Advanced probability theory

Lecture 5. Infinite Bernoulli convolutions

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1. Probability spaces, random variables and random elements.

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2. Distributions and moments.

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3. Infinite Bernoulli convolutions.

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1. Probability spaces, random variables and random elements.
2. Distributions and moments.
3. Infinite Bernoulli convolutions.
4. Distributional properties of infinite Bernoulli convolutions.

Probability spaces, random variables and random elements

Definition

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$*, where*

- Ω *is an arbitrary set called the set of elementary events;*
- $\mathcal{F}$ *is a* σ*-algebra of subsets of* Ω*, the elements of* $\mathcal{F}$ *are called events,* $\Omega \in \mathcal{F}$*;*
- $\mathbb{P}$ *is a measure on* $\mathcal{F}$ *such that* $\mathbb{P}\{\Omega\} = 1$*, such a measure is called probability measure.*

Probability spaces, random variables and random elements

Example

$\Omega = \{1, 2, \ldots, n\}$, $\mathcal{F} = 2^{\Omega}$, $\mathbb{P}\{A\} = \frac{\mathrm{card}(A)}{n}$.

Probability spaces, random variables and random elements

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Example

$\Omega = [0,\ 1]$, $\mathcal{F}$ is a σ-algebra of Lebesgue measurable subsets of $[0, 1]$, $\mathbb{P}$ is the Lebesgue measure.

Probability spaces, random variables and random elements

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be the real line with the Borel σ-algebra on it.

Definition

A random variable ξ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable mapping from Ω to $\mathbb{R}$, that is, a mapping $\xi : \Omega \mapsto \mathbb{R}$ such that

$$\{\omega : \xi(\omega) \in B\} \in \mathcal{F} \text{ for every } B \in \mathcal{B}(\mathbb{R}).$$

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Roughly speaking, a random variable is a numerical characteristic which assigns a real number to every elementary event $\omega \in \Omega$.

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Roughly speaking, a random variable is a numerical characteristic which assigns a real number to every elementary event $\omega \in \Omega$.

The requirement of measurability is essential for the following reason. With every $B \in \mathcal{B}(\mathbb{R})$ we can associate a real number $\mathbb{P}\{\omega : \xi(\omega) \in B\} \in [0, 1]$, the "probability that a numerical characteristic ξ" takes values in set B.

Probability spaces, random variables and random elements

The notion of a random variable encompasses such objects as a random number and a random vector. A more general notion of a random element allows one to consider such objects as random point, random function, random line, random convex set etc.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(X, \mathcal{X})$ *be an arbitrary space* X *with a* σ*-algebra* $\mathcal{X}$ *on it. A random element (or an* X*-valued random variable) is a mapping* $\xi : \Omega \mapsto X$ *which is* $(\mathcal{F}, \mathcal{X})$*-measurable.*

Example

If $(X, \mathcal{X}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then ξ is a d-dimensional random vector;

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If $(X, \mathcal{X}) = (\mathcal{C}(\mathbb{R}^d), \mathcal{B}(\mathcal{C}(\mathbb{R}^d)))$, then ξ is a random compact set in $\mathbb{R}^d$.

Distributions and moments

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(X, \mathcal{X})$ *be an arbitrary space* X *with a* σ*-algebra* $\mathcal{X}$ *on it. The distribution of a random element* $\xi : \Omega \mapsto X$ *is a function*

$$\mathcal{X} \ni B \mapsto \mathbb{P}\{\omega : \xi(\omega) \in B\} \in [0, 1].$$

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Definition

The distribution function of a random vector $\xi = (\xi_1, \xi_2, \ldots, \xi_d) : \Omega \mapsto \mathbb{R}^d$ *is a function*

$$\mathbb{R}^d \ni (x_1, x_2, \ldots, x_d) \mapsto \mathbb{P}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_d \leq x_d\} \in [0, 1].$$

Distributions and moments

Let $\xi : \Omega\mathbb{R}$ be a random variable. The Lebesgue integral (if it exists)

$$\mathbb{E}\xi := \int_\Omega \xi(\omega)\mathrm{d}\mathbb{P}$$

is called the **mathematical expectation** of a random variable ξ.

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If ξ is an X-valued random variable and $f : X \mapsto \mathbb{R}$ is an $\mathcal{X}$-measurable mapping, then $f(\xi)$ is a random variable and $\mathbb{E}f(\xi)$ is called the f-moment of an X-valued random variable ξ.

Distributions and moments

Proposition

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- *If* $c \in \mathbb{R}$ *and* ξ *is integrable, then* $\mathbb{E}(c\xi) = c\mathbb{E}\xi$.

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- *If* ξ *and* η *are non-negative OR* $|\xi|$ *and* $|\eta|$ *are integrable, then* $\mathbb{E}(\xi + \eta) = \mathbb{E}\xi + \mathbb{E}\eta$.

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- *If* $\xi = \eta$ $\mathbb{P}$*-almost surely and* $|\xi|$ *is integrable, then* $|\eta|$ *is integrable and* $\mathbb{E}\xi = \mathbb{E}\eta$.
- *Levi's theorem, Fatou's lemma, Lebesgue's dominated convergence theorem.*

Distributions and moments

Let ξ be a random variable. The distribution of ξ, that is, the probability measure $\mathcal{B}(\mathbb{R}) \ni B \mapsto \mathbb{P}\{\xi \in B\}$ (alternatively, the distribution function $x \mapsto \mathbb{P}\{\xi \le x\}$) possesses the Lebesgue decomposition into discrete, absolutely continuous and singular continuous parts.

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- $F_1(x) := (x \vee 1)\mathbb{1}_{x \ge 0}$, the distribution function of a uniform random variable on $[0, 1]$; it is absolutely continuous;
- $F_2(x) = \mathrm{Cantor's\ \ staircase}$, the distribution function of the so-called Cantor's distribution; it is singular continuous;
- $F_3(x) = \mathbb{1}_{x \ge 0}$, the distribution function of the degenerate random variable taking value 0 with probability one; it is discrete.

These are distribution functions of **pure type**. The distribution function of a mixed type is, for example, $x \mapsto (F_1(x) + F_2(x) + F_3(x))/3$.

Independence

Definition

A finite family of random variables $\xi_1, \xi_2, \dots, \xi_m$ *are called* **mutually** *independent, if for arbitrary Borel sets* B_i*,* $i = 1, \dots, m$*, it holds*

$$\mathbb{P}\{\xi_1 \in B_1, \dots, \xi_m \in B_m\} = \mathbb{P}\{\xi_1 \in B_1\} \times \cdots \times \mathbb{P}\{\xi_m \in B_m\}.$$

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Lemma

The distribution of $\xi_1 + \xi_2 + \cdots + \xi_m$, *where* $\xi_1, \xi_2, \ldots, \xi_m$ *are independent, is given by the convolution of distributions of* $\xi_1, \xi_2, \ldots, \xi_m$.

Infinite Bernoulli convolutions

Let $(\xi_k(\omega))_{k\in\mathbb{N}_0}$ be a sequence of mutually independent random variables (this means that members of every finite subfamily are mutually independent), and such that

$$\mathbb{P}\{\omega : \xi_k(\omega) = -1\} = \mathbb{P}\{\omega : \xi_k(\omega) = 1\} = 1/2.$$

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For $\lambda \in (0, 1)$, the series

$$I_\lambda(\omega) := \sum_{n=0}^{\infty} \xi_n(\omega)\lambda^n,$$

converges $\mathbb{P}$-almost surely (by the dominated convergence).

Infinite Bernoulli convolutions

Example

If $\lambda = 1/2$, then $I_{1/2}$ has the uniform distribution on $[-2, 2]$, that is

$$\mathbb{P}\{\omega : I_{1/2}(\omega) \le x\} = (x+2)/4, \quad x \in [-2, 2].$$

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If $\lambda = 1/3$, then $I_{1/3}$ has the (linearly transformed) Cantor's distribution:

$$\mathbb{P}\{\omega : (I_{1/3}(\omega) + 3/2)/3 \leq x\} = f(x),$$

where f is the standard Cantor's staircase.

Laws of pure types

Theorem (Jessen and Wintner, 1935)

The distribution of I_λ is of pure type. It is either absolutely continuous, or singular continuous with respect to the Lebesgue measure, depending of the value of $\lambda \in (0,1)$.

Proof.

Note that I_λ has the same distribution as $\xi + \lambda I_\lambda$, where ξ and I_λ are independent. Equivalently, upon setting $F_\lambda(B) := \mathbb{P}\{\omega : I_\lambda(\omega) \leq x\}$, $x \in \mathbb{R}$,

$$F_\lambda(x) = \frac{1}{2}\left(F_\lambda(\lambda^{-1}(x+1)) + F_\lambda(\lambda^{-1}(x-1))\right), \quad x \in \mathbb{R}.$$

□

Laws of pure types

$$G_\lambda(x) = \frac{1}{2}\left(G_\lambda(\lambda^{-1}(x+1)) + G_\lambda(\lambda^{-1}(x-1))\right), \quad x \in \mathbb{R}. \qquad (1)$$

This functional equation has a unique solution in the class of distribution functions, that is, nondecreasing functions with properties $G_\lambda(-\infty) = 0$ and $G_\lambda(+\infty) = 1$. Let F_λ be such a solution.

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This functional equation has a unique solution in the class of distribution functions, that is, nondecreasing functions with properties $G_\lambda(-\infty) = 0$ and $G_\lambda(+\infty) = 1$. Let F_λ be such a solution. If we drop the condition $G_\lambda(+\infty) = 1$, the solution to (**??**) is not unique but always has the form

$$G_\lambda(x) = cF_\lambda(x), \quad x \in \mathbb{R},$$

for some $c \geq 0$, in the class of bounded nondecreasing functions satisfying $G_\lambda(-\infty) = 0$.

Laws of pure types

Let $F_\lambda(x) = F_\lambda^{abs}(x) + F_\lambda^{sing}(x)$ be the Lebesgue decomposition. Note that

$$F_\lambda^{abs}(x) = \int_{-\infty}^{x} F'_\lambda(t)\mathrm{d}t, \quad F_\lambda^{abs}(-\infty) = 0, \quad F_\lambda^{abs}(+\infty) \leq 1.$$

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Therefore, $F_\lambda^{abs}(x)$ satisfies (1) and, thus,

$$F_\lambda^{abs}(x) = cF_\lambda(x), \quad x \in \mathbb{R},$$

where $c = F_\lambda^{abs}(+\infty)$. This is only possible if $c \in \{0, 1\}$ and we see that I_λ is either absolutely continuous or singular with respect to Lebesgue measure.

Laws of pure types

It is easy to see that the distribution of I_λ can not have a discrete component. Let x_0 be a point of discontinuity where the jump $p(x_0) := F_{\lambda(x_0)} - F_{\lambda(x_0-0)}$ is maximal. Assume that $p(x_0) > 0$. Then from the functional equation

$$F_\lambda(x) = \frac{1}{2}\left(F_\lambda(\lambda^{-1}(x+1)) + F_\lambda(\lambda^{-1}(x-1))\right), \quad x \in \mathbb{R},$$

we obtain that $p(\lambda^{-1}(x_0+1)) = p(x_0) = p(\lambda^{-1}(x_0-1))$.

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$$F_\lambda(x) = \frac{1}{2}\left(F_\lambda(\lambda^{-1}(x+1)) + F_\lambda(\lambda^{-1}(x-1))\right), \quad x \in \mathbb{R},$$

we obtain that $p(\lambda^{-1}(x_0+1)) = p(x_0) = p(\lambda^{-1}(x_0-1))$. Iterating this equality we obtain an infinite sequence of atoms of the same weight $p(x_0) > 0$. This is impossible.

Singularity for $\lambda < 1/2$

Theorem (Kershner and Wintner, 1935)

If $\lambda < 1/2$, the distribution of I_λ is singular continuous with respect to the Lebesgue measure.

Proof.

We need to show that there exist a Borel set A such that $\mathbb{P}\{I_\lambda \in A\} = 1$ and $\mu(A) = 0$.

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Let $(x_i)_{i=1,\ldots,2^{n+1}}$ be the set of all possible values of the sum $\sum_{k=0}^{n} \lambda^k \xi_k$. For each x_i construct an interval J_{x_i} of length $2\sum_{k\geq n+1} \lambda^k = \frac{2\lambda^{n+1}}{1-\lambda}$ centred at x_i.

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Set $O_n := \cup_{i=1}^{2^{n+1}} J_{x_i}$ and note that $\mathbb{P}\{I_\lambda \in O_n\} = 1$ for every $n \in \mathbb{N}$, since $\sum_{k\geq n+1} \lambda^k \xi_k \in \left[-\frac{\lambda^{n+1}}{1-\lambda}, \frac{\lambda^{n+1}}{1-\lambda}\right]$.

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Thus, $\mathbb{P}\{I_\lambda \in \cap_{n=1}^{\infty} O_n\} = 1$, and we may take $A = \cap_{n=1}^{\infty} O_n$. This set satisfies

$$\mu(A) \leq \mu(O_n) \leq \sum_{i=1}^{2^{n+1}} \mu(J_{x_i}) \leq 4(1-\lambda)^{-1}(2\lambda)^n.$$

Absolute continuity for $\lambda \geq 1/2$?

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Definition

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Definition

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Theorem (Erdős, 1939)

If $\lambda \neq 1/2$ and $1/\lambda$ is a PV-number, then the distribution of I_λ is singular continuous with respect to the Lebesgue measure.

Absolute continuity for $\lambda \geq 1/2$?

Lemma

Let θ be a PV-number, then $\mathrm{dist}(\theta^n, \mathbb{Z}) \leq \mathrm{const}\rho^n$ for some $\rho \in (0, 1)$.

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Lemma (Riemann-Lebesgue lemma)

If f is a density of an absolutely continuous distribution, then $\int_{\mathbb{R}} e^{itx} f(x) \mathrm{d}x \to 0$, as $t \to \infty$.

Absolute continuity for $\lambda \geq 1/2$?

Proof of Erdős' theorem.

We have

$$\phi_\lambda(t) := \mathbb{E}e^{itI_\lambda} = \prod_{n=0}^{\infty} \cos(\lambda^n t),$$

and thereupon, for $\theta := \lambda^{-1}$ and $N \geq 1$,

$$\phi_\lambda(\pi\theta^N) = \prod_{n=1}^{N} \cos(\pi\theta^n)\phi_\lambda(\pi).$$

Thus,

$$|\phi_\lambda(\pi\theta^N)| \geq \prod_{n=1}^{\infty} |\cos(\mathrm{const}\,\rho^n)||\phi_\lambda(\pi)| := \delta > 0,$$

for all $N \in \mathbb{N}$. This shows that $\phi_\lambda(t)$ does not tend to zero, as $t \to \infty$. $\square$