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ON THE LOCAL TIME OF A RECURRENT RANDOM WALK ON $\mathbb{Z}^2$

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Abstract. We prove a functional limit theorem for the number of visits by a planar random walk on $\mathbb{Z}^2$ with zero mean and finite second moment to the points of a fixed finite set $P \subset \mathbb{Z}^2$. The proof is based on the analysis of an accompanying random process with immigration at renewal epochs in case when the inter-arrival distribution has a slowly varying tail.

1. Introduction and summary of main results

Let $(\boldsymbol{\xi}_i)_{i\in\mathbb{N}}$ be a sequence of independent copies of a random vector $\boldsymbol{\xi}$, which takes values in $\mathbb{Z}^2$. Define a standard random walk

$$\boldsymbol{S}_0 := \boldsymbol{0} = (0,0), \quad \boldsymbol{S}_n = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \cdots + \boldsymbol{\xi}_n, \quad n \in \mathbb{N}, \tag{1.1}$$

and let

$$N_t(\boldsymbol{x}) := \sum_{1\le k\le t} \mathbb{1}_{\{\boldsymbol{S}_k = \boldsymbol{x}\}}, \quad \boldsymbol{x} \in \mathbb{Z}^2, \quad t \ge 0,$$

be the number of visits of the random walk $(\boldsymbol{S}_n)_{n\in\mathbb{N}}$ to a point $\boldsymbol{x}$ up to and including time t. We also put

$$T(\boldsymbol{x}) := \inf\{k \in \mathbb{N} : \boldsymbol{S}_k = \boldsymbol{x}\}, \quad \boldsymbol{x} \in \mathbb{Z}^2.$$

Throughout this paper we use bold symbols in formulas to indicate $\mathbb{R}^2$-valued (in particular, $\mathbb{Z}^2$-valued) objects and regular font for $\mathbb{R}$-valued (in particular, $\mathbb{Z}$-valued) objects. If the random vector $\boldsymbol{\xi}$ has the following distribution

$$\mathbb{P}\{\boldsymbol{\xi} = (0,1)\} = \mathbb{P}\{\boldsymbol{\xi} = (0,-1)\} = \mathbb{P}\{\boldsymbol{\xi} = (1,0)\} = \mathbb{P}\{\boldsymbol{\xi} = (-1,0)\} = 1/4, \tag{1.2}$$

then $(\boldsymbol{S}_n)_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is called simple symmetric planar random walk. Pólya's classic random walk theorem states that a simple symmetric planar random walk is recurrent, that is, it infinitely often returns to the origin with probability one, and therefore to any fixed point $\boldsymbol{x} \in \mathbb{Z}^2$:

$$\mathbb{P}\{T(\boldsymbol{x}) < \infty\} = 1 \quad \text{and} \quad \lim_{t\to+\infty} N_t(\boldsymbol{x}) = +\infty \quad \text{a.s.}, \quad \boldsymbol{x} \in \mathbb{Z}^2. \tag{1.3}$$

A more precise result on the rate of growth of $N_t(\boldsymbol{x})$ for a simple symmetric random walk follows from the next theorem which was first proved in papers [11, 12]. Throughout this paper we shall denote convergence of random processes in the Skorokhod space $D[0,\infty)$ of càdlàg functions endowed with the J_1-topology (respectively, M_1-topology) by $\Longrightarrow_{J_1}$

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(respectively, $\Longrightarrow_{M_1}$). Recall that a measurable real-valued function L, defined in a neighborhood of $+\infty$, is called slowly varying at $+\infty$ if for any $\lambda > 0$

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1,$$

see [1] for a comprehensive treatment of slow and regular variation. We also stipulate that $f(t) \sim g(t)$, $t \to \infty$, if and only if $\lim_{t\to\infty} f(t)/g(t) = 1$.

Theorem 1.1 (Kasahara (1985)). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent copies of a nonnegative real-valued random variable X, which satisfies*

$$\mathbb{P}\{X > t\} \ \sim\ \frac{1}{L(t)}, \quad t \to \infty, \tag{1.4}$$

for some nondecreasing function L, which is slowly varying at $+\infty$. Put $\widetilde{S}_0 := 0$, $\widetilde{S}_n := X_1 + X_2 + \cdots + X_n$, for $n \in \mathbb{N}$, and $\widetilde{\nu}(t) := \inf\{k \in \mathbb{N} : \widetilde{S}_k > t\}$, for $t \geq 0$. Then

$$\left(n^{-1}L(\widetilde{S}_{\lfloor nt\rfloor})\right)_{t\geq 0} \Longrightarrow_{J_1} \ (m(t))_{t\geq 0}, \quad n \to \infty, \tag{1.5}$$

where the random process $(m(t))_{t\geq 0}$ on the right-hand side is called extremal process and is defined by equality

$$m(t) := \max_{k:\ t_k\leq t} y_k, \quad t \geq 0.$$

Here (t_k, y_k) are the atoms of the Poisson point process on $[0,\infty) \times (0,\infty]$ with the intensity measure $\mathrm{d}t \times y^{-2}\mathrm{d}y$.

There is also convergence of inverse processes:

$$\left(t^{-1}\widetilde{\nu}(L^{\leftarrow}(tu)\right)_{u\geq 0} \Longrightarrow_{M_1} \ (m^{\leftarrow}(u))_{u\geq 0}, \quad t \to \infty, \tag{1.6}$$

where $L^{\leftarrow}$ is the generalized inverse function of L and

$$m^{\leftarrow}(t) := \inf\{y \geq 0 :\ m(y) > t\}, \quad t \geq 0,$$

is the inverse extremal process.

In combination with the classic result of Dvoretzky and Erdős [2], which shows that for a simple symmetric random walk

$$\mathbb{P}\{T(\mathbf{0}) > n\} \ \sim\ \frac{\pi}{\log n}, \quad n \to \infty, \tag{1.7}$$

Theorem 1.1 directly implies the functional limit theorem

$$\left(\frac{N_{n^t}(\mathbf{0})}{\log n}\right)_{t\geq 0} \Longrightarrow_{M_1} \ \left(\pi^{-1}m^{\leftarrow}(t)\right)_{t\geq 0}, \quad n \to \infty. \tag{1.8}$$

Indeed, convergence (1.8) follows immediately from the observation that the process $(N_t(\mathbf{0}) + 1)_{t\geq 0}$ has the same distribution as the process $(\widetilde{\nu}(t))_{t\geq 0}$ constructed using the random variable $X = T(\mathbf{0})$. Since for any fixed $\boldsymbol{x} \in \mathbb{Z}^2$ there is an equality of distributions:

$$(N_t(\boldsymbol{x}))_{t\geq 0} \stackrel{d}{=} \left(\mathbb{1}_{\{T(\boldsymbol{x})\leq t\}}(1 + N'_{t-T(\boldsymbol{x})}(\mathbf{0}))\right)_{t\geq 0},$$

where $(N'_t(\mathbf{0}))_{t\geq 0}$ is an independent copy of $(N_t(\mathbf{0}))_{t\geq 0}$, also independent of $T(\boldsymbol{x})$, the term $N_{n^t}(\mathbf{0})$ on the left-hand side of (1.8) can be replaced by $N_{n^t}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{Z}^2$. Thereby, the number of visits by a simple symmetric random walk to any fixed state $\boldsymbol{x} \in \mathbb{Z}^2$ satisfies the functional limit theorem

$$\left(\frac{N_{n^t}(\boldsymbol{x})}{\log n}\right)_{t\geq 0} \Longrightarrow_{M_1} \ \left(\pi^{-1}m^{\leftarrow}(t)\right)_{t\geq 0}, \quad n \to \infty. \tag{1.9}$$

The relation (1.9) is the starting point of our study. The purpose of this paper is to give a two-fold generalization of the limit theorem (1.9). First, it is well known, see, for example, T1 on p. 83 in [15], that an arbitrary two-dimensional random walk on the lattice $\mathbb{Z}^2$ with zero mean and finite second moment is recurrent, that is, equations (1.3) hold for $\boldsymbol{x} = \boldsymbol{0}$ and, more general, for all accessible states. Thus, the question of obtaining a counterpart of (1.9) for such random walks is natural. Secondly, it is interesting to study the weak convergence of *a random field* $(N_t(\boldsymbol{x}))_{t\geq 0,\boldsymbol{x}\in\mathbb{Z}^2}$.

To formulate our main result we need some additional definitions and notation. Recall that a state $\boldsymbol{x} \in \mathbb{Z}^2$ is called *accessible* for a random walk $(\boldsymbol{S}_n)_{n\in\mathbb{N}_0}$ if $\mathbb{P}\{\boldsymbol{S}_n = \boldsymbol{x}\} > 0$ for some $n \in \mathbb{N}_0$. A random walk is called *aperiodic* if the smallest additive subgroup of $\mathbb{Z}^2$, which contains a set of all accessible states, coincides with $\mathbb{Z}^2$. A random walk is called *strongly aperiodic* if for any $\boldsymbol{x} \in \mathbb{Z}^2$ the smallest additive subgroup of $\mathbb{Z}^2$, which contains a set $\boldsymbol{x} + \{\boldsymbol{y} \in \mathbb{Z}^2 : \mathbb{P}\{\boldsymbol{\xi} = \boldsymbol{y}\} > 0\}$, is equal to $\mathbb{Z}^2$.

Here is our main result.

Theorem 1.2. *Let $(\boldsymbol{S}_n)_{n\in\mathbb{N}_0}$ be a strongly aperiodic random walk on the lattice $\mathbb{Z}^2$, which is defined by equations* (1.1)*, and such that*

$$\mathbb{E}\boldsymbol{\xi} = \boldsymbol{0}, \quad \mathbb{E}\|\boldsymbol{\xi}\|^2 < \infty.$$

Then, for any fixed finite set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_p\} \subset \mathbb{Z}^2$, there is convergence

(1.10)
$$\left(\frac{N_{n^t}(\boldsymbol{x}_1)}{\log n}, \frac{N_{n^t}(\boldsymbol{x}_2)}{\log n}, \dots, \frac{N_{n^t}(\boldsymbol{x}_p)}{\log n}\right)_{t\geq 0} \Longrightarrow_{M_1} C_{\boldsymbol{\xi}}(m^{\leftarrow}(t), \dots, m^{\leftarrow}(t))_{t\geq 0}, \quad n \to \infty,$$

in the Skorokhod space $(D[0,\infty))^p$ endowed with the product M_1-topology, where

$$C_{\boldsymbol{\xi}} := (2\pi\sqrt{Q})^{-1}, \quad Q := \det(\mathrm{Cov}\,(\boldsymbol{\xi})) > 0 \tag{1.11}$$

and $\mathrm{Cov}\,(\boldsymbol{\xi})$ *denotes the covariance matrix of the random vector* $\boldsymbol{\xi}$*.*

A simple necessary and sufficient condition for strong aperiodicity of $(\boldsymbol{S}_n)_{n\in\mathbb{N}_0}$ in terms of the characteristic function of $\boldsymbol{\xi}$ has the following form, see Proposition P8 on p. 75 in [15]:

$$\left|\mathbb{E}e^{i\langle\boldsymbol{\theta},\boldsymbol{\xi}\rangle}\right| = 1 \quad \Longrightarrow \quad \text{all coordinates of } \boldsymbol{\theta} = (\theta_1, \theta_2) \text{ are multiples of } 2\pi.$$

Note that a simple symmetric random walk with steps distributed as (1.2), and for which

$$\mathbb{E}e^{i\langle\boldsymbol{\theta},\boldsymbol{\xi}\rangle} = \frac{\cos(\theta_1) + \cos(\theta_2)}{2},$$

is aperiodic, but not strongly aperiodic. Therefore, Theorem 1.2 is not directly applicable. Nevertheless, we have the following statement.

Proposition 1.3. *Let $(\boldsymbol{S}_n)_{n\in\mathbb{N}_0}$ be a simple symmetric random walk on $\mathbb{Z}^2$. Then the convergence* (1.10) *holds with $C_{\boldsymbol{\xi}} = \pi^{-1}$.*

Remark 1.4. *We emphasize that convergence* (1.10) *takes place exactly in the product M_1-topology, and does not hold in the product J_1-topology on $(D[0,\infty))^p$ even if $p = 1$, see Remark 2.2 in* [10]*. We also emphasize that due to the coordinate-wise monotonicity of the prelimit processes, convergence in the product M_1-topology is actually equivalent to convergence of finite-dimensional distributions, see Corollary 12.5.1 in* [16]*.*

The result of Theorem 1.2 is in some sense degenerate since the components of the limit process are the copies of the same process $(m^{\leftarrow}(t))_{t\geq 0}$. This suggests that a non-trivial limit process can be obtained by replacing a fixed set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_p\} \subset \mathbb{Z}^2$ by a set $\{\boldsymbol{x}_1^{(n)}, \boldsymbol{x}_2^{(n)}, \dots, \boldsymbol{x}_p^{(n)}\} \subset \mathbb{Z}^2$, in which the points move away from the origin as n

increases. The problem of deriving a limit theorem in this case is highly non-trivial even for a simple symmetric random walk, cannot be solved by the methods used in this work, and is the subject of our further research.

2. Proofs

2.1. Tail behavior of the first return to the origin by a strongly aperiodic recurrent random walk on $\mathbb{Z}^2$.

We start with proving an auxiliary lemma that generalizes relation (1.7) to the case of strongly aperiodic recurrent walks.

Lemma 2.1. *Under the conditions of Theorem 1.2 the following relation holds*

$$\mathbb{P}\{T(\mathbf{0})>n\}\ \sim\ \frac{1}{C_{\boldsymbol{\xi}}\log n},\quad n\to\infty. \tag{2.1}$$

Proof. The following argument is a modification of the proof by Dvoretzky and Erdős, see [2]. The starting point is the following equality, which holds for every $n\in\mathbb{N}$,

$$\begin{aligned}1&=\sum_{j=1}^{n}\mathbb{P}\{\text{the last visit to }\mathbf{0}\text{ up to time }n\text{ occured at time }j\}\\&\qquad\qquad+\mathbb{P}\{\text{there were no visits to }\mathbf{0}\text{ before and including time }n\}\\&=\sum_{j=1}^{n}\mathbb{P}\{\boldsymbol{S}_j=\mathbf{0}\}\mathbb{P}\{T(\mathbf{0})>n-j\}+\mathbb{P}\{T(\mathbf{0})>n\}\\&=\sum_{j=0}^{n}\mathbb{P}\{\boldsymbol{S}_j=\mathbf{0}\}\mathbb{P}\{T(\mathbf{0})>n-j\}.\end{aligned}\tag{2.2}$$

According to the local central limit theorem for strongly aperiodic two-dimensional random walks, see, for example, paragraph 6 in [14] or P9 on p. 75 in [15], we have

$$\mathbb{P}\{\boldsymbol{S}_n=\mathbf{0}\}\ \sim\ \frac{C_{\boldsymbol{\xi}}}{n},\quad n\to\infty. \tag{2.3}$$

Note that $C_{\boldsymbol{\xi}}\in(0,\infty)$ since for a strongly aperiodic random walks $Q>0$, see P7 on p. 74 in [15]. For large $n\in\mathbb{N}$, put $k(n):=n-\lfloor n/\log n\rfloor$ and write using (2.2)

$$\mathbb{P}\{T(\mathbf{0})>n-k(n)\}\sum_{j=0}^{k(n)}\mathbb{P}\{\boldsymbol{S}_j=\mathbf{0}\}+\sum_{j=k(n)+1}^{n}\mathbb{P}\{\boldsymbol{S}_j=\mathbf{0}\}\geq 1.$$

Sending $n\to\infty$ in both parts and using (2.3), we obtain

$$\liminf_{n\to\infty}\mathbb{P}\{T(\mathbf{0})>n-k(n)\}C_{\boldsymbol{\xi}}\log k(n)\geq 1.$$

Since $\log k(n)\sim\log(n-k(n))$, as $n\to\infty$, we have

$$\liminf_{n\to\infty}\mathbb{P}\{T(\mathbf{0})>n-k(n)\}C_{\boldsymbol{\xi}}\log(n-k(n))\geq 1. \tag{2.4}$$

Due to the fact that the sequence $n-k(n)=\lfloor n/\log n\rfloor$ runs over all sufficiently large integers, the equation (2.4) is equivalent to

$$\liminf_{n\to\infty}\mathbb{P}\{T(\mathbf{0})>n\}C_{\boldsymbol{\xi}}\log n\geq 1. \tag{2.5}$$

The upper bound is obtained more easily by using inequality

$$\mathbb{P}\{T(\mathbf{0})>n\}\sum_{j=0}^{n}\mathbb{P}\{\boldsymbol{S}_j=\mathbf{0}\}\leq 1,$$

from which it immediately follows that

$$\limsup_{n\to\infty} \mathbb{P}\{T(\mathbf{0})>n\}C_{\boldsymbol{\xi}}\log n \leq 1,$$

using relation (2.3). □

Remark 2.2. *In some cases the condition of strong aperiodicity appearing in Lemma 2.1 can be relaxed to aperiodicity. According to Proposition P1 on p. 42 in* [15]*, for an aperiodic recurrent walk, which is not strongly aperiodic, there exists an integer $s \geq 2$, such that $\mathbb{P}\{\boldsymbol{S}_{ns}=\mathbf{0}\}>0$ for all large enough $n\in\mathbb{N}$, and $\mathbb{P}\{\boldsymbol{S}_k=\mathbf{0}\}=0$ for all $k\in\mathbb{N}$ which are not multiples of s. If there is an analogue of relation* (2.3)*, that is, for some constant $C_{s,\boldsymbol{\xi}}>0$, it holds*

$$\mathbb{P}\{\boldsymbol{S}_{ns}=\mathbf{0}\} \sim \frac{C_{s,\boldsymbol{\xi}}}{n}, \quad n\to\infty, \tag{2.6}$$

then, upon repeating the proof of Lemma 2.1 with obvious modifications, it can be checked that

$$\mathbb{P}\{T(\mathbf{0})>n\} \sim \frac{1}{C_{s,\boldsymbol{\xi}}\log n}, \quad n\to\infty.$$

We have not been able to find in the literature simple conditions under which the relation (2.6) *holds with explicit form of the constant $C_{s,\boldsymbol{\xi}}$.*

2.2. **Random processes with immigration at renewal epochs and renewal shot noise processes.** The main tool that we shall use in the proof of Theorem 1.2 is the analysis of an auxiliary random processes with immigration and renewal shot noise processes. Let us proceed by recalling the definition, see, for example, [7].

Definition 2.3. Let $(Y_k,X_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of the pair (Y,X), where $Y=Y(\cdot)$ is a random process with paths in the Skorokhod space $D[0,\infty)$, and X is a real-valued positive random variable, possibly dependent on the process $Y(\cdot)$. The random process

$$Z(t) := \sum_{k\geq 0} Y_{k+1}(t-\widetilde{S}_k), \quad t\geq 0,$$

where $\widetilde{S}_0 := 0$, $\widetilde{S}_n := X_1+X_2+\cdots+X_n$, for $n\in\mathbb{N}$, is called *random process with immigration at renewal epochs.* If $\mathbb{P}\{Y(t)=h(t)\}=1$ for some non-random function $h\in D[0,\infty)$, then the process $Z(\cdot)$ is called *renewal shot noise process.*

The common distribution of random variables X_k's is called *inter-arrival* distribution.

Renewal shot noise processes and random processes with immigration at renewal epochs serve as mathematical models for many processes of cumulative nature in physics, chemistry, geology, insurance and other sciences. More detailed information about these processes and their applications can be found in the book [4].

Let us now explain how random processes with immigration at renewal epochs arise in the analysis of two-dimensional recurrent random walks. Fix a set of pairwise distinct points $(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_p)\subset\mathbb{Z}^2$ and consider the vector $(N_t(\boldsymbol{x}_1),\ldots,N_t(\boldsymbol{x}_p))^\top$. We have an obvious equality

$$\begin{bmatrix} N_t(\boldsymbol{x}_1)\\ N_t(\boldsymbol{x}_2)\\ \vdots \\ N_t(\boldsymbol{x}_p)\end{bmatrix} = \begin{bmatrix} N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_1)\\ N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_2)\\ \vdots \\ N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_p)\end{bmatrix} + \begin{bmatrix} N'_{(T(\mathbf{0}),t]}(\boldsymbol{x}_1)\\ N'_{(T(\mathbf{0}),t]}(\boldsymbol{x}_2)\\ \vdots \\ N'_{(T(\mathbf{0}),t]}(\boldsymbol{x}_p)\end{bmatrix}, \quad t\geq 0,$$

where $N'_{(T(\mathbf{0}),t]}(\boldsymbol{x})$ is the number of visits to the state $\boldsymbol{x}$ up to time t after the first return to $\mathbf{0}$. By definition this value is equal to zero if there was no return to the state $\mathbf{0}$ before time t. A random value $T(\mathbf{0})$, that is, the moment of the first return to the state $\mathbf{0}$, is a stopping time, and therefore from the strong Markov property we obtain a distributional equality of processes:

$$\begin{bmatrix} N_t(\boldsymbol{x}_1) \\ N_t(\boldsymbol{x}_2) \\ \vdots \\ N_t(\boldsymbol{x}_p) \end{bmatrix}_{t\geq 0} \overset{d}{=} \begin{bmatrix} N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_1) \\ N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_2) \\ \vdots \\ N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_p) \end{bmatrix}_{t\geq 0} + \begin{bmatrix} N''_{t-T(\mathbf{0})}(\boldsymbol{x}_1) \\ N''_{t-T(\mathbf{0})}(\boldsymbol{x}_2) \\ \vdots \\ N''_{t-T(\mathbf{0})}(\boldsymbol{x}_p) \end{bmatrix}_{t\geq 0},$$

where the vector-valued process $(N''_t(\boldsymbol{x}_1), N''_t(\boldsymbol{x}_2), \dots, N''_t(\boldsymbol{x}_p))^\top_{t\geq 0}$ does not depend on the pair

$$(T(\mathbf{0}), (N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_1), N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_2), \dots, N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_p))^\top_{t\geq 0}),$$

and has the same distribution as the vector process $(N_t(\boldsymbol{x}_1), N_t(\boldsymbol{x}_2), \dots, N_t(\boldsymbol{x}_p))^\top_{t\geq 0}$.

Fix an arbitrary set $\alpha_1, \alpha_2, \dots, \alpha_p$ of nonnegative real numbers and consider a linear combination

$$\bar{Z}(t) := \sum_{i=1}^{p} \alpha_i N_t(\boldsymbol{x}_i), \quad t \geq 0. \tag{2.7}$$

According to the Cramér–Wold device, in order to prove Theorem 1.2 and Proposition 1.3 it is enough to verify

$$\left(\frac{\bar{Z}(n^t)}{\log n}\right)_{t\geq 0} \Longrightarrow_{M_1} C_{\boldsymbol{\xi}} \left(\sum_{i=1}^{p} \alpha_i\right) (m^{\leftarrow}(t))_{t\geq 0}. \tag{2.8}$$

From the aforementioned arguments it follows that $(\bar{Z}(t))_{t\geq 0}$ is a random process with immigrations at renewal epochs generated by the pair $(\bar{Y}, T(\mathbf{0}))$, where

$$\bar{Y}(t) := \sum_{i=1}^{p} \alpha_i N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_i), \quad t \geq 0. \tag{2.9}$$

Note that components in the pair $(\bar{Y}, T(\mathbf{0}))$ are dependent.

Summarizing, we have shown that in order to prove Theorem 1.2 and Proposition 1.3 it is enough to verify that a random process with immigration at renewal epochs, defined by equalities (2.7) and (2.9), satisfies the functional limit theorem (2.8).

In the past decade there was a splash of activity around random processes with immigration at renewal epochs and renewal shot noise processes. Many studies were concerned with derivation of various limit theorems for these processes, as $t \to \infty$, in different asymptotic regimes, see, for example, [3, 4, 5, 6, 7, 8, 13] and references therein. In the listed papers a more or less complete asymptotic theory of random process with immigration at renewal epochs was constructed in a case when the inter-arrival distribution belongs to the domain of attraction of a stable law. However, a feature of the process $(\bar{Z}(t))_{t\geq 0}$, defined by (2.7), is that the inter-arrival distribution has a slowly varying tail according to Lemma 2.1. Therefore, in our situation the inter-arrival distribution does not belong to the domain of attraction of any stable law. We know only one paper [10] which deals with renewal shot noise processes constructed using inter-arrival distribution having slowly varying tail.

2.3. **Analysis of the process $\bar{Z}(t)$.** To prove (2.8), we use the decomposition

$$\bar{Z}(t) = \sum_{k\geq 0} \left(\bar{Y}_{k+1}(t - \widetilde{S}_k) - h(t - \widetilde{S}_k)\right) \mathbb{1}_{\{\widetilde{S}_k \leq t\}} + \sum_{k\geq 0} h(t - \widetilde{S}_k)\mathbb{1}_{\{\widetilde{S}_k \leq t\}}$$
$$=: \bar{Z}_1(t) + \bar{Z}_2(t),$$

where

$$h(t) := \mathbb{E}\bar{Y}(t) := \sum_{i=1}^{p} \alpha_i \mathbb{E} N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_i), \quad t \geq 0.$$

Without loss of generality we assume that at least one coefficient among $\alpha_1, \ldots, \alpha_p$ is not equal to zero.

Note that $N_{T(\mathbf{0})}(\mathbf{0}) = 1$, and, therefore, $\lim_{t\to\infty} \mathbb{E}N_{T(\mathbf{0})\wedge t}(\mathbf{0}) = 1$. According to Proposition P3 on p. 112 in [15], for an arbitrary $\boldsymbol{x} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$,

$$\mathbb{E}N_{T(\mathbf{0})}(\boldsymbol{x}) = 1,$$

thus, from the Lebesgue dominated convergence theorem

$$\lim_{t\to\infty} \mathbb{E}N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \mathbb{Z}^2. \tag{2.10}$$

We emphasize that a rather counter-intuitive fact that $\mathbb{E}N_{T(\mathbf{0})}(\boldsymbol{x}) = 1$ does not depend on x is the reason why the limit process in Theorem 1.2 is degenerate.

Using formula (2.10) it follows that

$$\lim_{t\to\infty} h(t) = \sum_{i=1}^{p} \alpha_i > 0,$$

in particular, the function h is nonnegative and slowly varying at infinity. According to Theorem 1.1 from [10] in combination with Lemma 2.1 (for the proof of Theorem 1.2) or with formula (1.7) (for the proof of Proposition 1.3), we have

$$\left(\frac{\bar{Z}_2(n^t)}{\log n}\right)_{t\geq 0} \overset{f.d.d.}{\longrightarrow} C_{\boldsymbol{\xi}} \left(\sum_{i=1}^{p} \alpha_i\right) (m^{\leftarrow}(t))_{t\geq 0}. \tag{2.11}$$

Assume that we proved

$$\frac{\bar{Z}_1(n^t)}{\log n} \overset{P}{\longrightarrow} 0, \quad n \to \infty,$$

for every fixed $t \geq 0$, which is equivalent to

$$\frac{\bar{Z}_1(n)}{\log n} \overset{P}{\longrightarrow} 0, \quad n \to \infty. \tag{2.12}$$

Then from (2.11) and (2.12) we get

$$\left(\frac{\bar{Z}(n^t)}{\log n}\right)_{t\geq 0} \overset{f.d.d.}{\longrightarrow} C_{\boldsymbol{\xi}} \left(\sum_{i=1}^{p} \alpha_i\right) (m^{\leftarrow}(t))_{t\geq 0}.$$

This convergence is equivalent to (2.8) since the prelimit processes are nondecreasing, see Corollary 12.5.1 in [16].

It remains to check convergence (2.12). To this end we need an auxiliary lemma. Note that in this lemma the condition of strong aperiodicity is not required.

Lemma 2.4. *For an arbitrary aperiodic recurrent random walk $(\boldsymbol{S}_n)_{n\in\mathbb{N}_0}$ on the lattice $\mathbb{Z}^2$ and for any point $\boldsymbol{x} \in \mathbb{Z}^2$ we have*

$$\mathbb{E}(N_{T(\mathbf{0})}(\boldsymbol{x}))^2 < \infty.$$

Proof. The statement of the lemma is obvious if $\boldsymbol{x}=\mathbf{0}$. Assume that $\boldsymbol{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}$. Then

$$\begin{aligned}
\mathbb{E}(N_{T(\mathbf{0})}(\boldsymbol{x}))^2 &= \mathbb{E}\left(\sum_{i=1}^{T(\mathbf{0})}\mathbb{1}_{\{\boldsymbol{S}_i=\boldsymbol{x}\}}\right)^2 = \mathbb{E}\left(\sum_{i=1}^{\infty}\mathbb{1}_{\{\boldsymbol{S}_i=\boldsymbol{x},T(\mathbf{0})>i\}}\right)^2\\
&= \sum_{i=1}^{\infty}\mathbb{P}\{\boldsymbol{S}_i=\boldsymbol{x},T(\mathbf{0})>i\}+2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{P}\{\boldsymbol{S}_i=\boldsymbol{x},\boldsymbol{S}_j=\boldsymbol{x},T(\mathbf{0})>j\}\\
&= \mathbb{E}N_{T(\mathbf{0})}(\boldsymbol{x})\\
&\qquad +2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{P}\{\boldsymbol{S}_i=\boldsymbol{x},\boldsymbol{\xi}_{i+1}+\dots+\boldsymbol{\xi}_j=\mathbf{0},\boldsymbol{S}_1\neq\mathbf{0},\dots,\boldsymbol{S}_j\neq\mathbf{0}\}\\
&= 1+2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{P}\{\boldsymbol{S}_i=\boldsymbol{x},\boldsymbol{S}_1\neq\mathbf{0},\dots,\boldsymbol{S}_{i-1}\neq\mathbf{0}\}\\
&\qquad \times\mathbb{P}\{\boldsymbol{\xi}_{i+1}+\dots+\boldsymbol{\xi}_j=\mathbf{0},\boldsymbol{\xi}_{i+1}\neq-\boldsymbol{x},\dots,\boldsymbol{\xi}_{i+1}+\dots+\boldsymbol{\xi}_j\neq-\boldsymbol{x}\}\\
&= 1+2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{P}\{\boldsymbol{S}_i=\boldsymbol{x},T(\mathbf{0})>i\}\mathbb{P}\{\boldsymbol{S}_{j-i}=\mathbf{0},T(-\boldsymbol{x})>j-i\}\\
&= 1+2\left(\sum_{i=1}^{\infty}\mathbb{P}\{\boldsymbol{S}_i=\boldsymbol{x},T(\mathbf{0})>i\}\right)\left(\sum_{j=1}^{\infty}\mathbb{P}\{\boldsymbol{S}_j=\mathbf{0},T(-\boldsymbol{x})>j\}\right).
\end{aligned}$$

The first series is just $\mathbb{E}N_{T(\mathbf{0})}(\boldsymbol{x})$, hence is equal to 1. The second series is the expectation of the number of visits to the state $\mathbf{0}$ (excluding time 0) before the first visit to the state $-\boldsymbol{x}$. The latter random variable has a geometric distribution on $\mathbb{N}_0$ with success probability $\mathbb{P}\{T(\mathbf{0})>T(-\boldsymbol{x})\}>0$. Thus, we obtained that

$$\mathbb{E}(N_{T(\mathbf{0})}(\boldsymbol{x}))^2=1+\frac{2\mathbb{P}\{T(\mathbf{0})<T(-\boldsymbol{x})\}}{\mathbb{P}\{T(\mathbf{0})>T(-\boldsymbol{x})\}},\quad \boldsymbol{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}.$$

□

Proof of formula (2.12). By Chebyshev's inequality it is enough to check that

$$\mathrm{Var}\,(\bar{Z}_1(t))=\mathbb{E}\bar{Z}_1^2(t)=O(\log t),\quad t\to\infty. \tag{2.13}$$

Let us introduce the notation

$$v(t):=\mathrm{Var}\,\bar{Y}(t)=\mathbb{E}(\bar{Y}(t)-h(t))^2,\quad t\geq 0,$$

and write

$$\begin{aligned}
\mathbb{E}\bar{Z}_1^2(t) &= \mathbb{E}\left(\sum_{k\geq 0}\left(\bar{Y}_{k+1}(t-\widetilde{S}_k)-h(t-\widetilde{S}_k)\right)\mathbb{1}_{\{\widetilde{S}_k\leq t\}}\right)^2\\
&= \mathbb{E}\left(\sum_{k\geq 0}v(t-\widetilde{S}_k)\mathbb{1}_{\{\widetilde{S}_k\leq t\}}\right)\\
&\quad +2\mathbb{E}\left(\sum_{i<j}\left(\bar{Y}_{i+1}(t-\widetilde{S}_i)-h(t-\widetilde{S}_i)\right)\left(\bar{Y}_{j+1}(t-\widetilde{S}_j)-h(t-\widetilde{S}_j)\right)\mathbb{1}_{\{\widetilde{S}_j\leq t\}}\right).
\end{aligned}$$

It is easy to see that the second expectation is equal to zero. Indeed,

$$\mathbb{E}\left(\sum_{i<j}\Big(\bar{Y}_{i+1}(t-\widetilde{S}_i)-h(t-\widetilde{S}_i)\Big)\Big(\bar{Y}_{j+1}(t-\widetilde{S}_j)-h(t-\widetilde{S}_j)\Big)\mathbb{1}_{\{\widetilde{S}_j\le t\}}\right)$$
$$=\sum_{i<j}\mathbb{E}\Big(\mathbb{E}\Big(\Big(\bar{Y}_{i+1}(t-\widetilde{S}_i)-h(t-\widetilde{S}_i)\Big)$$
$$\times\Big(\bar{Y}_{j+1}(t-\widetilde{S}_j)-h(t-\widetilde{S}_j)\Big)\mathbb{1}_{\{\widetilde{S}_j\le t\}}\Big)\Big|(Y_1,X_1),\dots,(Y_j,X_j)\Big)=0$$

due to the fact that $\bar{Y}_{j+1}$ does not depend on $(Y_1,X_1),\dots,(Y_j,X_j))$.

From Lemma 2.4, under the assumptions of Theorem 1.2 or Proposition 1.3, we obtain

$$0\le v(t)=\mathrm{Var}\left(\sum_{i=1}^{p}\alpha_i N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_i)\right)\le C'_p\sum_{i=1}^{p}\alpha_i^2\,\mathrm{Var}\,N_{T(\mathbf{0})\wedge t}(\boldsymbol{x}_i)\le C'<\infty,\quad t\ge 0,$$

where the constant C'_p depends only on p, whereas the constant C' depends on p, $\alpha_1,\alpha_2,\dots,\alpha_p$ and the distribution of $\boldsymbol{\xi}$. Thus,

$$\mathbb{E}\bar{Z}_1^2(t)=\mathbb{E}\left(\sum_{k\ge 0}v(t-\widetilde{S}_k)\mathbb{1}_{\{\widetilde{S}_k\le t\}}\right)$$
$$\le C'\mathbb{E}\left(\sum_{k\ge 0}\mathbb{1}_{\{\widetilde{S}_k\le t\}}\right)\le C'(1+\mathbb{E}N_t(\mathbf{0}))=O(\log t),\quad t\to\infty,$$

where we used that $\mathbb{E}N_t(\mathbf{0})\sim\mathrm{const}\cdot\log t$ as $t\to\infty$, see, for example, Proposition 2.1 in [9]. □

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