Advanced probability theory

Lecture 2. The Lebesgue integral

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July 16, 2021

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3. The integral for measurable functions.
4. Convergence in measure and almost everywhere.
5. Convergence theorems.

Measurable functions

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The mapping f is called $(\mathfrak{S}_X, \mathfrak{S}_Y)$-measurable if $A \in \mathfrak{S}_Y$ implies $f^{-1}(A) := \{x \in X : f(x) \in A\} \in \mathfrak{S}_X$.

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Example

If $X = Y = \mathbb{R}$ and $\mathfrak{S}_X = \mathfrak{S}_Y$ is the family of all open subsets of $\mathbb{R}$, then f is $(\mathfrak{S}_X, \mathfrak{S}_Y)$-measurable iff f is continuous.

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Example

If $X = Y = \mathbb{R}$ and $\mathfrak{S}_X = \mathfrak{S}_Y$ is the Borel σ-algebra (minimial σ-algebra which contains all intervals), then $(\mathfrak{S}_X, \mathfrak{S}_Y)$-measurable function is called Borel measurable.

Measurable functions

Definition

Assume that on X we are given with a σ-additive measure μ defined on a σ-algebra $\mathfrak{S}_\mu =: \mathfrak{S}_X$. Assume that $Y = \mathbb{R}^d$ and $\mathfrak{S}_Y$ is the Borel σ-algebra. The function $f : X \mapsto \mathbb{R}^d$ is called μ-measurable if for every Borel set $A \subset \mathbb{R}^d$ the preimage $f^{-1}(A) \in \mathfrak{S}_\mu$.

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Theorem

Let X, Y, Z be arbitrary sets with given families $\mathfrak{S}_X, \mathfrak{S}_Y$ and $\mathfrak{S}_Z$, respectively. If $f : X \mapsto Y$ is $(\mathfrak{S}_X, \mathfrak{S}_Y)$-measurable and $g : Y \mapsto Z$ is $(\mathfrak{S}_Y, \mathfrak{S}_Z)$-measurable, then $g \circ f : X \mapsto Z$ is $(\mathfrak{S}_X, \mathfrak{S}_Z)$-measurable.

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Corollary

The composition of a Borel measurable function and a μ-measurable function is μ-measurable.

Measurable functions

Theorem

A function $f : X \mapsto \mathbb{R}$ *is* μ*-measurable iff for every* $c \in \mathbb{R}$ *the set* $\{x : f(x) < c\}$ *is* μ*-measurable.*

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The sum, difference, product and quotient of two μ-measurable function if μ-measurable.

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The sum, difference, product and quotient of two μ*-measurable function if* μ*-measurable.*

Theorem

If $f_n : X \mapsto \mathbb{R}$ *is* μ*-measurable for every* $n \in \mathbb{N}$*, and* $f_n(x) \to f(x)$ *for every* $x \in X$*, then* f *is* μ*-measurable.*

Equivalence

Definition

Two functions defined on the same μ-measurable set E are called μ-equivalent if $\mu(\{x \in E : f(x) \neq g(x)\}) = 0$.

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Theorem

If f is μ-measurable, and g and f are μ-equivalent, then g is also measurable.

Proposition

If f and g are continuous and μ-equivalent, then $f \equiv g$.

Lebesgue integral for simple functions

Definition

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Theorem

A function which attains at most countable number of values $y_1, y_2, \ldots, y_n, \ldots$ is μ-measurable iff all sets

$$\{x \in X : f(x) = y_k\}, \quad k \in \mathbb{N},$$

are μ-measurable.

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Theorem

A function f is μ-measurable iff it is the uniform limit of a sequence of simple μ-measurable functions.

Lebesgue integral for simple functions

Let f be a simple μ-measurable function which attains values $y_1, y_2, \ldots,$, $y_i \neq y_j$, for $i \neq j$. Let A be a μ-measurable set. Put

$$\int_A f(x)\mathrm{d}\mu = \sum_n y_n \mu(A_n), \quad A_n := \{x \in A : f(x) = y_n\}$$

and assume that the series on the right-hand side is convergent.

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Definition

A simple function f is called μ-integrable or μ-summable if the above series converges absolutely. Its value is called the Lebesgue integral of f over the set A.

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Proposition

The Lebesgue integral of a simple function has the following properties:

- $\int_A (f(x) + g(x))\mathrm{d}\mu = \int_A f(x)\mathrm{d}\mu + \int_A g(x)\mathrm{d}\mu;$
- $\int_A cf(x)\mathrm{d}\mu = c\int_A f(x)\mathrm{d}\mu;$
- $\left|\int_A f(x)\mathrm{d}\mu\right| \leq sup_{x\in A}|f(x)|\mu(A).$

Lebesgue integral for measurable functions

Definition

A function f is called integrable over the set A if there exists a sequence of simple functions integrable over the set A and which converges to f uniformly on A. The limit

$$I := \lim_{n\to\infty} \int_A f_n(x)\mathrm{d}\mu =: \int_A f(x)\mathrm{d}\mu$$

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In order for the above definition to be correct we must check that:

(A) the above limit exists for arbitrary uniformly convergence to f sequence f_n;

(B) the limit does not depend on the choice of the sequence f_n;

(C) for simple function this definition is consistent with the definition for simple functions.

Lebesgue integral for measurable functions

Part (A) follows from

$$\left| \int_A f_n(x) \mathrm{d}\mu - \int_A f_m(x) \mathrm{d}\mu \right| \leq \mu(A) \sup_{x \in A} |f_n(x) - f_m(x)|,$$

which we have proved for simple measurable functions.

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For part (B) consider two given sequences $(f_n(x))$ and $(f_n^*(x))$ which both converge to the same limit $f(x)$. If the limits I for these two sequences where different, then the limit I would not exists for the sequence $f_1(x), f_1^*(x), f_2(x), f_2^*(x), \ldots$.

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Part (C) becomes obvious upon taking $f_n(x) = f(x)$ for all $n \in \mathbb{N}$.

Properties of the Lebesgue integral

Let A be a μ-measurable set and f, g two μ-measurable functions. Then:

(i) $\int_A 1 \mathrm{d}\mu = \mu(A)$;

(ii) for every $c \in \mathbb{R}$, $\int_A cf(x) \mathrm{d}\mu = c \int_A f(x) \mathrm{d}\mu$;

(iii) $\int_A (f(x) + g(x)) \mathrm{d}\mu = \int_A f(x) \mathrm{d}\mu + \int_A g(x) \mathrm{d}\mu$;

(iv) if $f(x) \geq 0$, then $\int_A f(x) \mathrm{d}\mu \geq 0$;

(v) if $\mu(A) = 0$, then $\int_A f(x) \mathrm{d}\mu = 0$;

(vi) if f and g are μ-equivalent, then $\int_A f(x) \mathrm{d}\mu = \int_A g(x) \mathrm{d}\mu$;

(vii) if ϕ is integrable and $|f(x)| \leq \phi(x)$ μ-almost everywhere, then f is also integrable;

(viii) there integrals $\int_A f(x) \mathrm{d}\mu$ and $\int_A |f(x)| \mathrm{d}\mu$ exist or do no exist simultaneously.

σ-additivity of the Lebesgue integral

Theorem

If $A = \cup_{n=1}^{\infty} A_n$ and $A_i \cap A_j = \varnothing$ for $i =\neq j$, then

$$\int_A f(x)\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{A_n} f(x)\mathrm{d}\mu.$$

Furthermore, if the integral in the lhs exists then all integrals on the rhs also exists and the series converges.

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Theorem

If $A = \cup_{n=1}^{\infty} A_n$ and $A_i \cap A_j = \varnothing$ for $i = \neq j$. The convergence of the series

$$\sum_{n=1}^{\infty} \int_{A_n} |f(x)| \mathrm{d}\mu$$

implies that f is μ-integrable and

$$\int_A f(x) \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{A_n} f(x) \mathrm{d}\mu.$$

Convergence in measure

Definition

The sequence of measurable functions $(f_n(x))$ *converges in* μ*-measure to a measurable function* $f(x)$ *if, for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| > \varepsilon\} = 0.$$

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Theorem

If a sequence $(f_n(x))$ converges μ-almost everywhere to a function f, then it converges to f in μ-measure.

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Theorem

If a sequence $(f_n(x))$ *converges* μ*-almost everywhere to a function* f*, then it converges to* f *in* μ*-measure.*

The converse is not true, but ...

Theorem

Let $(f_n(x))$ *converges to* f *in* μ*-measure, then there exist a subsequence* $(f_{n_k}(x))$ *which converges to* f μ*-almost everywhere, as* $k \to \infty$.

Convergence theorems

Theorem (B. Levi's theorem)

Assume that μ-almost everywhere on A,

$$f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots,$$

and $\sup_{n \in \mathbb{N}} \int_A f_n(x) \mathrm{d}\mu < \infty$. *Then, the limit* $f(x) := \lim_{n \to \infty} f_n(x)$ *exists μ-almost everywhere and*

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$$\lim_{n\to\infty} \int_A f_n(x)\mathrm{d}\mu = \int_A f(x)\mathrm{d}\mu.$$

Corollary

Assume that $\psi_n(x) \ge 0$ *and* $\sum_{n=1}^{\infty} \int_A \psi_n(x)\mathrm{d}\mu < \infty$. *Then the series* $\sum_{n=1}^{\infty} \psi_n(x)$ *converges μ-almost everywhere and*

$$\int_A \sum_{n=1}^{\infty} \psi_n(x)\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_A \psi_n(x)\mathrm{d}\mu.$$

Convergence theorems

Theorem (Fatou's lemma)

Let $(f_n(x))$ be a sequence of non-negative μ-measurable functions. Then

$$\liminf_{n\to\infty} \int_A f_n(x)\mathrm{d}\mu \ \geq \ \int_A \liminf_{n\to\infty} f_n(x)\mathrm{d}\mu.$$

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Theorem (Lebesgue dominated convergence theorem)

Assume that $(g_n(x))$ be a sequence of measurable functions, $g_n(x) \to g(x)$ and $\int_A g(x)\mathrm{d}\mu < \infty$. Let $(f_n(x))$ be another sequence of measurable functions such that $f_n(x) \to f(x)$. Then

$$\int_A f_n(x)\mathrm{d}\mu \to \int_A f(x)\mathrm{d}\mu.$$