T\sqrt{s(t)}(\nu(\lfloor s(t)\rfloor)+1)}{t}\alpha t^{\alpha}\ell(t/(\nu(\lfloor s(t)\rfloor)+1))/(\nu(\lfloor s(t)\rfloor)+1)^{1+\alpha},\quad t\to\infty.
\end{aligned}$$

Since ν is regularly varying both in the settings of Theorem 1.1 and Theorem 1.3, the relation $s(t)\sim t$, as $t\to\infty$, implies $\nu(\lfloor s(t)\rfloor)+1\sim\nu(t)$ and thereupon

$$P_2(s(t))\ \sim\ \mathrm{const}\cdot\frac{t^{\alpha-1/2}\ell(t/\nu(t))}{(\nu(t))^{\alpha}},\quad t\to\infty.$$

If $\nu=r_i$, then the index of regular variation of ν is $\alpha/(\alpha+1)<1/2$, see Remark 1.2 and thereupon

$$\lim_{t\to\infty}\frac{t^{\alpha-1/2}\ell(t/r_i(t))}{r_i^{\alpha}(t)}=0=\lim_{t\to\infty}\frac{t^{\alpha-1/2}\ell(t/r_i(t))}{r_i^{\alpha}(t)c(t)}. \tag{32}$$

If $\nu=w\in\mathscr{W}_{RV}^{r}$, that is, we are in settings of Theorem 1.3, then

$$\lim_{t\to\infty}\frac{t^{\alpha-1/2}\ell(t/w(t))}{w^{\alpha}(t)c(t)}=\lim_{t\to\infty}\frac{w(t)}{t^{1/2}}=0. \tag{33}$$

holds as well, since we assume $w(t)=o(r(t))$ and $r(t)=o(\sqrt{t})$, as $t\to\infty$.

In order to estimate $P_1(t)$ we decompose it as follows: for fixed $0<a<1<A$,

$$P_1(t)=\left(\int_{(0,a\nu(\lfloor t\rfloor)/t)}+\int_{[a\nu(\lfloor t\rfloor)/t,A\nu(\lfloor t\rfloor)/t]}+\int_{(A\nu(\lfloor t\rfloor)/t,\infty)}\right)\cdots\ =:\ P_{11}(t)+P_{12}(t)+P_{13}(t).$$

As far as $P_{12}(t)$ is concerned we have, by the stochastic monotonicity of $\mathrm{Poi}(\lambda)$ in parameter λ,

$$P_{12}(t)\leq\int_{[a\nu(\lfloor t\rfloor)/t,A\nu(\lfloor t\rfloor)/t]}\mathbb{P}\{\nu(\lfloor t\rfloor)-\mathrm{Poi}'(2AT\nu(\lfloor t\rfloor)/\sqrt{t})\leq\mathrm{Poi}((t-T\sqrt{t})x)\leq\nu(\lfloor t\rfloor)-1\}\mathrm{d}(-\mathbb{E}\rho(x)).$$

Pick $M\in\mathbb{N}$ so large that $\alpha<M/(M+2)$ and note that

$$\begin{aligned}P_{12}(t)&\leq \mathbb{P}\{\mathrm{Poi}'(2ATv(\lfloor t\rfloor)/\sqrt{t})\geq M\}\int_{[av(\lfloor t\rfloor)/t,Av(\lfloor t\rfloor)/t]}\mathrm{d}(-\mathbb{E}\rho(x))\\&+\sum_{j=1}^{M-1}\mathbb{P}\{\mathrm{Poi}'(2ATv(\lfloor t\rfloor)/\sqrt{t})=j\}\sum_{i=1}^{j}\int_{[av(\lfloor t\rfloor)/t,Av(\lfloor t\rfloor)/t]}\mathbb{P}\{\mathrm{Poi}((t-T\sqrt{t})x)=v(\lfloor t\rfloor)-i\}\mathrm{d}(-\mathbb{E}\rho(x))\\&\leq \mathrm{const}\cdot\left(\frac{v(t)}{\sqrt{t}}\right)^{M}\mathbb{E}\rho(av(\lfloor t\rfloor)/t)\\&+\mathrm{const}\cdot\sum_{j=1}^{M-1}\left(\frac{v(t)}{\sqrt{t}}\right)^{j}\sum_{i=1}^{j}\int_{(0,\infty)}\mathbb{P}\{\mathrm{Poi}((t-T\sqrt{t})x)=v(\lfloor t\rfloor)-i\}\mathrm{d}(-\mathbb{E}\rho(x)),\end{aligned}$$

where the bound $\mathbb{P}\{\mathrm{Poi}(\lambda)\geq M\}\leq\lambda^M$ has been utilized for the estimate of $\mathbb{P}\{\mathrm{Poi}'(2ATv(\lfloor t\rfloor)/\sqrt{t})\geq M\}$. The first term goes to zero, as $t\to\infty$, by (31) and the choice of M. By Lemma 2.3 every integral in the second term is $O(t^{\alpha}\ell(t/v(t))/(v(t))^{\alpha+1})$ (with a possibly dependent on $j=1,\dots,M$ constant in the Landau symbol). Thus, by (32) and (33) all summands in the second term tend to zero upon division by $c(t)$, as $t\to\infty$.

For the estimates of $P_{11}(t)$ and $P_{13}(t)$ we employ known bounds for Poisson tail probabilities borrowed from [4]. By part (ii) of Proposition 1 in [4], we have, for large enough $t>0$,

$$\begin{aligned}P_{11}(t)&\leq\int_{(0,av(\lfloor t\rfloor)/t)}\mathbb{P}\{v(\lfloor t\rfloor)\leq\mathrm{Poi}((t+T\sqrt{t})x)\}\mathrm{d}(-\mathbb{E}\rho(x)),\\&\leq\int_{(0,av(\lfloor t\rfloor)/t)}\left(1-\frac{(t+T\sqrt{t})x}{1+v(\lfloor t\rfloor)}\right)^{-1}\mathbb{P}\{\mathrm{Poi}((t+T\sqrt{t})x)=v(\lfloor t\rfloor)\}\mathrm{d}(-\mathbb{E}\rho(x))\\&\leq\mathrm{const}\cdot\int_{(0,av(\lfloor t\rfloor)/t)}\mathbb{P}\{\mathrm{Poi}((t+T\sqrt{t})x)=v(\lfloor t\rfloor)\}\mathrm{d}(-\mathbb{E}\rho(x)),\end{aligned}$$

and this converges to zero exponentially fast, which is readily seen upon integration by parts, the bound

$$\left|\frac{\mathrm{d}\mathbb{P}\{\mathrm{Poi}(\lambda)=j\}}{\mathrm{d}\lambda}\right|\leq\mathbb{P}\{\mathrm{Poi}(\lambda)=j\}+\mathbb{P}\{\mathrm{Poi}(\lambda)=j-1\},\quad j\in\mathbb{N},$$

and (9). Similarly, by Proposition 1(i) in [4]

$$\begin{aligned}P_{13}(t)&\leq\int_{(Av(\lfloor t\rfloor)/t,\infty)}\mathbb{P}\{\mathrm{Poi}((t-T\sqrt{t})x)\leq v(\lfloor t\rfloor)\}\mathrm{d}(-\mathbb{E}\rho(x))\\&\leq\int_{(Av(\lfloor t\rfloor)/t,\infty)}\left(1-\frac{v(\lfloor t\rfloor)}{(t+T\sqrt{t})x}\right)^{-1}\mathbb{P}\{\mathrm{Poi}((t+T\sqrt{t})x)=v(\lfloor t\rfloor)\}\mathrm{d}(-\mathbb{E}\rho(x))\\&\leq\mathrm{const}\cdot\int_{(Av(\lfloor t\rfloor)/t,\infty)}\mathbb{P}\{\mathrm{Poi}((t+T\sqrt{t})x)=v(\lfloor t\rfloor)\}\mathrm{d}(-\mathbb{E}\rho(x)),\end{aligned}$$

and this also converges to zero in view of (12). This finishes the proof of (30) and of Theorem 1.1.

Using similar arguments it can be checked that (27) implies (6).

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