Advanced probability theory

Lecture 3. Operations on measures and integrals

Alexander Marynych
http://do.unicyb.kiev.ua/marynych/

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1. The Radon–Nykodym theorem.

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2. Products of measure spaces and Fubini's theorem.
3. Push-forward of measures and change of variables formula.
4. Convolution and its properties.

Signed measures

Let μ be a σ-additive set function (non necessarily non-negative!) defined on a σ-algebra $\mathfrak{M}$ with identity E.

Theorem (Hahn decomposition)

There exist disjoint sets X_+ and $X_- \in \mathfrak{M}$ such that $E = X_+ \cup X_-$ and for all $A \in \mathfrak{M}$, one has

$$\mu(A \cap X_-) \leq 0 \quad \textit{and} \quad \mu(A \cap X_+) \geq 0.$$

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Corollary (Jordan-Hahn decomposition)

The set functions

$$\mu_+(A) := \mu(A \cap X_+) \quad \textit{and} \quad \mu_-(A) := -\mu(A \cap X_-)$$

are non-negative σ-additive measures and $\mu = \mu^+ - \mu^-$.

The Radon-Nykodym theorem

Let f be a μ-measurable function, where μ is a non-negative measure. The set function

$$\nu(A) := \int_A f(x) \mathrm{d}\mu$$

is countably additive,

$$\nu_+(A) = \int_A f_+(x) \mathrm{d}\mu \quad \text{and} \quad \nu_-(A) = \int_A f_-(x) \mathrm{d}\mu.$$

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The function f is called the density (or the Radon-Nykodym density) of a signed-measure ν with respect to μ and is denoted by $\mathrm{d}\nu/\mathrm{d}\mu$.

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Definition

The set-function $|\mu| := \mu_+ + \mu_-$, which is a σ-additive measure, is called **the total variation measure** *of μ.*

Absolute continuity and singularity

Definition

Let μ, ν be σ-additive set functions on the σ-algebra $\mathfrak{M}$ with identity E.

- *The measure ν is called absolutely continuous with respect to μ if $|\nu|(A) = 0$ for every set $A \in \mathfrak{M}$ such that $|\mu|(A) = 0$. This is denoted by $\nu \ll \mu$.*
- *The measure ν is called singular with respect to μ if there exists a set $\Omega \in \mathfrak{M}$ such that*

$$|\mu|(\Omega) = 0 \quad \text{and} \quad |\nu|(E \setminus \Omega) = 0.$$

  This is denoted by $\nu \perp \mu$.

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The (signed) measure ν is absolutely continuous with respect to the (signed) measure μ iff there exists a function f such that

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Corollary (Lebesgue decomposition)

Let ν and μ be two (signed) finite measure on a σ-algebra $\mathfrak{M}$. Then there exists a measure μ_0 on $\mathfrak{M}$ and a μ-integrable function f such that

$$\nu(A) = \int_A f(x)\mathrm{d}\mu + \nu_0 \quad \textit{and} \quad \nu_0 \perp \mu.$$

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$$\nu(A) = \int_A f_\nu(x) \mathrm{d}\lambda = \int_A f_\nu(x) \mathrm{d}\nu + \int_A f_\nu(x) \mathrm{d}\mu.$$

for a function f_ν. Clearly, we have $0 \leq f_\nu(x) \leq 1$ for $(\mu + \nu)$-almost all x, and, thus, for ν-almost all x.

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$$U := \{x : f_\nu(x) = 1\} \quad \text{and} \quad V := \{x : 0 \leq f_\nu(x) < 1\}.$$

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$$U := \{x : f_\nu(x) = 1\} \quad \text{and} \quad V := \{x : 0 \leq f_\nu(x) < 1\}.$$

Then

$$\nu(U) = \int_U f_\nu(x)\mathrm{d}\mu + \int_U f_\nu(x)\mathrm{d}\nu = \mu(U) + \nu(U),$$

and therefore, $\mu(U) = 0$. For every measurable set A, put

$$\nu_0(A) := \nu(U \cap A) \quad \text{and} \quad \nu_1(A) := \nu(V \cap A).$$

It is clear that $\nu_0 \perp \mu$.

Proof of the Lebesgue decomposition

Let us check that $\nu_1 \ll \mu$. If $\mu(A) = 0$, then

$$\int_{A\cap V} \mathrm{d}\nu = \nu(A \cap V) = \int_{A\cap V} f_\nu(x)\mathrm{d}\mu + \int_{A\cap V} f_\nu(x)\mathrm{d}\nu$$
$$= \int_{A\cap V} f_\nu(x)\mathrm{d}\nu,$$

and therefore $\int_{A\cap V}(1 - f_\nu(x))\mathrm{d}\nu = 0$. This implies, $\nu(A \cap V) = \nu_1(A) = 0$.

Cantor measure

Let $K \subset [0,1]$ be a standard Cantor set and consider the complement

$$[0,1] \setminus K = \left(\frac{1}{3}, \frac{2}{3}\right) \bigcup \left(\frac{1}{9}, \frac{2}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right) \bigcup \cdots$$

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An interval of length $(1/3)^n$ in this decomposition is said to have rank n. There are $(1/3)^{n-1}$ intervals of rank n and we enumerate them from left to right. We say that a point $t \in [0,1]$ is a point of the first kind if it belongs to the **union of closures** of the above intervals. Other points are said to be of the second kind.

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Define a function $f : [0, 1] \mapsto [0, 1]$ first for the points of the first kind as follows: on k-th interval of rank n (and at its endpoints) put $f(t) := \frac{2k-1}{2^n}$, $k = 1, 2, \dots, 2^{n-1}$.

Cantor measure

Let t^* be a point of the second kind. Then there exist the sequences of points of the first kind such that $t_n \uparrow t^*$ and $t'_n \downarrow t^*$. Furthermore, by monotonicity

$$\lim_{n\to\infty} f(t_n) \uparrow A \quad \text{and} \quad \lim_{n\to\infty} f(t'_n) \downarrow B.$$

and the limits A and B coincide. Put $f(t^*) = A = B$. The function $f : [0,1] \mapsto [0,1]$ is continuous and nondecreasing on $[0,1]$.

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Proposition

The Lebesgue-Stieltjes measure ν defined on $[0,1]$ by its distribution function $\nu([0,a]) := f(a)$, $a \in [0,1]$ is continuous and singular with respect to the Lebesgue measure on $[0,1]$.

Proof.

The claim follows immediately since the Lebesgue measure of $[0,1] \setminus K$ is 0, whereas $\nu([0,1] \setminus K) = \nu([0,1]) = 1$. □

Fubini's theorem

Suppose that μ and ν are finite non-negative measures on σ-algebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with units E_1 and E_2, respectively. For $A \subset E_1 \times E_2$, $x \in E_1$ and $y \in E_2$, set

$$A_x := \{y : (x, y) \in A\}, \quad A_y := \{x : (x, y) \in A\}.$$

Theorem

Assume that $A \subset E_1 \times E_2$ is measurable with respect to $\mu \otimes \nu$. Then

- *for μ-almost all x, the set A_x is ν-measurable and the function $x \mapsto \nu(A_x)$ is μ-measurable;*
- *for ν-almost all y, the set A_y is μ-measurable and the function $y \mapsto \mu(A_y)$ is ν-measurable;*
- *one has*

$$(\mu \otimes \nu)(A) = \int_{E_1} \nu(A_x) \mathrm{d}\mu = \int_{E_2} \mu(A_y) \mathrm{d}\nu.$$

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Corollary

The previous theorem holds true also for σ-finite measures μ and ν.

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Let ν be the Lebesgue measure on $\mathbb{R}$ and f be a non-negative μ-integrable function on E_2 with a σ-finite measure μ. Then

$$\int_{E_2} f(x)\mathrm{d}\mu = (\mu \otimes \lambda)(\{(x, y) : 0 \leq y \leq f(x)\}).$$

Proof.

The set $A := \{(x, y) : (x, y) : 0 \leq y \leq f(x)\}$ is measurable with respect to $\mu \otimes \lambda$ and $\lambda(A_x) = f(x)$. □

Fubini's theorem

The following important result is called Fubini's theorem

Theorem

Let μ and ν be σ-finite non-negative measures on the spaces E_1 and E_2, respectively. Assume that $f : E_1 \times E_2 \mapsto \mathbb{R}$ is $\mu \otimes \nu$-integrable. Then the function $y \mapsto f(x,y)$ is ν-integrable for μ-almost all x, the function $x \mapsto f(x,y)$ is μ-integrable for ν-almost all y. Furthermore, the functions

$$x \mapsto \int_{E_2} f(x,y)\mathrm{d}\nu \quad \textit{and} \quad y \mapsto \int_{E_1} f(x,y)\mathrm{d}\mu$$

are integrable with respect to μ and ν, respectively and

$$\begin{aligned}\int_{E_1 \times E_2} f(x,y)\mathrm{d}(\mu \otimes \nu) &= \int_{E_2} \left(\int_{E_1} f(x,y)\mathrm{d}\mu \right) \mathrm{d}\nu \\ &= \int_{E_1} \left(\int_{E_2} f(x,y)\mathrm{d}\nu \right) \mathrm{d}\mu.\end{aligned}$$

Example

Let $A = [-1,1]^2$ and $f(x,y) = \frac{xy}{(x^2+y^2)^2}$. Then

$$\int_{[-1,1]} f(x,y)\mathrm{d}x = 0, \quad y \neq 0,$$

and

$$\int_{[-1,1]} f(x,y)\mathrm{d}y = 0, \quad x \neq 0.$$

Thus,

$$\int_{[-1,1]} \int_{[-1,1]} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{[-1,1]} \int_{[-1,1]} f(x,y)\mathrm{d}y\mathrm{d}x = 0;$$

but the Lebesgue integral over A does not exist. Indeed,

$$\int_A |f(x,y)|\mathrm{d}x\mathrm{d}y \geq \int_{(x,y):x^2+y^2\leq 1} |f(x,y)|\mathrm{d}x\mathrm{d}y = \infty.$$

Images of measure under mappings

Let $f : X \mapsto Y$ be a $(\mathfrak{M}_X, \mathfrak{M}_Y)$-measurable mapping between spaces X and Y with σ-algebras $\mathfrak{M}_X$ and $\mathfrak{M}_Y$, respectively.

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$$\mu \circ f^{-1}(B) := \mu(f^{-1}(B)), \quad B \in \mathfrak{M}_Y$$

defines a measure on $\mathfrak{M}_Y$, for every measure μ on $\mathfrak{M}_X$.

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$$\mu \circ f^{-1}(B) := \mu(f^{-1}(B)), \quad B \in \mathfrak{M}_Y$$

defines a measure on $\mathfrak{M}_Y$, for every measure μ on $\mathfrak{M}_X$. Indeed, if $B = \cup_n B_n$ and $B_i \cap B_j = \varnothing$, $i \neq j$, then

$$\mu \circ f^{-1}(B) = \mu(f^{-1}(\cup_n B_n)) = \mu(\cup_n f^{-1}(B_n)) = \sum_n \mu(f^{-1}(B_n)).$$

Definition

The measure $\mu \circ f^{-1}$ is called **push-forward** *of μ under mapping f.*

Images of measure under mappings

Theorem

Let μ be a non-negative measure. A $\mathfrak{M}_Y$ measurable function $g : X \mapsto \mathbb{R}$ is $\mu \circ f^{-1}$-integrable iff $g \circ f$ is μ-integrable. Furthermore,

$$\int_Y g(y) \mathrm{d}\mu \circ f^{-1} = \int_{M_X} g(f(x)) \mathrm{d}\mu.$$

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Example

For a measurable function f, put $\Phi_f(t) := \mu(\{x : f(x) \leq t\})$, $t \in \mathbb{R}$, so $\Phi_f(t) = \mu \circ f^{-1}((-\infty, t])$. We have

$$\int_X \psi(f(x))\mathrm{d}\mu = \int_{\mathbb{R}} \psi(t)\mathrm{d}\Phi_f(t),$$

where on the right-hand side we have the Lebesgue-Stieltjes integral. If Φ_f is continuous and μ is a probability measure, then the image of μ under $\Phi_f \circ f$ is the Lebesgue measure on $[0, 1]$.

Convolutions of measures

Definition

Let μ and ν be two bounded Borel measures on $\mathbb{R}^d$. The convolution $\nu * \nu$ of μ and ν is defined as the push-forward of the measure $\mu \otimes \nu$ under the mapping $(x, y) \mapsto x + y$.

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For every Borel set B one has

$$(\mu * \nu)(B) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_{\{x+y \in B\}} \mathrm{d}\mu \mathrm{d}\nu$$
$$= \int_{\mathbb{R}^d} \mu(B - y) \mathrm{d}\nu = \int_{\mathbb{R}^d} \nu(B - y) \mathrm{d}\mu.$$

Convolutions of measures

Lemma

Let $F_\mu(x_1, x_2, \ldots, x_d) := \mu((-\infty, x_1] \times (-\infty, x_2] \times \cdots \times (-\infty, x_d])$ *be the distribution function of the measure* μ *and* F_ν *and* $F_{\mu*\nu}$ *be defined in a similar way. Then*

$$F_{\mu*\nu}(x_1, x_2, \ldots, x_d) = \int_{\mathbb{R}^d} F_\mu(x_1 - y_1, x_2 - y_2, \ldots, x_d - y_d) \mathrm{d}F_\nu(y_1, y_2, \ldots, y_d).$$

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Proposition

The convolution operation has the following properties:

- $\mu * \nu = \nu * \mu$;
- $(\mu * \nu) * \theta = \mu * (\nu * \theta)$;
- $(\mu + \nu) * \theta = \mu * \theta + \nu * \theta$;
- *the measure* $\mathrm{Id}(B) = \mathbb{1}_{0 \in B}$ *satisfies* $\mu * \mathrm{Id} = \mathrm{Id} * \mu = \mu$.

Convolutions of measures

For a (possibly signed) finite Borel measure on $\mathbb{R}^d$ let $\|\mu\| := \mu_+(\mathbb{R}^d) + \mu_-(\mathbb{R}^d)$ denote the total variation of μ on $\mathbb{R}^d$. Then

$$\|\mu * \nu\| \leq \|\mu\|\|\nu\|.$$

Theorem

The family of finite Borel measures on $\mathbb{R}^d$ endowed with the total variation norm $\| \cdot \|$ is a Banach algebra.