Advanced probability theory

Lecture 4. Lebesgue integral as a function of set.

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Lecture 3

1. Monotone functions.

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2. Functions of bounded variation.

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2. Functions of bounded variation.
3. Absolutely continuous function.

Monotone functions

From the elementary calculus we know that, if f is continuous and F is continuously differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x) \quad \text{and} \quad \int_a^b F'(t)\mathrm{d}t = F(b) - F(a).$$

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- Does the first formula hold true for Lebesgue integrable functions?
- What is the largest class of functions satisfying the second relation?

Lebesgue–Stieltjes measures

Properties of monotone functions:

- Every monotone function can have only discontinuities of the first kind.
- Every monotone function can have at most countable number of discontinuities.
- Every right-continuous nondecreasing function f can be represented as a sum of a continuous nondecreasing function and a step function of the form

$$h(x) := \sum_{n: x_n \leq x} h_n,$$

  where $\{x_n\}$ is a set of discontinuities of f and $h_n := f(x_n + 0) - f(x_n - 0)$.

Differentiability of monotone functions

Theorem (Lebesgue theorem)

A monotone function f defined on an interval $[a, b]$ is differentiable almost everywhere on this interval with respect to the Lebesgue measure on $[a, b]$.

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Set

$$d(x, x_0) := \frac{f(x) - f(x_0)}{x - x_0}, \quad x \neq x_0. \tag{1}$$

- Λ_r is the upper limit in (1) as $x \to x_0 + 0$;
- Λ_l is the upper limit in (1) as $x \to x_0 - 0$;
- λ_r is the lower limit in (1) as $x \to x_0 + 0$;
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A function is differentiable at x_0 if all four values coincide.

Proof of the Lebesgue theorem

Definition

Let g be a continuous function on $[a, b]$. We say that a point $x_0 \in (a, b)$ is **right-invisible** *if there exists $x \in (x_0, b]$ such that $g(x_0) < g(x)$.*

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Lemma

For every continuous function g the set of right-invisible points is open, and, thus, can be represented as at most countable union of open intervals (a_k, b_k) (some of them might be semi-open and include either a or b). It also holds $g(a_k) \leq g(b_k)$.

Proof of the Lebesgue theorem

Definition

Let g be a function on $[a, b]$ which has only discontinuities of the first kind. We say that a point $x_0 \in (a, b)$ is **right-invisible** *if there exists $x \in (x_0, b]$ such that*
$\max\{g(x_0 - 0), g(x_0), g(x_0 + 0)\} < g(x)$.

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$\max\{g(x_0 - 0), g(x_0), g(x_0 + 0)\} < g(x)$.

Lemma

For every function g which has only discontinuities of the first kind the set of right-invisible points is open, and, thus, can be represented as at most countable union of open intervals (a_k, b_k) (some of them might be semi-open and include either a or b). It also holds $g(a_k) \leq g(b_k)$.

Proof of the Lebesgue theorem

Lemma

Let a measurable set $A \subset [a, b]$ *be such that for every* $(\alpha, \beta) \subset [a, b]$ *it holds* $\mu(A \cap (\alpha, \beta)) \leq \rho(\beta - \alpha)$*, where* $\rho \in [0, 1)$*. Then* $\mu(A) = 0$*.*

Proof.

Let $\mu(A) = t$. For every $\varepsilon > 0$ there exists an open set $G = \cup_m (a_m, b_m)$ such that $A \subset G$ and $\sum_m (b_m - a_m) < t + \varepsilon$. Put $t_m := \mu(A \cap (a_m, b_m))$ and note that $t = \sum_m t_m$. By the assumption $t_m \leq \rho(b_m - a_m)$, and therefore

$$t \leq \rho \sum_m (b_m - a_m) < \rho(t + \varepsilon).$$

This is possible only if $t = 0$. □

Proof of the Lebesgue theorem

Corollary

For every measurable and integrable on $[a, b]$ function f the function

$$x \mapsto \int_a^x f(x) \mathrm{d}\mu$$

has a finite derivative μ-almost everywhere on $[a, b]$.

Proof.

It follows immediately from the Lebesgue theorem, since the above function can be represented as the difference of two monotone functions.

$$\int_a^x f(x) \mathrm{d}\mu = \int_a^x f_+(x) \mathrm{d}\mu - \int_a^x f_-(x) \mathrm{d}\mu.$$

□

Functions of bounded variation

Definition

A function $f : [a, b] \mapsto \mathbb{R}$ *is called* **a function of bounded variation** *if*

$$\sup_P \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty,$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \cdots < x_n = b$ *of* $[a, b]$. *The value of the above supremum is called* **the total variation** *of* f *over the interval* $[a, b]$ *and is denoted by* $V_a^b(f)$.

Functions of bounded variation

Proposition

The following relations hold true:

- $V_a^b(\alpha f) = |\alpha| V_a^b(f)$;
- $V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)$;
- *if* $a < b < c$, *then* $V_a^b(f) + V_b^c(f) = V_a^c(f)$;
- *the function* $x \mapsto V_a^x(f)$ *is non-decreasing;*
- *the function* $x \mapsto V_a^x(f) - f(x)$ *is non-decreasing.*

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Theorem

Every function of bounded variation can be represented as a difference of two monotone functions. Thus, every function of bounded variation is differentiable μ*-almost everywhere.*

Theorem

For every measurable and integrable on $[a, b]$ function f the equality

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\mathrm{d}\mu = f(x)$$

holds μ-almost everywhere on $[a, b]$.

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The last question of this kind is: for which functions F it holds that

$$F(x) = F(a) + \int_a^x +F'(x)\mathrm{d}x.$$

Note that F must be of bounded variation. But this is not sufficient (Cantors' staircase)!

Absolutely continuous functions

Definition

A function $f : [a, b] \mapsto \mathbb{R}$ *is called absolutely continuous if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every finite family of open intervals* (a_k, b_k)*,* $k = 1, \dots, n$*, such that*

$$\sum_{k=1}^{n}(b_k - a_k) < \delta,$$

it holds $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$*.*

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Remark

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Example

The Cantor staircase is continuous (even uniformly continuous) and monotone, but not absolutely continuous.

Absolutely continuous functions

Proposition

Absolutely continuous functions have the following properties:

- *Absolutely continuous functions have bounded variation.*
- *The family of absolutely continuous functions if closed with respect to linear combinations.*
- *Every absolutely continuous function can be represented as a difference of two non-decreasing absolutely continuous functions.*

Absolutely continuous functions

Theorem

The function $F(x) = \int_a^x f(t)\mathrm{d}\mu$, *where* f *is a Lebesgue integrable function is absolutely continuous.*

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Lemma

If f *is a* μ*-integrable function on* A*, then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left| \int_B f(x)\mathrm{d}\mu \right| < \varepsilon,$$

for every measurable set $B \subset A$ *such that* $\mu(B) < \delta$*.*

Absolutely continuous functions

Theorem

The derivative $f(x) = F'(x)$ *of an absolutely continuous function* $F : [a, b] \mapsto \mathbb{R}$ *is Lebesgue integrable on* $[a, b]$ *and for every* $x \in [a, b]$

$$\int_a^x f(t)\mathrm{d}t = F(x) - F(a).$$

Lebesgue decomposition

Recall that every monotone function can be represented as a sum of a monotone step function and a monotone continuous function.

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where H is a step function of bounded variation and ϕ is a continuous function of bounded variation.

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$$f(x) = H(x) + \phi(x),$$

where H is a step function of bounded variation and ϕ is a continuous function of bounded variation. Put

$$\psi(x) := \int_a^x \phi'(t)\mathrm{d}\mu, \quad \chi(x) := \phi(x) - \psi(x).$$

If ϕ is absolutely continuous, $\chi(x) \equiv 0$. But in general, we only know that χ is a continuous function of bounded variation such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\chi(x) = \phi'(x) - \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \phi'(t)\mathrm{d}t = 0$$

holds μ-almost everywhere.

Lebesgue decomposition

Theorem

Every function of bounded variation f, can be represented as a sum of three components: a step-function (discrete component), absolutely continuous function (absolutely continuous component) and a singular function (singular component). If f is non-decreasing, then all three components are also non-decreasing. In particular, every distribution function of a (possibly, signed) measure μ, that is

$$t \mapsto \mu((-\infty, t]), \quad t \in \mathbb{R}$$

on $\mathbb{R}$ can be represented in this way.