Advanced probability theory

Lecture 1. Measures and their extensions

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6. Lebesgue and Lebesgue–Stieltjes measures.

The notion of a measure $\mu(A)$ of a set A is a natural extension of:

- the length $\ell(\Delta)$ of an interval $\Delta \subset \mathbb{R}$;
- the area $S(F)$ of a planar set $A \subset \mathbb{R}^2$;
- the volume $V(G)$ of a solid body in $\mathbb{R}^3$;
- the increment $\phi(b) - \phi(a)$ of a non-decreasing function ϕ on an interval $(a, b]$;
- the integral of a non-negative function taken over some subset of $\mathbb{R}, \mathbb{R}^2, \ldots$.

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Let us consider a family $\mathfrak{S}$ of subsets of $\mathbb{R}^2$ defined by the pair of double inequalities:

$$a \leq x \leq b, \quad a < x \leq b, \quad a \leq x < b \quad \text{or} \quad a < x < b$$

and

$$c \leq y \leq d, \quad c < y \leq d, \quad c \leq y < d \quad \text{or} \quad c < y < d.$$

Such sets are called rectangles.

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Such sets are called rectangles.
Let $m : \mathfrak{S} \mapsto [0, \infty)$ be the function (pre-measure, measure) defined by the following two rules:

- $m(\varnothing) = 0$;
- for a non-empty rectangle R (closed, open, semi-open) given by the quadruple a, b, c, d, put $m(R) = (b - a)(d - c)$.

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Area of planar sets

It is easy to check that m possesses the following property: if $\mathfrak{S} \ni P = \cup_{i=1}^{m} P_i$, where $P_i \cap P_j = \varnothing$ for $i \neq j$ and $P_i \in \mathfrak{S}$, then

$$m(P) = \sum_{i=1}^{m} m(P_i).$$

This is called (finite) **additivity** of m.

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This is called (finite) **additivity** of m.
Our goal is to extend the definition of m from the family $\mathfrak{S}$ to a larger class of planar sets.

Definition

A set $A \subset \mathbb{R}^2$ is called **elementary** *if it can be represented as a finite union of pairwise disjoint rectangles.*

Theorem

The union, intersection, difference and symmetric difference of two elementary sets is again an elementary set. We say that the family of elementary sets form a **ring of sets**.

Area of planar sets

The family of elementary sets will be denoted by $\mathfrak{R}(\mathfrak{S})$. Define of function m' on $\mathfrak{R}(\mathfrak{S})$ by the following rule: if $A = \cup_{i=1}^{m} P_i$ for some disjoint $P_i \in \mathfrak{S}$, $i = 1, \ldots, m$, then

$$m'(A) := \sum_{i=1}^{m} m(P_i).$$

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We must check that this definition does not depend on the way we partition A. Indeed, if

$$A = \cup_i P_i = \cup_j Q_j, \quad P_i, Q_j \in \mathfrak{S},$$

then $P_i = \cup_j (P_i \cap Q_j)$, $Q_j = \cup_i (Q_j \cap P_i)$ and, therefore,

$$m'(A) = \sum_i m(P_i) = \sum_i \sum_j m(P_i \cap Q_j) = \sum_j m(Q_j),$$

where we have used that $P_i \cap Q_j$ is a rectangle and m is additive for rectangles.

Area of planar sets

The function m' is nonnegative and additive, and m' and m coincide on rectangles. We say that m' is an **extension** of m to $\mathfrak{R}(\mathfrak{S})$.

Theorem

Let $A \in \mathfrak{R}(\mathfrak{S})$ be an elementary set and $\{A_n\} \subset \mathfrak{R}(\mathfrak{S})$ be at most countable family of elementary sets such that $A \subset \cup_i A_i$, then m' is **subbadditive**, *that is, $m'(A) \leq \sum_i m'(A_i)$.*

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Corollary

Let $A \in \mathfrak{R}(\mathfrak{S})$ be an elementary set and $\{A_n\} \subset \mathfrak{R}(\mathfrak{S})$ be a countable family of elementary sets such that $A = \cup_i A_i$, then $m'(A) = \sum_i m'(A_i)$.

This property is called **countable additivity or σ-additivity**.

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Yes! The corresponding construction is called the **Lebesgue extension**.
Suppose for the time being that all elementary sets that we consider lie in $E := [0, 1] \times [0, 1]$.

Definition

The outer measure of a set $A \subset E$ is the number

$$\mu^*(A) := \inf_{A \subset \cup P_k} \sum_k m(P_k),$$

where the infimum is taken over all at most countable coverings of A by rectangles P_k.

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- If A is an elementary set, then $m'(A) = \mu^*(A)$. Indeed, if $P_1, P_2, \ldots, P_k$ are the rectangles comprising A, then by definition of m'

$$m'(A) = \sum_{i=1}^{k} m(P_i).$$

  Therefore, $\mu^*(A) \leq m'(A)$. On the other hand, by Theorem 3 $m'(A) \leq \sum_{i=1}^{k} m(P_i)$.

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Therefore, $\mu^*(A) \leq m'(A)$. On the other hand, by Theorem 3 $m'(A) \leq \sum_{i=1}^{k} m(P_i)$.

Theorem

If $A \subset \cup_i A_i$, then $\mu^*(A) \leq \sum_i \mu^*(A_i)$, where $\{A_i\}$ is at most countable family of subsets of $\mathbb{R}^2$.

Area of planar sets

Definition

A set $A \subset E$ is called measurable (in Lebesgue sense) if, for every $\varepsilon > 0$, there exist an elementary set $B = B(\varepsilon)$ such that

$$\mu^*(A \Delta B) < \varepsilon.$$

The class of measurable set is denoted by $\mathfrak{M}_E$. The function μ^* restricted to $\mathfrak{M}_E$ is called **Lebesgue measure** *and is denoted by μ.*

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Theorem

If A, B are measurable, then $E \setminus A$, $A \cup B$, $A \cap B$, $A \setminus B$ and $A \Delta B$ are measurable.

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If $A_1, A_2, \ldots, A_n$ are pairwise disjoint measurable sets, then

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

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If $A_1, A_2, \ldots, A_n, \ldots$ *are pairwise disjoint measurable sets, then*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Algebras and semi-algebras of sets

Definition

A non-empty family $\mathfrak{S}$ *of sets is called* **semiring** *if* $\varnothing \in \mathfrak{S}$, $\mathfrak{S}$ *is closed under intersections and has the following property: if* $A_1, A \in \mathfrak{S}$ *and* $A_1 \subset A$, *then there exists* $A_2, \ldots, A_k \in \mathfrak{S}$ *such that*

$$A = \cup_{i=1}^{k} A_i.$$

Definition

A function μ *defined on a semiring* $\mathfrak{S}_\mu$ *is called* **measure** *(sometimes, premeasure), if*

- *for every* $A \in \mathfrak{S}_\mu$, $\mu(A) \in [0, \infty)$;
- $\mu(A)$ *is additive, that is, if* $A = \cup_{i=1}^{n} A_i$ *and* $A_i \cap A_j = \varnothing$, $i \neq j$,

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i).$$

Rectangles on $\mathbb{R}^2$ form a semi-ring and m is a measure on it!

Algebras and semi-algebras of sets

Definition

A non-empty family $\mathfrak{R}$ of sets is called **ring** *if, for every $A, B \in \mathfrak{R}$, it holds $A \Delta B \in \mathfrak{R}$ and $A \cap B \in \mathfrak{R}$.*

$$A \cup B = (A \Delta B) \Delta (A \cap B), \quad A \setminus B = A \Delta (A \cap B), \quad \varnothing = A \setminus A.$$

Theorem

If $\mathfrak{S}$ is a semiring, then the minimal ring $\mathfrak{R}(\mathfrak{S})$, which contains $\mathfrak{S}$ coincides with a family of sets A admitting the following finite decompositions

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Elementary sets form a minimal ring containing all rectangles.

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Theorem

For every measure m defined on a semiring $\mathfrak{S}_m$ there exists a unique extension m' of m to the minimal ring $\mathfrak{R}(\mathfrak{S}_m)$.

Additivity and countable additivity of measures

The planar Lebesgue measure which we constructed before possesses the property of countable additivity: if $A = \cup_{k=1}^{\infty} A_k$, $A_i \cap A_j = \varnothing$, $i \neq j$, then

$$m'(A) = \sum_{k=1}^{\infty} m'(A_k). \tag{1}$$

Definition

A measure m' is called σ-additive is (1) *holds for all A and $A_1, A_2, \ldots$ from the domain of definition of m'.*

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A measure m' is called σ-additive is (1) holds for all A and $A_1, A_2, \ldots$ from the domain of definition of m'.

In general additivity does not imply σ-additivity!

Example

Let $X = \mathbb{Q} \cap [0, 1]$ and $\mathfrak{S}_m$ is a semiring comprised of intersections of X with intervals of the form $(a, b]$, $[a, b)$, (a, b) and $[a, b]$. For every $A_{a,b} \in \mathfrak{S}_m$, put $m(A_{ab}) = b - a$. The measure m is additive but not σ-additive: $m(X) = 1$, but $m(X) \neq \sum_{x \in X} m(\{x\}) = 0$.

Additivity and countable additivity of measures

Lemma

If a measure m defined on a semiring $\mathfrak{S}_m$ is σ-additive, then the unique extension μ of m to $\mathfrak{R}(\mathfrak{S}_m)$ is also σ-additive.

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Lemma

Let μ be defined on a ring $\mathfrak{R}$ and is σ-additive on $\mathfrak{R}$. For $A, A_1, A_2, \ldots \in \mathfrak{R}$, it holds

(a) *If $\cup_{k=1}^{\infty} A_k \subset A$, $A_i \cap A_j = \varnothing$, $i \neq j$, then $\sum_{k=1}^{\infty} m(A_k) \leq m(A)$.*

(b) *If $\cup_{k=1}^{\infty} A_k \supset A$, $A_i \cap A_j = \varnothing$, $i \neq j$, then $\sum_{k=1}^{\infty} m(A_k) \geq m(A)$.*

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(b) *If $\cup_{k=1}^{\infty} A_k \supset A$, $A_i \cap A_j = \varnothing$, $i \neq j$, then $\sum_{k=1}^{\infty} m(A_k) \geq m(A)$.*

Part (a) does not require σ-additivity! But it is essential for part (b).

Outer measure and Lebesgue extension

Let m be an (additive) measure on a semiring $\mathfrak{S}_m$. As we have seen it can be uniquely extended to an additive measure on the ring $\mathfrak{R}(\mathfrak{S}_m)$.
If m is also σ-additive, then it can be further extended to a much larger class of sets (we have also seen this in our planar example)!
This procedure is called **Lebesgue extension**.

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If m is also σ-additive, then it can be further extended to a much larger class of sets (we have also seen this in our planar example)! This procedure is called **Lebesgue extension**.

It will be more convenient for us to assume that the initial semiring contains the identity, that is, a set $E \in \mathfrak{S}_m$ such that $A \cap E = A$. Then the ring $\mathfrak{R}(\mathfrak{S}_m)$ also contains the identity. A ring of sets with identity is called **algebra** of sets.

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Thus, from now on let us assume that we are given with a σ-finite measure m on a semiring $\mathfrak{S}_m$ with an identity.

Outer measure and Lebesgue extension

Definition

The outer measure of a set $A \subset E$ is the number

$$\mu^*(A) = \inf \sum_n m(B_n),$$

where the infimum is taken over all coverings of A by at most countable families $\{B_n\} \subset \mathfrak{S}_m$.

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Theorem

If $A \subset \cup_n A_n$, then $\mu^*(A) \leq \sum_n \mu^*(A_n)$.

Outer measure and Lebesgue extension

Definition

The set $A \subset E$ is called (Lebesgue) measurable if for every $\varepsilon > 0$, there exist a set $B = B_\varepsilon \in \mathfrak{R}(\mathfrak{S}_m)$ (not necessarily in $\mathfrak{S}_m$!) such that

$$\mu^*(A \Delta B) < \varepsilon.$$

The function μ^* restricted to the family $\mathfrak{M}$ of measurable sets is called **Lebesgue** *measure and $*$ in the notation is suppressed.*

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Note that all sets from $\mathfrak{R}(\mathfrak{S}_m)$ are measurable and $\mu(A) = m(A)$ for all $A \in \mathfrak{R}(\mathfrak{S}_m)$.

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Definition

A ring of sets which is closed with respect to countable unions and intersections is called σ-ring of sets. A σ-ring of sets with identity is called σ-algebra of sets.

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Definition

A ring of sets which is closed with respect to countable unions and intersections is called σ-ring of sets. A σ-ring of sets with identity is called σ-algebra of sets.The Lebesgue extension of m is a function μ defined on $\mathfrak{M}$ which coincides with μ^* on $\mathfrak{M}$.

Some examples

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If the domain of definition $\mathfrak{S}_m$ of a measure m is NOT a semi-ring, the extension might not be unique!

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Example

Let $X = C$ be a circle in $\mathbb{R}^2$ of length 1. The set of arcs (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$ and $[\alpha, \beta]$, $\alpha, \beta \in [0, 2\pi]$, is a semiring. The pre-measure m is defined by putting $m(arc) = \beta - \alpha$, $\beta \geq \alpha$.

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Proposition

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Let $X = C$ be a circle in $\mathbb{R}^2$ of length 1. The set of arcs (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$ and $[\alpha, \beta]$, $\alpha, \beta \in [0, 2\pi]$, is a semiring. The pre-measure m is defined by putting $m(arc) = \beta - \alpha$, $\beta \geq \alpha$. Let $t \in \mathbb{R} \setminus \mathbb{Q}$. Points $a, b \in C$ are equivalent if a can be turned into b using a rotation by $nt\pi$, $n \in \mathbb{Z}$. Let $\Phi_0 \subset C$ be a set comprised of unique representatives of every class. Let Φ_n be a set obtained from Φ_0 using the rotation $nt\pi$. It is clear, that

$$C = \bigcup_{n \in \mathbb{Z}} \Phi_n, \quad \Phi_n \cap \Phi_m = \varnothing, \quad n \neq m. \tag{2}$$

If Φ_0 were measurable, then all Φ_n would be also measurable and $m(\Phi_0) = m(\Phi_n)$, $n \in \mathbb{Z}$. This contradicts (2).

Finite and σ-finite measure

Recall that from the very beginning we assume existence of $E \in \mathfrak{S}_m$, such that $A \cap E = A$ for all $A \in \mathfrak{S}_m$. In particular, this means that $m(E)$ must be finite ∞. This makes many important sets (strips in $\mathbb{R}^2$, entire plane etc.) non-measurable.

Finite and σ-finite measure

Recall that from the very beginning we assume existence of $E \in \mathfrak{S}_m$, such that $A \cap E = A$ for all $A \in \mathfrak{S}_m$. In particular, this means that $m(E)$ must be finite ∞. This makes many important sets (strips in $\mathbb{R}^2$, entire plane etc.) non-measurable.

This can be overcome by assuming that the space X can be represented as a countable union of sets having finite measures. Let

$$X = \bigcup_{k=1}^{\infty} X_k, \quad X_i \cap X_j = \varnothing, \quad i \neq j, \quad m(X_k) < \infty. \qquad (3)$$

Then it can be checked that the family $\mathfrak{M}_{X_k} := \{C : C \subset X_k, C \text{ is measurable}\}$ is a σ-algebra.
The measure m satisfying (3) is called σ-finite.

Finite and σ-finite measure

Theorem

The family $\mathfrak{U}$ such that $A \in \mathfrak{U}$ iff it can be represented in the form

$$A = \bigcup_{k=1}^{\infty} A_k, \quad A_k \in \mathfrak{M}_{X_k}, \quad k \in \mathbb{N},$$

is a σ-algebra. The function $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \sum_{k=1}^{\infty} m(A_k)$$

is a σ-additive measure (which may take values $+\infty$).
Neither $\mathfrak{U}$ nor $\tilde{\mu}$ depend on the partition of X satisfying (3)*.*

Lebesgue–Stieltjes measures

Let $[a, b] \subseteq \mathbb{R}$ be a fixed segment of the real line and $F : [a, b] \mapsto \mathbb{R}$ be a right-continuous nondecreasing bounded function.

Lebesgue–Stieltjes measures

Let $[a, b] \subseteq \mathbb{R}$ be a fixed segment of the real line and $F : [a, b] \mapsto \mathbb{R}$ be a right-continuous nondecreasing bounded function.
Let $\mathfrak{S}$ be a semiring of intervals $(\alpha, \beta), (\alpha, \beta], [\alpha, \beta)$ and $[\alpha, \beta]$ in $[a, b]$. Define a measure m on $\mathfrak{S}$ by putting

$$m((\alpha, \beta]) = F(\beta) - F(\alpha), \quad m([\alpha, \beta]) = F(\beta) - F(\alpha - 0),$$
$$m(([\alpha, \beta)) = F(\beta - 0) - F(\alpha - 0), \quad m((\alpha, \beta)) = F(\beta - 0) - F(\alpha).$$

Using Lebesgue's method the measure m can be extended to a σ-algebra which contains all intervals (Borel σ-algebra).

Definition

The measure μ_F defined as the Lebesgue extension of m to the σ-algebra of measurable sets (including all Borel sets) is called Lebesgue–Stieltjes measure corresponding to the function F.

Lebesgue–Stieltjes measures

Properties of monotone functions:

- Every monotone function can have only discontinuities of the first kind.
- Every monotone function can have at most countable number of discontinuities.
- Every right-continuous nondecreasing function f can be represented as a sum of a continuous f nondecreasing function and a step function of the form

$$h(x) := \sum_{n : x_n \leq x} h_n,$$

where $\{x_n\}$ is a set of discontinuities of f and $h_n := f(x_n + 0) - f(x_n - 0)$.

Discrete Lebesgue–Stieltjes measures

Assume that the function F is purely discrete, that is,

$$F(x) = \sum_{n:x_n \leq x} h_n,$$

for some at most countable set $\{x_n\} \subset [a, b]$.

Discrete Lebesgue–Stieltjes measures

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$$F(x) = \sum_{n:x_n \leq x} h_n,$$

for some at most countable set $\{x_n\} \subset [a, b]$. Then, all subsets of $[a, b]$ are measurable and

$$\mu_F(A) := \sum_{n:x_n \in A} h_n, \quad A \subset [a, b].$$

Absolutely continuous Lebesgue–Stieltjes measures

Assume that the function F possesses a derivative $f(x) = F'(x)$ for all $x \in [a, b]$ which is also Riemann integrable. Then

$$F(\beta) - F(\alpha) = \int_{\alpha}^{\beta} f(x) \mathrm{d}x.$$

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The above two possibilities do not exhaust all class of monotone functions, since there exists continuous functions which are not differentiable! We shall discuss this later on upon introducing the notion of the Lebesgue integral.

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To be continued...