GENERALISED CONVEXITY WITH RESPECT TO FAMILIES OF AFFINE MAPS

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ABSTRACT. The standard convex closed hull of a set is defined as the intersection of all images, under the action of a group of rigid motions, of a half-space containing the given set. In this paper we propose a generalisation of this classical notion, that we call a $(K,\mathbb{H})$-hull, and which is obtained from the above construction by replacing a half-space with some other convex closed subset K of the Euclidean space, and a group of rigid motions by a subset $\mathbb{H}$ of the group of invertible affine transformations. The main focus is put on the analysis of $(K,\mathbb{H})$-convex hulls of random samples from K.

1. INTRODUCTION

Let $\mathbb{H}$ be a nonempty subset of the product $\mathbb{R}^d \times \mathbb{GL}_{\mathrm{d}}$, where $\mathbb{GL}_{\mathrm{d}}$ is a group of all invertible linear transformations of $\mathbb{R}^d$. We regard elements of $\mathbb{H}$ as invertible affine transformations of $\mathbb{R}^d$ by identifying $(x,g) \in \mathbb{H}$ with a mapping

$$\mathbb{R}^d \ni y \mapsto g(y+x) \in \mathbb{R}^d,$$

which first translates the argument y by the vector x and then applies the linear transformation $g \in \mathbb{G}$ to the translated vector.

Fix a convex closed set K in $\mathbb{R}^d$ which is distinct from the whole space. For a given set $A \subseteq \mathbb{R}^d$ consider the set

$$\operatorname{conv}_{K,\mathbb{H}}(A) := \bigcap_{(x,g)\in\mathbb{H}:\, A\subseteq g(K+x)} g(K+x),$$

where $g(B) := \{gz : z \in B\}$ and $B+x := \{z+x : z \in B\}$, for $g \in \mathbb{G}$, $x \in \mathbb{R}^d$, and $B \subseteq \mathbb{R}^d$. If there is no $(x,g) \in \mathbb{H}$ such that $g(K+x)$ contains A, the intersection on the right-hand side is taken over the empty family and we stipulate that $\operatorname{conv}_{K,\mathbb{H}}(A) = \mathbb{R}^d$. The set $\operatorname{conv}_{K,\mathbb{H}}(A)$ is by definition the intersection of all images of K under affine transformations from $\mathbb{H}$, which contain A. We call the set $\operatorname{conv}_{K,\mathbb{H}}(A)$, which is easily seen to be closed and convex, the $(K,\mathbb{H})$-hull of A; the set A is said to be $(K,\mathbb{H})$-convex if it coincides with its $(K,\mathbb{H})$-hull. This definition of the convex hull fits the abstract convexity concept described in [1].

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In what follows we assume that $\mathbb{H}$ contains the pair $(0,I)$, where I is the unit matrix. If $A \subseteq K$, then

$$\operatorname{conv}_{K,\mathbb{H}}(A) \subseteq K. \tag{1.1}$$

It is obvious that a larger family $\mathbb{H}$ results in a smaller $(K,\mathbb{H})$-hull, that is, if $\mathbb{H} \subseteq \mathbb{H}_1$, then

$$\operatorname{conv}_{K,\mathbb{H}_1}(A) \subseteq \operatorname{conv}_{K,\mathbb{H}}(A), \quad A \subseteq \mathbb{R}^d.$$

In particular, if $\mathbb{H} = \{0\} \times \{I\}$ and $A \subseteq K$, then

$$\operatorname{conv}_{K,\{0\}\times\{I\}}(A) = K,$$

so equality in (1.1) is possible. Since $(K,\mathbb{H})$-hull is always a convex closed set which contains A,

$$\operatorname{conv}(A) \subseteq \operatorname{conv}_{K,\mathbb{H}}(A), \quad A \subseteq \mathbb{R}^d,$$

where conv denotes the operation of taking conventional closed convex hull. If K is a fixed closed half-space, $\mathbb{H} = \mathbb{R}^d \times \mathbb{SO}_{\mathrm{d}}$, where $\mathbb{SO}_{\mathrm{d}}$ is the special orthogonal group, then $\operatorname{conv}_{K,\mathbb{H}}(A) = \operatorname{conv}(A)$, since every closed half-space can be obtained as an image of a fixed closed half-space under a rigid motion. The equality here can also be achieved for arbitrary closed convex set K and bounded $A \subseteq \mathbb{R}^d$ upon letting $\mathbb{H} = \mathbb{R}^d \times \mathbb{GL}_{\mathrm{d}}$, see Proposition 2.5 below.

A nontrivial example, when $\operatorname{conv}_{K,\mathbb{H}}(A)$ differs both from $\operatorname{conv}(A)$ and from K, is as follows. If K is a closed ball of a fixed radius and $\mathbb{H} = \mathbb{R}^d \times \{I\}$, then $\operatorname{conv}_{K,\mathbb{H}}(A)$ is known in the literature as the ball hull of A, and, more generally, if K is an arbitrary convex body (that is, a convex compact set with nonempty interior) and $\mathbb{H} = \mathbb{R}^d \times \{I\}$, then $\operatorname{conv}_{K,\mathbb{H}}(A)$ is called the K-hull of A, see [6] and [10]. It is also clear from the definition that further nontrivial examples could be obtained from this case by enlarging the family of linear transformations involved in $\mathbb{H}$.

In the above examples the set $\mathbb{H}$ takes the form $\mathbb{T} \times \mathbb{G}$ for some $\mathbb{T} \subseteq \mathbb{R}^d$ and $\mathbb{G} \subseteq \mathbb{GL}_{\mathrm{d}}$. We implicitly assume this, whenever we specify $\mathbb{T}$ and $\mathbb{G}$. Furthermore, many interesting examples arise if $\mathbb{T}$ is a linear subspace of $\mathbb{R}^d$ and $\mathbb{G}$ is a subgroup of $\mathbb{GL}_{\mathrm{d}}$.

The paper is organised as follows. In Section 2 we analyse some basic properties of $(K,\mathbb{H})$-hulls and show how various known hulls in *convex geometry* can be obtained as particular cases of our construction. However, the main focus in our paper will be put on probabilistic aspects of $(K,\mathbb{H})$-hulls. As for many other models of *stochastic geometry*, we shall study $(K,\mathbb{H})$-hulls of random samples from K as the size of the sample tends to infinity. In Section 3 we introduce a random closed set which can be thought of as a variant of the Minkowski difference between the set K and the $(K,\mathbb{H})$-hull of a random sample from K. The limit theorems for this object are formulated and proved in Section 4. Various properties of the limiting random closed set are studied in Section 5. A number of examples for various choices of K and $\mathbb{H}$ are presented in Section 6. Some technical results related mostly to the convergence in distribution of random closed sets are collected in the Appendix.

2. $(K,\mathbb{H})$-HULLS OF SUBSETS OF $\mathbb{R}^d$

We first show that the $(K,\mathbb{H})$-hull is an idempotent operation.

Proposition 2.1. *If $A \subseteq K$, then*

$$\operatorname{conv}_{K,\mathbb{H}}\big(\operatorname{conv}_{K,\mathbb{H}}(A)\big) = \operatorname{conv}_{K,\mathbb{H}}(A).$$

Proof. We only need to show that the left-hand side is a subset of the right-hand one. Assume that z belongs to the left-hand side, and $z \notin g(K+x)$ for at least one $(x,g) \in \mathbb{H}$ such that $A \subseteq g(K+x)$. The latter implies $\operatorname{conv}_{K,\mathbb{H}}(A) \subseteq g(K+x)$, so that $\operatorname{conv}_{K,\mathbb{H}}(\operatorname{conv}_{K,\mathbb{H}}(A))$ is also a subset of $g(K+x)$, which is a contradiction. □

It is easy to see that $\operatorname{conv}_{K,\mathbb{H}}(K) = K$ and $\operatorname{conv}_{K,\mathbb{H}}(A)$ is equal to the intersection of all $(K,\mathbb{H})$-convex sets containing A.

For each $A \subseteq \mathbb{R}^d$, denote

$$K \ominus_{K,\mathbb{H}} A := \big\{(x,g) \in \mathbb{H} : A \subseteq g(K+x)\big\}.$$

If $\mathbb{H} = \mathbb{R}^d \times \{I\}$, the set $\big\{x \in \mathbb{R}^d : (-x,I) \in (K \ominus_{K,\mathbb{H}} A)\big\}$ is the usual Minkowski difference

$$K \ominus A := \{x \in \mathbb{R}^d : A + x \subseteq K\}$$

of K and A, see p. 146 in [14]. By definition of $(K,\mathbb{H})$-hulls

$$\operatorname{conv}_{K,\mathbb{H}}(A) = \bigcap_{(x,g)\in K\ominus_{K,\mathbb{H}}A} g(K+x),$$

and, therefore, A is $(K,\mathbb{H})$-convex if and only if

$$A = \bigcap_{(x,g)\in K\ominus_{K,\mathbb{H}}A} g(K+x).$$

The following is a counterpart of Proposition 2.2 in [10].

Lemma 2.2. *For every $A \subseteq \mathbb{R}^d$, we have*

$$K \ominus_{K,\mathbb{H}} A = K \ominus_{K,\mathbb{H}} \big(\operatorname{conv}_{K,\mathbb{H}}(A)\big).$$

Proof. Since $A \subseteq \operatorname{conv}_{K,\mathbb{H}}(A)$, the right-hand side is a subset of the left-hand one. Let $(x,g) \in K \ominus_{K,\mathbb{H}} A$. Then $A \subseteq g(K+x)$, and, therefore, $\operatorname{conv}_{K,\mathbb{H}}(A) \subseteq g(K+x)$. The latter means that $(x,g) \in K \ominus_{K,\mathbb{H}} (\operatorname{conv}_{K,\mathbb{H}}(A))$. □

Now we shall investigate how $\operatorname{conv}_{K,\mathbb{H}}(A)$ looks for various choice of K and $\mathbb{H}$, in particular, how various known hulls (conventional, spherical, conical, etc.) can be derived as particular cases of our model. In order to proceed, we recall some basic notions of convex geometry. Let K be a convex closed set and let

$$h(K,u) := \sup\big\{\langle x,u\rangle : x \in K\big\}, \quad u \in \mathbb{R}^d$$

denote the support function of K in the direction u, where $\langle x,u\rangle$ is the scalar product. Put

$$\mathrm{dom}(K) := \{u \in \mathbb{R}^d : h(K,u) < \infty\}$$

and note that $\mathrm{dom}(K) = \mathbb{R}^d$ for compact sets K. The cone $\mathrm{dom}(L)$ is sometimes called the barrier cone of L, see the end of Section 2 in [13]. For $u \in \mathrm{dom}(K)$, let $H(K,u)$, $H^-(K,u)$ and $F(K,u)$ denote the support plane, supporting halfspace and support set of K with outer normal vector $u \neq 0$, respectively. Formally,

$$H(K,u) := \{x \in \mathbb{R}^d : \langle x,u\rangle = h(K,u)\}, \quad H^-(K,u) := \{x \in \mathbb{R}^d : \langle x,u\rangle \le h(K,u)\}$$

and $F(K,u) = H(K,u) \cap K$. We shall also need notions of supporting and normal cones of K at a point $v \in K$. The supporting cone at $v \in K$ is defined by

$$S(K,v) := \mathrm{cl}\Big(\bigcup_{\lambda>0} \lambda(K-v)\Big),$$

where cl is the topological closure, see p. 81 in [14]. If $v \in F(K,u)$ for some $u \in \mathrm{dom}(K)$, then

$$S(K,v) + v \subseteq H^-(K,u). \tag{2.1}$$

For v which belong to the boundary ∂K of K, the normal cone $N(K,v)$ to K at v is defined by

$$N(K,v) := \{u \in \mathbb{R}^d \setminus \{0\} : v \in H(K,u)\} \cup \{0\}.$$

Proposition 2.3. *Let K be a convex closed set, and let $\mathbb{H} = \mathbb{T} \times \mathbb{G}$, where $\mathbb{T} = \mathbb{R}^d$, and $\mathbb{G} = \{\lambda I : \lambda > 0\}$ is the group of all scaling transformations. If $A \subseteq K$, then*

$$\mathrm{conv}_{K,\mathbb{H}}(A) = \bigcap_{x\in\mathbb{R}^d, v\in\partial K, A\subseteq S(K,v)+v+x} \big(S(K,v)+v+x\big), \tag{2.2}$$

that is, $\mathrm{conv}_{K,\mathbb{H}}(A)$ is the intersection of all translations of supporting cones to K that contain A.

Proof. If $A \subseteq \lambda K + x$, then $A \subseteq S(K,v)+v+x$ for any $v \in K$. Hence, we only need to show that the right-hand side of (2.2) is contained in the left-hand one. Assume that $z \in S(K,v)+v+x$ for all $v \in \partial K$ and $x \in \mathbb{R}^d$ such that $A \subseteq S(K,v)+v+x$, but $z \notin \lambda_0 K + y_0$ for some $\lambda_0 > 0$ and $y_0 \in \mathbb{R}^d$ with $A \subseteq \lambda_0 K + y_0$. By the separating hyperplane theorem, see Theorem 1.3.4 in [14], there exists a hyperplane $H_0 \subseteq \mathbb{R}^d$ such that $\lambda_0 K + y_0 \subseteq H_0^-$ and $z \in H_0^+$, where $H_0^\pm$ are the open half-spaces bounded by H_0. Let u_0 be the unit outer normal vector to H_0^- and note that $u_0 \in \mathrm{dom}(K)$. Choose an arbitrary v_0 from the support set $F(K,u_0)$. Since $\lambda_0 K = \lambda_0(K-v_0)+\lambda_0 v_0 \subseteq S(K,v_0)+\lambda_0 v_0$, we have $A \subseteq S(K,v_0)+\lambda_0 v_0 + y_0$. However,

$$\begin{aligned} S(K,v_0)+\lambda_0 v_0 + y_0 &= S(\lambda_0 K, \lambda_0 v_0) + \lambda_0 v_0 + y_0 \\ &\subseteq H^-(\lambda_0 K_0, u_0) + y_0 = H^-(\lambda_0 K_0 + y_0, u_0) \subseteq H_0^- \cup H_0, \end{aligned}$$

where the penultimate inclusion follows from (2.1). Thus, $z \notin S(K,v_0)+v_0+x_0$ with $x_0 = (\lambda_0 - 1)v_0 + y_0$, and $S(K,v_0)+v_0+x_0$ contains A. The obtained contradiction completes the proof. □

Corollary 2.4. *If K is a smooth convex body (meaning that the normal cone at each boundary point is one-dimensional), $\mathbb{T} = \mathbb{R}^d$, and $\mathbb{G} = \{\lambda I : \lambda > 0\}$ is the group of all scaling transformations, then* $\operatorname{conv}_{K,\mathbb{H}}(A) = \operatorname{conv}(A)$*, for all $A \subseteq K$.*

Proof. Since K is a smooth convex body, $\operatorname{dom}(K) = \mathbb{R}^d$ and its supporting cone at each boundary point is equal to the supporting half-space. The convex hull of A is exactly the intersection of all such half-spaces. □

The next result formalises an intuitively obvious fact that the $(K,\mathbb{H})$-hull of a bounded set A coincides with $\operatorname{conv}(A)$ for arbitrary K provided $\mathbb{H}$ is sufficiently rich, in particular, if $\mathbb{H} = \mathbb{R}^d \times \mathbb{GL}_d$.

Proposition 2.5. *Let K be a convex closed set, $\mathbb{T} = \mathbb{R}^d$, and let $\mathbb{G}$ be the group of all scaling and orthogonal transformations of $\mathbb{R}^d$, that is,*

$$\mathbb{G} = \big\{x \mapsto \lambda g(x) : \lambda > 0,\ g \in \mathbb{SO}_d\big\},$$

where $\mathbb{SO}_d$ is the special orthogonal group of $\mathbb{R}^d$. Then $\operatorname{conv}_{K,\mathbb{H}}(A) = \operatorname{conv}(A)$ *for all bounded $A \subseteq \mathbb{R}^d$.*

Proof. It is clear that $\operatorname{conv}(A) \subseteq \operatorname{conv}_{K,\mathbb{H}}(A)$. In the following we prove the opposite inclusion. Since $A \subseteq g(K+x)$ if and only if $\operatorname{conv}(A) \subseteq g(K+x)$, we have $\operatorname{conv}_{K,\mathbb{H}}(A) = \operatorname{conv}_{K,\mathbb{H}}(\operatorname{conv}(A))$ and there is no loss of generality in assuming that A is compact and convex. Further, by passing to subspaces, it is possible to assume that K has a nonempty interior. Take a point $z \in \mathbb{R}^d \setminus A$. We need to show that there exists a pair $(x,g) \in \mathbb{R}^d \times \mathbb{G}$ (depending on z) such that $z \notin g(K+x)$ and $A \subseteq g(K+x)$. By the separating hyperplane theorem, see Theorem 1.3.4 in [14], there exists a hyperplane $H \subseteq \mathbb{R}^d$ such that $A \subseteq H^-$ and $z \in H^+$. If K is compact, Theorem 2.2.5 in [14] implies that the boundary of K contains at least one point at which the supporting cone is a closed half-space. This holds also for convex closed K, which is not necessarily bounded, by taking intersections of K with a growing family of closed Euclidean balls. Let $v \in \partial K$ be such a point. After applying appropriate translation x_0 and orthogonal transformation $g_0 \in \mathbb{SO}_d$, we may assume that the supporting cone $S(g_0(K+x_0), g_0(v+x_0))$ is the closure of H^-. Thus,

$$A \subseteq \bigcup_{\lambda>0} \lambda\big(g_0(K-v)\big) \quad \text{and} \quad z \notin \bigcup_{\lambda>0} \lambda\big(g_0(K-v)\big).$$

It remains to show that there exists a $\lambda_0 > 0$ such that $A \subseteq \lambda_0(g_0(K-v))$. Assume that for every $n \in \mathbb{N}$ there exists an $a_n \in A$ such that $a_n \notin n(g_0(K-v))$. Since A is compact, there is a subsequence (a_{n_j}) converging to $a \in A \subseteq H^-$, as $j \to \infty$. Thus, there exists a $\lambda_0 > 0$ such that a lies in the interior of $\lambda_0(g_0(K-v))$. Hence, $a_{n_j} \in \lambda_0(g_0(K-v))$ for all sufficiently large j, which is a contradiction. □

Remark 2.6. For unbounded sets A the claim of Proposition 2.5 is false in general. As an example, one can take $d=2$, A is a closed half-space and K is an acute closed wedge. Thus, $\operatorname{conv}_{K,\mathbb{H}}(A)=\mathbb{R}^2$, whereas $\operatorname{conv}(A)=A$.

Proposition 2.7. *Let $\mathbb{G}=\mathbb{GL}_{\mathrm{d}}$ be the general linear group, $\mathbb{T}=\{0\}$, and let $K=B_1$ be the unit ball in $\mathbb{R}^d$. Then, for arbitrary compact set $A\subseteq\mathbb{R}^d$, it holds $\operatorname{conv}_{K,\mathbb{H}}(A)=\operatorname{conv}(A\cup\check{A})$, where $\check{A}=\{-z: z\in A\}$.*

Proof. The images of the unit ball under the elements of $\mathbb{GL}_{\mathrm{d}}$ are all ellipsoids centered at 0. Since each of these ellipsoids is origin symmetric and convex, it is clear that $\operatorname{conv}_{K,\mathbb{H}}(A)\supseteq\operatorname{conv}(A\cup\check{A})$. Let us prove the converse inclusion. Since replacing A by the convex compact set $\operatorname{conv}(A\cup\check{A})$ does not change its $(K,\mathbb{H})$-hull, it suffices to assume that A is an origin symmetric convex compact set. Let us take some $z\notin A$. We need to construct an ellipsoid F centered at the origin and such that $A\subseteq F$, whereas $z\notin F$. By the separating hyperplane theorem, see Theorem 1.3.4 in [14], there exists an affine hyperplane $H\subseteq\mathbb{R}^d$ such that $A\subseteq H^-$ and $z\in H^+$, where $H^\pm$ are open half-spaces bounded by H. Let $x=(x_1,\dots,x_d)$ be the coordinate representation of a generic point in $\mathbb{R}^d$. After applying an orthogonal transformation, we may assume that the hyperplane H is $\{x_1=a\}$ for some $a>0$. Then, $A\subseteq\{|x_1|<a\}$, while $z\in\{x_1>a\}$. Hence, $A\subseteq\{x\in\mathbb{R}^d: |x_1|\leq a-\varepsilon, x_2^2+\cdots+x_d^2\leq R^2\}=: D$ for sufficiently small $\varepsilon>0$ and sufficiently large $R>0$. Clearly, there is an ellipsoid F centered at 0, containing D and contained in the strip $\{|x_1|<a\}$. By construction, we have $A\subseteq F$ and $z\notin F$, and the proof is complete. □

Our next example deals with conical hulls.

Proposition 2.8. *Let $\mathbb{T}=\{0\}$, $\mathbb{G}=\mathbb{SO}_{\mathrm{d}}$ be the special orthogonal group, and let K be the closed half-space in $\mathbb{R}^d$ such that $0\in\partial K$. If $A\subseteq K$, then*

$$\operatorname{conv}_{K,\mathbb{H}}(A)=\operatorname{cl}(\operatorname{pos}(A)),$$

where

$$\operatorname{pos}(A):=\Big\{\sum_{i=1}^m\alpha_i x_i:\ \alpha_i\geq 0,\ x_i\in A, m=1,2,\dots\Big\}$$

is the positive (or conical) hull of A.

Proof. By definition, $\operatorname{conv}_{K,\mathbb{H}}(A)$ is the intersection of all closed half-spaces which contain the origin on the boundary, because every such half-space is an image of K under some orthogonal transformation. Since $\operatorname{cl}(\operatorname{pos}(A))$ is the intersection of *all* convex closed cones which contain A, $\operatorname{cl}(\operatorname{pos}(A))\subseteq\operatorname{conv}_{K,\mathbb{H}}(A)$. On the other hand, every convex closed cone is the intersection of its supporting half-spaces, see Corollary 1.3.5 in [14]. Since all these supporting half-spaces contain the origin on the boundary, $\operatorname{cl}(\operatorname{pos}(A))$ is the intersection of *some* family of half-spaces containing the origin on the boundary, which means that $\operatorname{conv}_{K,\mathbb{H}}(A)\subseteq\operatorname{cl}(\operatorname{pos}(A))$. □

The next corollary establishes connections with a probabilistic model studied recently in [7]. We shall return to this model in Section 6. Let

$$B_1^+ := \{(x_1, x_2, \dots, x_d) : x_1^2 + \dots + x_d^2 \le 1, x_1 \ge 0\} \tag{2.3}$$

be the unit upper half-ball in $\mathbb{R}^d$, and let

$$\mathbb{S}_+^{d-1} := \{(x_1, x_2, \dots, x_d) : x_1^2 + \dots + x_d^2 = 1, x_1 \ge 0\}$$

be the unit upper $(d-1)$-dimensional half-sphere. Further, let $\pi : \mathbb{R}^d \setminus \{0\} \to \mathbb{S}^{d-1}$ be the mapping $\pi(x) = x/\|x\|$.

Corollary 2.9. *Let $K = B_1^+$, $\mathbb{G} = \mathbb{SO}_\mathrm{d}$ and $\mathbb{T} = \{0\}$. Then, for arbitrary $A \subseteq K$, it holds*

$$\operatorname{conv}_{K,\mathbb{H}}(A) = \operatorname{cl}(\operatorname{pos}(A)) \cap B_1. \tag{2.4}$$

Furthermore, if $A \ne \{0\}$, then

$$\operatorname{conv}_{K,\mathbb{H}}(A) \cap \mathbb{S}^{d-1} = \operatorname{cl}\big(\operatorname{pos}(A)\big) \cap \mathbb{S}^{d-1} = \operatorname{cl}\big(\operatorname{pos}(\pi(A \setminus \{0\}))\big) \cap \mathbb{S}^{d-1}, \tag{2.5}$$

which is the closed spherical hull of the set $\pi(A \setminus \{0\}) \subseteq \mathbb{S}^{d-1}$.

Proof. Note that $g(B_1^+) = g(H_0^+) \cap B_1$, for every $g \in \mathbb{SO}_\mathrm{d}$, where H_0^+ is defined by

$$H_0^+ = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_1 \ge 0\}. \tag{2.6}$$

Thus,

$$\begin{aligned}\operatorname{conv}_{K,\mathbb{H}}(A) &= \bigcap_{g \in \mathbb{SO}_\mathrm{d}, A \subseteq g(B_1^+)} g(B_1^+) \\ &= \bigcap_{g \in \mathbb{SO}_\mathrm{d}, A \subseteq g(H_0^+)} \big(g(H_0^+) \cap B_1\big) = \Bigg(\bigcap_{g \in \mathbb{SO}_\mathrm{d}, A \subseteq g(H_0^+)} g(H_0^+)\Bigg) \cap B_1,\end{aligned}$$

where we have used that $A \subseteq B_1$. By Proposition 2.8 the right-hand side is $\operatorname{cl}(\operatorname{pos}(A)) \cap B_1$. The first equation in (2.5) is a direct consequence of (2.4), while the second one follows from $\operatorname{pos}(A) = \operatorname{pos}(A \setminus \{0\}) = \operatorname{pos}(\pi(A \setminus \{0\}))$. ☐

3. $(K, \mathbb{H})$-HULLS OF RANDOM SAMPLES FROM K

From now on we additionally assume that $K \in \mathscr{K}^d_{(0)}$, that is, K is a convex compact set in $\mathbb{R}^d$ which contains the origin in the interior. Fix a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$. For $n \in \mathbb{N}$, let $\Xi_n := \{\xi_1, \xi_2, \dots, \xi_n\}$ be a sample of n independent copies of a random variable ξ uniformly distributed on K. Put

$$Q_n := \operatorname{conv}_{K,\mathbb{H}}(\Xi_n)$$

and

$$\mathbb{X}_{K,\mathbb{H}}(\Xi_n) := \{(x, g) \in \mathbb{H} : \Xi_n \subseteq g(K + x)\} = K \ominus_{K,\mathbb{H}} \Xi_n = K \ominus_{K,\mathbb{H}} Q_n, \tag{3.1}$$

where the last equality follows from Lemma 2.2.

We start with a simple lemma which shows that, for every $n \in \mathbb{N}$, $\mathbb{X}_{K,\mathbb{H}}(\Xi_n)$ is a random closed subset of $\mathbb{H}$ equipped with the relative topology induced by $\mathbb{R}^d \times \mathsf{M}_d$, see Appendix for the definition of a random closed set. Here and in what follows M_d denotes the set of $d \times d$ matrices with real entries. Note that Q_n is a.s. closed by definition as intersection of closed sets.

Lemma 3.1. *For all $n \in \mathbb{N}$, $\mathbb{X}_{K,\mathbb{H}}(\Xi_n)$ is a random closed set in $\mathbb{H}$.*

Proof. Let $\mathbb{X}_\xi := \{(x,g) \in \mathbb{H} : \xi \in g(K+x)\}$. For each compact set $L \subseteq \mathbb{H}$, we have

$$\big\{\omega \in \Omega :\ \mathbb{X}_{\xi(\omega)} \cap L \neq \varnothing\big\} = \big\{\omega \in \Omega :\ \xi(\omega) \in LK\big\}, \tag{3.2}$$

where $LK := \{g(z+x) : (x,g) \in L, z \in K\}$. Note that LK is a compact set, hence it is Borel, and the event on the right-hand side of (3.2) is measurable. Thus, in view of (3.2), $\mathbb{X}_\xi$ is a random closed set in the sense of Definition 1.1.1 in [12]. Hence,

$$\mathbb{X}_{K,\mathbb{H}}(\Xi_n) = \mathbb{X}_{\xi_1} \cap \cdots \cap \mathbb{X}_{\xi_n}$$

is also a random closed set, being a finite intersection of random closed sets, see Theorem 1.3.25 on [12]. $\square$

We are interested in the asymptotic properties of $\mathbb{X}_{K,\mathbb{H}}(\Xi_n)$, as $n \to \infty$. Note that the sequence of sets (Q_n) is increasing in n and, for every $n \in \mathbb{N}$, $P_n := \operatorname{conv}(\Xi_n) \subseteq Q_n$. Since P_n converges almost surely to K in the Hausdorff metric, as $n \to \infty$, the sequence (Q_n) also converges almost surely to K. Since the sequence of sets $(\mathbb{X}_{K,\mathbb{H}}(\Xi_n))$ is decreasing in n,

$$\mathbb{X}_{K,\mathbb{H}}(\Xi_n) \downarrow (K \ominus_{K,\mathbb{H}} K) = \big\{(x,g) \in \mathbb{H} : K \subseteq g(K+x)\big\} \quad \text{a.s. as} \quad n \to \infty. \tag{3.3}$$

Since we assume $(0,I) \in \mathbb{H}$, the set $K \ominus_{K,\mathbb{H}} K$ contains $(0,I)$. However, the set $K \ominus_{K,\mathbb{H}} K$ may contain other points, e.g., all $(0,g) \in \mathbb{H}$ such that $K \subseteq gK$.

It is natural to ask whether it is possible to renormalise, in an appropriate sense, the set $\mathbb{X}_{K,\mathbb{H}}(\Xi_n)$ such that it would converge to a random limit? Before giving a rigorous answer to this question we find it more instructive to explain our approach informally. While doing this, we shall also recollect necessary concepts, and introduce some further notation.

First of all, note that

$$\mathbb{X}_{K,\mathbb{H}}(\Xi_n) = \mathbb{X}_{K,\mathbb{R}^d \times \mathbb{GL}_d}(\Xi_n) \cap \mathbb{H} \quad \text{and} \quad K \ominus_{K,\mathbb{H}} K = (K \ominus_{K,\mathbb{R}^d \times \mathbb{GL}_d} K) \cap \mathbb{H}.$$

Thus, we can first focus on the special case $\mathbb{H} = \mathbb{R}^d \times \mathbb{GL}_d$ and then derive the corresponding result for arbitrary $\mathbb{H}$ by taking intersections. Denote

$$\mathbb{X}_n := \mathbb{X}_{K,\mathbb{R}^d \times \mathbb{GL}_d}(\Xi_n).$$

In order to quantify the convergence in (3.3) and derive a meaningful limit theorem for $\mathbb{X}_n$, we shall pass to tangent spaces. The vector space M_d is a tangent space to the Lie group $\mathbb{GL}_d$ at I and is, actually, the Lie algebra of $\mathbb{GL}_d$. However, for our purposes the multiplicative structure of the Lie algebra is not needed and we use only its linear structure as of a vector space over $\mathbb{R}$. Let

$$\exp : \mathsf{M}_d \to \mathbb{GL}_d$$

be the standard matrix exponent, and let $\mathbb{V}$ be a sufficiently small neighbourhood of I in $\mathbb{GL}_d$, where the exponent is bijective, see, for example, Theorem 2.8 in [4]. Finally, let $\log : \mathbb{V} \to \mathsf{M}_d$ be its inverse and define mappings $\widetilde{\log} : \mathbb{R}^d \times \mathbb{V} \to \mathbb{R}^d \times \log \mathbb{V}$ and $\widetilde{\exp} : \mathbb{R}^d \times \log \mathbb{V} \to \mathbb{R}^d \times \mathbb{V}$ by

$$\widetilde{\log}(x,g) = (x, \log g), \quad \widetilde{\exp}(x,h) = (x, \exp h), \quad x \in \mathbb{R}^d,\ g \in \mathbb{V},\ h \in \log \mathbb{V}.$$

Using the above notation, we can write

$$\begin{aligned}\widetilde{\log}(\mathbb{X}_n \cap (\mathbb{R}^d \times \mathbb{V})) &= \big\{(x,C) \in \mathbb{R}^d \times \log \mathbb{V} : \Xi_n \subseteq \exp(C)(K+x)\big\} \\ &= \big\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : \Xi_n \subseteq \exp(C)(K+x)\big\} \cap (\mathbb{R}^d \times \log \mathbb{V}) = \mathfrak{X}_n \cap (\mathbb{R}^d \times \log \mathbb{V}),\end{aligned}$$

where we set

$$\mathfrak{X}_n := \big\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : \Xi_n \subseteq \exp(C)(K+x)\big\}.$$

In the definition of $\mathfrak{X}_n$ the space $\mathbb{R}^d \times \mathsf{M}_d$ should be regarded as a tangent vector space at $(0,I)$ to the Lie group of all invertible affine transformations of $\mathbb{R}^d$. Similarly to Lemma 3.1, it is easy to see that $\mathfrak{X}_n$ is a random closed set in $\mathbb{R}^d \times \mathsf{M}_d$. Note that $\mathfrak{X}_n$ may be unbounded (in the product of the standard norm on $\mathbb{R}^d$ and some matrix norm on M_d) and, in general, is not convex.

We shall prove below, see Theorem 4.1, that the sequence $(n\mathfrak{X}_n)$ converges in distribution to a nondegenerate random set $\check{\mathfrak{Z}}_K = \{-z : z \in \mathfrak{Z}_K\}$ as random closed sets, see the Appendix for necessary formalities. We pass from the random set $\mathfrak{Z}_K$ defined at (4.1) to its reflected variant to simplify later notation. Moreover, for arbitrary convex compact subset $\mathfrak{K}$ in $\mathbb{R}^d \times \mathsf{M}_d$ which contains the origin, the sequence of random sets $(n\mathfrak{X}_n \cap \mathfrak{K})$ converges in distribution to $\check{\mathfrak{Z}}_K \cap \mathfrak{K}$ on the space of compact subsets of $\mathbb{R}^d \times \mathsf{M}_d$ endowed with the usual Hausdorff metric.

Since $\mathbb{V}$ contains the origin in its interior, there exists an $n_0 \in \mathbb{N}$ such that

$$n(\mathbb{R}^d \times \log \mathbb{V}) \supseteq \mathfrak{K} \quad \text{and} \quad \mathbb{R}^d \times \mathbb{V} \supseteq \widetilde{\exp}(\mathfrak{K}/n), \quad n \geq n_0. \tag{3.4}$$

Hence,

$$n\mathfrak{X}_n \cap \mathfrak{K} = n\big(\mathfrak{X}_n \cap (\mathbb{R}^d \times \log \mathbb{V})\big) \cap \mathfrak{K}, \quad n \geq n_0.$$

and, therefore, $n\widetilde{\log}(\mathbb{X}_n \cap (\mathbb{R}^d \times \mathbb{V})) \cap \mathfrak{K}$ converges in distribution to $\check{\mathfrak{Z}}_K \cap \mathfrak{K}$, as $n \to \infty$. In particular, the above arguments show that the limit does not depend on the choice of $\mathbb{V}$.

Let us now explain the case of arbitrary $\mathbb{H} \subseteq \mathbb{R}^d \times \mathbb{GL}_d$ containing $(0,I)$ and introduce assumptions that we shall impose on $\mathbb{H}$. Assume that there exist the following objects:

- a neighbourhood $\mathfrak{U} \subseteq \mathbb{R}^d \times \log \mathbb{V}$ of $(0,0)$ in $\mathbb{R}^d \times \mathsf{M}_d$;
- a neighbourhood $\mathbb{U} \subseteq \mathbb{R}^d \times \mathbb{V}$ of $(0,I)$ in $\mathbb{R}^d \times \mathbb{GL}_d$;
- a convex closed cone $\mathfrak{C}_{\mathbb{H}}$ in $\mathbb{R}^d \times \mathsf{M}_d$ with the apex at $(0,0)$;

such that

$$\mathbb{H}\cap\mathbb{U}=\widetilde{\exp}(\mathfrak{C}_{\mathbb{H}}\cap\mathfrak{U}). \tag{3.5}$$

Informally speaking, condition (3.5) means that locally around $(0,I)$ the set $\mathbb{H}$ is an image of a convex cone in the tangent space $\mathbb{R}^d\times\mathsf{M}_d$ under the extended exponential map $\widetilde{\exp}$. The most important particular cases arise when $\mathbb{H}$ is the product of a linear space $\mathbb{T}$ in $\mathbb{R}^d$ and a Lie subgroup $\mathbb{G}$ of $\mathbb{GL}_\mathrm{d}$. In this situation $\mathfrak{C}_{\mathbb{H}}$ is a linear subspace of $\mathbb{R}^d\times\mathsf{M}_d$ which is the direct sum of $\mathbb{T}$ and the Lie algebra $\mathfrak{G}$ of $\mathbb{G}$. Furthermore, $\mathfrak{C}_{\mathbb{H}}$ is a tangent space to $\mathbb{H}$ (regarded as a product of smooth manifolds) at $(0,I)$. In a more general class of examples, we allow $\mathfrak{C}_{\mathbb{H}}$ to be the direct sum of $\mathbb{T}$ and an arbitrary linear subspace of M_d, which is not necessarily a Lie algebra. In the latter case, the second component of $\mathbb{H}$ is not a Lie subgroup of $\mathbb{G}$. Furthermore, $\mathfrak{C}_{\mathbb{H}}$ can be a proper cone, that is, not a linear subspace, so the second components of $\mathbb{H}$ do not form a group as well. For example, assume that $d=2$ and $\mathbb{H}=\{0\}\times\mathbb{G}$, where

$$\mathbb{G}=\left\{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\lambda_1,\lambda_2\in(0,1]\right\}.$$

Then (3.5) holds for appropriate $\mathbb{U}$ and $\mathfrak{U}$ upon choosing

$$\mathfrak{C}_{\mathbb{H}}=\{0\}\times\left\{\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix},\mu_1,\mu_2\leq 0\right\}.$$

This example is important for the analysis of $(K,\mathbb{H})$-hulls because we naturally want to exclude transformations that enlarge K. Examples of a different kind, where $\mathbb{H}$ is not a Lie subgroup, arise by taking $\mathfrak{C}_{\mathbb{H}}$ to be an arbitrary linear subspace of $\mathbb{R}^d\times\mathsf{M}_d$ which is not a Lie subalgebra.

For an arbitrary convex compact subset $\mathfrak{K}$ of $\mathbb{R}^d\times\mathsf{M}_d$ which contains the origin, the set $\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}}$ is also convex compact and contains the origin. Furthermore, there exists an $n_0\in\mathbb{N}$ such that

$$\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}}\subseteq n(\mathfrak{C}_{\mathbb{H}}\cap\mathfrak{U})=n\widetilde{\log}(\mathbb{H}\cap\mathbb{U})\subseteq n\widetilde{\log}(\mathbb{H}\cap(\mathbb{R}^d\times\mathbb{V})),\quad n\geq n_0.$$

Since $\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}}$ is a convex compact set which contains the origin,

$$\begin{aligned}
&n\widetilde{\log}\Big(\mathbb{X}_{K,\mathbb{H}}(\Xi_n)\cap(\mathbb{R}^d\times\mathbb{V})\Big)\cap(\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}})\\
&\quad=n\widetilde{\log}\Big(\mathbb{X}_n\cap(\mathbb{H}\cap(\mathbb{R}^d\times\mathbb{V}))\Big)\cap(\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}})\\
&\quad=n\mathfrak{X}_n\cap n\widetilde{\log}(\mathbb{H}\cap(\mathbb{R}^d\times\mathbb{V}))\cap(\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}})=n\mathfrak{X}_n\cap(\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}})
\end{aligned}$$

converges to $\check{\mathfrak{Z}}_K\cap\mathfrak{K}\cap\mathfrak{C}_{\mathbb{H}}$, as $n\to\infty$. The limit here is also independent of $\mathbb{V}$.

Let us make a final remark in this informal discussion by connecting the convergence of the sequence $(n\widetilde{\log}\left(\mathbb{X}_n\cap(\mathbb{R}^d\times\mathbb{V})\right))$ and relation (3.3). The above argument demonstrates that $\check{\mathfrak{Z}}_K$ necessarily contains a nonrandom set

$$R_K:=\liminf_{n\to\infty}\Big(n\widetilde{\log}((K\ominus_{K,\mathbb{R}^d\times\mathbb{GL}_\mathrm{d}}K)\cap(\mathbb{R}^d\times\mathbb{V}))\Big)$$

$$= \bigcup_{k\geq 1} \bigcap_{n\geq k} \bigcap_{y\in K} \Big\{ (x,C) \in \mathbb{R}^d \times \mathsf{M}_d : y \in \exp(C/n)(K+x/n) \Big\}, \tag{3.6}$$

which is unbounded. As we shall show, the set R_K is, indeed, contained in the recession cone of $\check{\mathfrak{Z}}_K$ which we identify in Proposition 5.1 below.

4. Limit theorems for $\mathbb{X}_{K,\mathbb{H}}(\Xi_n)$

Recall that $N(K,x)$ denotes the normal cone to K at $x \in \partial K$, where $K \in \mathscr{K}^d_{(0)}$. By $\operatorname{Nor}(K)$ we denote the normal bundle, that is, a subset of $\partial K \times \mathbb{S}^{d-1}$, which is the family of $(x, N(K,x) \cap \mathbb{S}^{d-1})$ for $x \in \partial K$. It is known, see p. 84 in [14], that K has the unique outer unit normal $u_K(x)$ at $x \in \partial K$ for almost all points x with respect to the $(d-1)$-dimensional Hausdorff measure $\mathscr{H}^{d-1}$. Denote the set of such points by $\Sigma_1(K)$, so $\Sigma_1(K) := \{x \in \partial K : \dim N(K,x) = 1\}$.

Let $\Theta_{d-1}(K,\cdot)$ be the generalised curvature measure of K, see Section 4.2 in [14]. The following formula, which is a consequence of Theorem 3.2 in [5], can serve as a definition and is very convenient for practical purposes. If W is a Borel subset of $\mathbb{R}^d \times \mathbb{S}^{d-1}$, then

$$\Theta_{d-1}\big(K, (\partial K \times \mathbb{S}^{d-1}) \cap W\big) = \int_{\Sigma_1(K)} \mathbf{1}_{\{(x,u_K(x))\in W\}} \mathrm{d}\mathscr{H}^{d-1}(x).$$

In particular, this formula implies that the support of $\Theta_{d-1}(K,\cdot)$ is a subset of $\operatorname{Nor}(K)$ and its total mass is equal to the surface area of K.

Let $\mathscr{P}_K := \sum_{i\geq 1} \delta_{(t_i,\eta_i,u_i)}$ be the Poisson process on $(0,\infty) \times \operatorname{Nor}(K)$ with intensity measure μ being the product of Lebesgue measure on $(0,\infty)$ normalised by $V_d(K)^{-1}$ and the measure $\Theta_{d-1}(K,\cdot)$. If K is strictly convex, $\mathscr{P}_K$ can be equivalently defined as a Poisson process $\{(t_i, F(K,u_i), u_i), i \geq 1\}$, where $\{(t_i,u_i), i \geq 1\}$ is the Poisson process on $(0,\infty) \times \mathbb{S}^{d-1}$ with intensity being the product of the Lebesgue measure on the half-line normalised by $V_d(K)^{-1}$ and the surface area measure $S_{d-1}(K,\cdot) := \Theta_{d-1}(K, \mathbb{R}^d \times \cdot)$ of K.

The notion of convergence of random closed sets in distribution with respect to the Fell topology is recalled in the Appendix. For $A \in \mathbb{R}^d \times \mathsf{M}_d$, denote by

$$\check{A} := \big\{(-x,-C) : (x,C) \in A\big\}$$

the reflection of A with respect to the origin in $\mathbb{R}^d \times \mathsf{M}_d$.

Theorem 4.1. *Assume that $K \in \mathscr{K}^d_{(0)}$, and let $\mathfrak{F}$ be a convex closed set in $\mathbb{R}^d \times \mathsf{M}_d$ which contains the origin. The sequence of random closed sets $((n\mathfrak{X}_n) \cap \mathfrak{F})_{n\in\mathbb{N}}$ converges in distribution in the space of closed subsets of $\mathbb{R}^d \times \mathsf{M}_d$ endowed with the Fell topology to a random convex closed set $\check{\mathfrak{Z}}_K \cap \mathfrak{F}$, where*

$$\mathfrak{Z}_K := \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : \langle C\eta + x, u\rangle \leq t\Big\}. \tag{4.1}$$

Remark 4.2. While $\mathfrak{X}_n$ is not convex in general, the set $\check{\mathfrak{Z}}_K$ from (4.1) is almost surely convex as an intersection of convex sets.

Letting $\mathfrak{F} = \mathbb{R}^d \times \mathsf{M}_d$ in Theorem 4.1 shows that $n\mathfrak{X}_n$ converges in distribution to $\check{\mathfrak{Z}}_K$. If $\mathfrak{F} = \mathfrak{K}$ is a convex compact set which contains the origin in $\mathbb{R}^d \times \mathsf{M}_d$, the theorem covers the setting of Section 3. Taking into account the discussion there, we obtain the following.

Corollary 4.3. *Assume that $K \in \mathscr{K}^d_{(0)}$, and let $\mathbb{H}$ be a subset of $\mathbb{R}^d \times \mathbb{GL}_\mathrm{d}$ which satisfies* (3.5). *Then, the sequence of random closed sets*

$$n\widetilde{\log}\Big(\mathbb{X}_{K,\mathbb{H}}(\Xi_n) \cap (\mathbb{R}^d \times \mathbb{V})\Big) \cap \mathfrak{C}_{\mathbb{H}}, \quad n \in \mathbb{N}, \tag{4.2}$$

converges in distribution in the space of closed subsets of $\mathbb{R}^d \times \mathsf{M}_d$ endowed with the Fell topology to a random convex closed set $\check{\mathfrak{Z}}_K \cap \mathfrak{C}_{\mathbb{H}}$.

The subsequent proof of Theorem 4.1 heavily relies on a series of auxiliary results on the properties of the Fell topology and convergence of random closed sets, which are collected in the Appendix. We encourage the readers to acquaint themselves with the Appendix before proceeding further.

We start the proof of Theorem 4.1 with an auxiliary result which is an extension of Theorem 5.6 in [10]. Let $\mathscr{K}^d_0$ be the space of convex compact sets in $\mathbb{R}^d$ containing the origin and endowed with the Hausdorff metric. Let L^o denote the polar set to a convex closed set L, that is,

$$L^o := \{x \in \mathbb{R}^d : h(L,x) \le 1\}. \tag{4.3}$$

In what follows we shall frequently use the relation

$$[0,t^{-1}u]^o = H_u^-(t), \quad t > 0, \quad u \in \mathbb{S}^{d-1}, \tag{4.4}$$

where

$$H_u^-(t) := \{x \in \mathbb{R}^d : \langle x,u\rangle \le t\}, \quad t \in \mathbb{R}, \quad u \in \mathbb{S}^{d-1}.$$

From Theorem 5.6 in [10] we know that

$$\sum_{k=1}^n \delta_{n^{-1}(K-\xi_k)^o} \xrightarrow{\mathrm{d}} \sum_{(t,\eta,u)\in\mathscr{P}_K} \delta_{[0,t^{-1}u]} \quad \text{as} \quad n \to \infty,$$

where the convergence is understood as the convergence in distribution on the space of point measures on[1] $\mathscr{K}^d_0 \setminus \{0\}$ endowed with the vague topology. The limiting point process consists of random segments $[0,x]$ with $x = t^{-1}u$ derived from the first and third coordinates of $\mathscr{P}_K$. Regarding ξ_k as a mark of $n^{-1}(K-\xi_k)^o$ for $k = 1,\dots,n$ we have the following convergence of marked point processes, which strengthens the above mentioned result from [10].

Lemma 4.4. *Assume that $K \in \mathscr{K}^d_{(0)}$. Then*

$$\sum_{k=1}^n \delta_{(n^{-1}(K-\xi_k)^o,\xi_k)} \xrightarrow{\mathrm{d}} \sum_{(t,\eta,u)\in\mathscr{P}_K} \delta_{([0,t^{-1}u],\eta)} \quad \textit{as} \quad n \to \infty, \tag{4.5}$$

[1]It would be more precise to write $\mathscr{K}^d_0 \setminus \{\{0\}\}$ instead of $\mathscr{K}^d_0 \setminus \{0\}$, but we prefer the latter notation for the sake of notational simplicity.

where the convergence is understood as the convergence in distribution on the space of point measures on $(\mathscr{K}_0^d \setminus \{0\}) \times \mathbb{R}^d$ *endowed with the vague topology.*

Proof. Let $p(\partial K, \cdot)$ be the metric projection on K, that is, $p(\partial K, x)$ is the set of closest to x points on ∂K. We start by noting that for the limiting Poisson process the following equality holds for all $L \in \mathscr{K}_0^d \setminus \{0\}$ and every Borel $R \subseteq \mathbb{R}^d$

$$\begin{aligned}
\mathbf{P}\big\{[0,t^{-1}u] \subseteq L \text{ or } \eta \notin p(\partial K, R) \text{ for all } (t,\eta,u) \in \mathscr{P}_K\big\} \\
&= \exp\big(-\mu(\{(t,\eta,u) : [0,t^{-1}u] \not\subseteq L, \eta \in p(\partial K,R)\}\big) \\
&= \exp\big(-\mu(\{(t,\eta,u) : L^o \not\subseteq H_u^-(t), \eta \in p(\partial K,R)\}\big) \\
&= \exp\big(-\mu(\{(t,\eta,u) : h(L^o,u) > t, \eta \in p(\partial K,R)\}\big) \\
&= \exp\Big(-\frac{1}{V_d(K)}\int_{\mathrm{Nor}(K)} \mathbf{1}_{\{x \in p(\partial K,R)\}} h(L^o,u)\Theta_{d-1}(K, \mathrm{d}x \times \mathrm{d}u)\Big).
\end{aligned}$$

According to Proposition 7.9 in the Appendix, see, in particular, Eq. (7.15), we need to show that, for every $L \in \mathscr{K}_0^d \setminus \{0\}$ and every Borel $R \subseteq \mathbb{R}^d$, as $n \to \infty$,

$$n\mathbf{P}\{n^{-1}(K-\xi)^o \not\subseteq L, \xi \in R\} \longrightarrow \frac{1}{V_d(K)}\int_{\mathrm{Nor}(K)} \mathbf{1}_{\{x \in p(\partial K,R)\}} h(L^o,u)\Theta_{d-1}(K, \mathrm{d}x \times \mathrm{d}u). \tag{4.6}$$

Note that

$$\begin{aligned}
\mathbf{P}\{n^{-1}(K-\xi)^o \not\subseteq L, \xi \in R\} &= \mathbf{P}\{n^{-1}L^o \not\subseteq (K-\xi), \xi \in R\} \\
&= \mathbf{P}\{\xi \notin K \ominus n^{-1}L^o, \xi \in R\} = \frac{V_d(R \cap (K \setminus (K \ominus n^{-1}L^o)))}{V_d(K)}.
\end{aligned}$$

Applying Theorem 1 in [9] with $C = p(\partial K, R)$, $A = K$, $P = B = W = \{0\}$, $Q = -(L^o)$ and $\varepsilon = n^{-1}$, we obtain (4.6). The proof is complete. □

Applying continuous mapping theorem to convergence (4.5) and using Lemma 7.2(ii), we obtain the convergence of marked point processes

$$\sum_{k=1}^{n} \delta_{(n(K-\xi_k),\, \xi_k)} \xrightarrow{\mathrm{d}} \sum_{(t,\eta,u) \in \mathscr{P}_K} \delta_{(H_u^-(t),\, \eta)} \quad \text{as} \quad n \to \infty. \tag{4.7}$$

Proof of Theorem 4.1. According to Lemma 7.3 in the Appendix it suffices to show that $(n\mathfrak{X}_n) \cap \mathfrak{F} \cap \mathfrak{K}$ converges to $\check{\mathfrak{Z}}_K \cap \mathfrak{F} \cap \mathfrak{K}$ for arbitrary compact convex subset $\mathfrak{K}$ of $\mathbb{R}^d \times \mathsf{M}_d$, which contains the origin in its interior and then pass to the limit $\mathfrak{K} \uparrow (\mathbb{R}^d \times \mathsf{M}_d)$. It holds

$$\begin{aligned}
(n\mathfrak{X}_n) \cap \mathfrak{K} &= \Big\{(x,C) \in \mathfrak{K} : \Xi_n \subseteq \exp(C/n)(K + x/n)\Big\} \\
&= \bigcap_{k=1}^{n} \Big\{(x,C) \in \mathfrak{K} : \xi_k \in \exp(C/n)(K + x/n)\Big\} \\
&= \bigcap_{k=1}^{n} \Big\{(x,C) \in \mathfrak{K} : \exp(-C/n)\xi_k \in K + x/n\Big\}
\end{aligned}$$

$$= \bigcap_{k=1}^{n} \Big\{ (x,C) \in \mathfrak{K} : \big(n(\exp(-C/n) - I)\big)\xi_k - x \in n(K - \xi_k) \Big\}.$$

Let

$$a_m := \sup_{n \geq m} \sup_{(x,C)\in\mathfrak{K}, y\in K} \big\| \big(n(\exp(-C/n) - I\big)y + Cy \big\|, \quad m \in \mathbb{N}.$$

Note that $a_m \to 0$, as $m \to \infty$, because $n\big(\exp(-C/n) - I\big) \to -C$ locally uniformly in C, as $n \to \infty$, the set $\mathfrak{K}$ is compact in $\mathbb{R}^d \times \mathsf{M}_d$, and K is compact in $\mathbb{R}^d$.

Let B_{a_m} be the closed ball of radius a_m in $\mathbb{R}^d$ centred at the origin. For each $m \in \mathbb{N}$ and $n \geq m$, we have

$$\mathfrak{Y}^-_{m,n} \subseteq \big((n\mathfrak{X}_n) \cap \mathfrak{K}\big) \subseteq \mathfrak{Y}^+_{m,n}, \tag{4.8}$$

where

$$\mathfrak{Y}^+_{m,n} := \bigcap_{k=1}^{n} \big\{ (x,C) \in \mathfrak{K} : -C\xi_k - x \in n(K - \xi_k) + B_{a_m} \big\}$$

and

$$\mathfrak{Y}^-_{m,n} := \bigcap_{k=1}^{n} \big\{ (x,C) \in \mathfrak{K} : -C\xi_k - x + B_{a_m} \subseteq n(K - \xi_k) \big\}. \tag{4.9}$$

The advantage of lower and upper bounds in (4.8) is the convexity of $\mathfrak{Y}^{\pm}_{m,n}$, which makes their analysis simpler. We aim to apply Lemma 7.8 from the Appendix with $Y^{\pm}_{n,m} = \mathfrak{Y}^{\pm}_{m,n} \cap \mathfrak{F}$ and $X_n = (n\mathfrak{X}_x) \cap \mathfrak{K} \cap \mathfrak{F}$. Let us start with analysis of $\mathfrak{Y}^+_{m,n}$ which is simpler.

Let $\mathfrak{L}$ be a compact subset of $\mathfrak{K}$. Denote

$$\begin{aligned}
M^+_m(\mathfrak{L}) &:= \big\{ (L,y) \in \mathscr{K}^d_0 \times \mathbb{R}^d : -Cy - x \in L^o + B_{a_m} \text{ for all } (x,C) \in \mathfrak{L} \big\}, \\
M^-_m(\mathfrak{L}) &:= \big\{ (L,y) \in \mathscr{K}^d_0 \times \mathbb{R}^d : -Cy - x + B_{a_m} \subseteq L^o \text{ for all } (x,C) \in \mathfrak{L} \big\}, \\
M(\mathfrak{L}) &:= \big\{ (L,y) \in \mathscr{K}^d_0 \times \mathbb{R}^d : -Cy - x \in L^o \text{ for all } (x,C) \in \mathfrak{L} \big\}.
\end{aligned}$$

Then

$$\mathbf{P}\big\{ \mathfrak{L} \subseteq \mathfrak{Y}^{\pm}_{m,n} \big\} = \mathbf{P}\big\{ (n^{-1}(K - \xi_i)^o, \xi_i) \in M^{\pm}_m(\mathfrak{L}), i = 1, \ldots, n \big\}.$$

By Lemma 4.4, the point process $\{(n^{-1}(K - \xi_i)^o, \xi_i), i = 1, \ldots, n\}$ converges in distribution to the Poisson process $\big\{([0, t^{-1}u], \eta) : (t, \eta, u) \in \mathscr{P}_K\big\}$. The sets $M(\mathfrak{L})$, $M^{\pm}_m(\mathfrak{L})$ are continuity sets for the distribution of the limiting Poisson process. Indeed, for each $(t, \eta, u) \in (0, \infty) \times \mathrm{Nor}(K)$,

$$\begin{aligned}
&\Big\{ ([0, t^{-1}u], \eta) \in \partial M^+_m(\mathfrak{L}) \Big\} \\
&= \Big\{ -C\eta - x \in H^-_u(t + a_m) \text{ for all } (x,C) \in \mathfrak{L} \Big\} \setminus \Big\{ -C\eta - x \in \mathrm{Int}\, H^-_u(t + a_m) \text{ for all } (x,C) \in \mathfrak{L} \Big\} \\
&= \Big\{ \langle -C\eta - x, u \rangle \leq t + a_m \text{ for all } (x,C) \in \mathfrak{L} \Big\} \setminus \Big\{ \langle -C\eta - x, u \rangle < t + a_m \text{ for all } (x,C) \in \mathfrak{L} \Big\} \\
&= \Big\{ \langle -C\eta - x, u \rangle \leq t + a_m \text{ for all } (x,C) \in \mathfrak{L} \text{ and } \langle -C\eta - x, u \rangle = t + a_m \text{ for some } (x,C) \in \mathfrak{L} \Big\},
\end{aligned}$$

where Int denotes the topological interior. Since the probability of the latter event for some $(t, \eta, u) \in \mathscr{P}_K$ vanishes, it follows that $M^+_m(\mathfrak{L})$ is a continuity set for the Poisson point process

$\{([0,t^{-1}u],\eta):(t,\eta,u)\in\mathscr{P}_K\}$. Letting $a_m=0$, we obtain that $M(\mathfrak{L})$ is also a continuity set. The argument for $M_m^-(\mathfrak{L})$ is similar by replacing a_m with $(-a_m)$.

Thus, for all $m\in\mathbb{N}$,

$$\mathbf{P}\left\{\mathfrak{L}\subseteq\mathfrak{Y}_{m,n}^+\right\}\to\mathbf{P}\left\{\left\{([0,t^{-1}u],\eta):(t,\eta,u)\in\mathscr{P}_K\right\}\subseteq M_m^+(\mathfrak{L})\right\}=\mathbf{P}\left\{\mathfrak{L}\subseteq\mathfrak{Y}_m^+\right\}\quad\text{as}\quad n\to\infty,$$

where

$$\mathfrak{Y}_m^+:=\bigcap_{(t,\eta,u)\in\mathscr{P}_K}\Big\{(x,C)\in\mathfrak{K}:-C\eta-x\in H_u^-(t+a_m)\Big\}.$$

The random closed sets $\mathfrak{Y}_{m,n}^+$ and $\mathfrak{Y}_m^+$ are convex and almost surely contain a neighbourhood of the origin in $\mathbb{R}^d\times\mathsf{M}_d$, hence, are regular closed, see the Appendix for the definition. By Theorem 7.5 applied to the space $\mathbb{R}^d\times\mathsf{M}_d$, the random convex set $\mathfrak{Y}_{m,n}^+$ converges in distribution to $\mathfrak{Y}_m^+$. Since $\mathfrak{Y}_{m,n}^+\overset{\mathrm{d}}{\longrightarrow}\mathfrak{Y}_m^+$, as $n\to\infty$, and the involved sets almost surely contain the origin in their interiors, Corollary 7.7 yields that $(\mathfrak{Y}_{m,n}^+\cap\mathfrak{F})\overset{\mathrm{d}}{\longrightarrow}(\mathfrak{Y}_m^+\cap\mathfrak{F})$, as $n\to\infty$, for each convex closed set $\mathfrak{F}$ which contains the origin in $\mathbb{R}^d\times\mathsf{M}_d$ (not necessarily as an interior point). Thus, we have checked part (i) of Lemma 7.8.

We proceed with checking part (ii) of Lemma 7.8 with $Y_m^-=\mathfrak{Y}_m^-$, where

$$\mathfrak{Y}_m^-:=\bigcap_{(t,\eta,u)\in\mathscr{P}_K}\Big\{(x,C)\in\mathfrak{K}:-C\eta-x\in H_u^-(t-a_m)\Big\}.$$

Note that the random sets $\mathfrak{Y}_{m,n}^-$ and $\mathfrak{Y}_m^-$ may be empty and otherwise not necessarily contain the origin. We need to check (7.14), which in our case reads as follows

$$\mathbf{P}\left\{\mathfrak{Y}_{m,n}^-\cap\mathfrak{F}\cap\mathfrak{L}\neq\varnothing,0\in\mathfrak{Y}_{m,n}^-\right\}\to\mathbf{P}\left\{\mathfrak{Y}_m^-\cap\mathfrak{F}\cap\mathfrak{L}\neq\varnothing,0\in\mathfrak{Y}_m^-\right\}\quad\text{as}\quad n\to\infty,\tag{4.10}$$

for all compact sets $\mathfrak{L}$ which are continuity sets of $\mathfrak{Y}_m^-\cap\mathfrak{F}$. We shall prove (4.10) for all compact sets $\mathfrak{L}$ in $\mathbb{R}^d\times\mathsf{M}_d$. To this end, we shall employ Lemma 7.6 and divide the derivation (4.10) into several steps, each devoted to checking one condition of Lemma 7.6.

STEP 1. Let us check that, for sufficiently large $n\in\mathbb{N}$,

$$\mathbf{P}\left\{(0,0)\in\mathfrak{Y}_{m,n}^-\right\}=\mathbf{P}\left\{(0,0)\in\operatorname{Int}\mathfrak{Y}_{m,n}^-\right\}>0,\quad m\in\mathbb{N}.$$

Since the interior of a finite intersection is intersection of the interiors, and using independence, it suffices to check this for each of the set which appear in the intersection in (4.9). If $(0,0)$ belongs to $Y_k:=\left\{(x,C)\in\mathfrak{K}:-C\xi_k-x+B_{a_m}\subseteq n(K-\xi_k)\right\}$, then $B_{a_m}\subseteq n(K-\xi_k)$. Since ξ_k is uniform on K, we have

$$\mathbf{P}\{B_{a_m}\subseteq n(K-\xi_k)\}=\mathbf{P}\{B_{a_m}\subseteq n\operatorname{Int}(K-\xi_k)\}$$

for all n. If $B_{a_m}\subseteq n\operatorname{Int}(K-\xi_k)$, then $-C\xi_k-x+B_{a_m}\subseteq n(K-\xi_k)$ for all x and C from a sufficiently small neighbourhood of the origin in $\mathbb{R}^d\times\mathsf{M}_d$. Furthermore,

$$\mathbf{P}\left\{(0,0)\in\mathfrak{Y}_{m,n}^-\right\}=\mathbf{P}\{B_{a_m}\subseteq n(K-\xi_k),k=1,\dots,n\}=\mathbf{P}\left\{B_{a_m/n}\subseteq K\ominus\Xi_n\right\}>0$$

for all sufficiently large n.

STEP 2. Let us check that, for each $m \in \mathbb{N}$,

$$\mathbf{P}\{(0,0) \in \mathfrak{Y}_m^-\} = \mathbf{P}\{(0,0) \in \operatorname{Int}\mathfrak{Y}_m^-\} > 0.$$

The equality above follows from the observation that the origin lies on the boundary of $\{(x,C) \in \mathfrak{K} : -C\eta - x \in H_u^-(t-a_m)\}$ only if $t = a_m$, which happens with probability zero. Furthermore, $(0,0) \in \mathfrak{Y}_m^-$ if $t \geq a_m$ for all $(t,\eta,u) \in \mathscr{P}_K$, which has positive probability.

STEP 3. By a similar argument as we have used for $\mathfrak{Y}_{m,n}^+$, for every compact subset $\mathfrak{L}$ of $\mathfrak{K}$ and $m \in \mathbb{N}$, it holds

$$\mathbf{P}\{\mathfrak{L} \subseteq \mathfrak{Y}_{m,n}^-\} \to \mathbf{P}\{\{([0,t^{-1}u],\eta) : (t,\eta,u) \in \mathscr{P}_K\} \subseteq M_m^-(\mathfrak{L})\} = \mathbf{P}\{\mathfrak{L} \subseteq \mathfrak{Y}_m^-\} \quad \text{as} \quad n \to \infty.$$

Summarising, we have checked all conditions of Lemma 7.6. This finishes the proof of (4.10) and shows that all the condition of part (ii) of Lemma 7.8 hold. It remains to note that

$$(\mathfrak{Y}_m^+ \cap \mathfrak{F}) \downarrow (\check{\mathfrak{Z}}_K \cap \mathfrak{K} \cap \mathfrak{F}), \quad (\mathfrak{Y}_m^- \cap \mathfrak{F}) \uparrow (\check{\mathfrak{Z}}_K \cap \mathfrak{K} \cap \mathfrak{F}) \quad \text{a.s. as} \quad m \to \infty,$$

in the Fell topology, and

$$\lim_{m\to\infty} \mathbf{P}\{0 \in \mathfrak{Y}_m^-\} = 1.$$

Thus, by Lemma 7.8 $(n\mathfrak{X}_n) \cap \mathfrak{K} \cap \mathfrak{F}$ converges in distribution to $\check{\mathfrak{Z}}_K \cap \mathfrak{K} \cap \mathfrak{F}$, as $n \to \infty$. By Lemma 7.3, $(n\mathfrak{X}_n) \cap \mathfrak{F}$ converges in distribution to $\check{\mathfrak{Z}}_K \cap \mathfrak{F}$. □

5. PROPERTIES OF THE SET $\mathfrak{Z}_K$

5.1. **Boundedness and the recession cone.** The random set $\mathfrak{Z}_K$ is a subset of the product space $\mathbb{R}^d \times \mathsf{M}_d$. The latter space can be turned into the real Euclidean vector space with the inner product given by

$$\langle (x,C_1),(y,C_2)\rangle_1 := \langle x,y\rangle + \operatorname{Tr}(C_1C_2^\top), \quad x,y \in \mathbb{R}^d, \quad C_1,C_2 \in \mathsf{M}_d,$$

where Tr denotes the trace of a square matrix and $A^\top$ is the transpose of $A \in \mathsf{M}_d$. In terms of this inner product the set $\mathfrak{Z}_K$ can be written as

$$\mathfrak{Z}_K = \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : \langle (x,C),(u,\eta\otimes u)\rangle_1 \leq t\} = \bigcap_{(t,\eta,u)\in\mathscr{P}_K} H^-_{(u,\eta\otimes u)}(t), \qquad (5.1)$$

where $H^-_{(u,\eta\otimes u)}(t)$ is a closed half-space of $\mathbb{R}^d \times \mathsf{M}_d$ containing the origin, and $\eta \otimes u$ is the tensor product of η and u. The boundaries of $H^-_{(u,\eta\otimes u)}(t)$, $(t,\eta,u) \in \mathscr{P}_K$, constitute a Poisson process on the affine Grassmannian of hyperplanes in $\mathbb{R}^d \times \mathsf{M}_d$ called a Poisson hyperplane tessellation. The random set obtained as the intersection of the half-spaces $H^-_{(u,\eta\otimes u)}(t)$, $(t,\eta,u) \in \mathscr{P}_K$, is called the zero cell, see Section 10.3 in [15]. The intensity measure of this tessellation is the measure on the affine Grassmannian obtained as the product of the Lebesgue measure on $\mathbb{R}_+$ (normalised by $V_d(K)$) and the measure ν_K obtained as the push-forward of the generalised curvature measure $\Theta_{d-1}(K,\cdot)$ under the map $\operatorname{Nor}(K) \ni (x,u) \mapsto (u, x\otimes u) \in \mathbb{R}^d \times \mathsf{M}_d$. If, for example, $K = B_1$ is the unit ball, ν_K is the push-forward of the $(d-1)$-dimensional Hausdorff measure on the unit sphere $\mathbb{S}^{d-1}$ by the map $u \mapsto (u, u\otimes u)$. For a strictly convex and smooth body K, the positive cone generated by $\{x\otimes u : (x,u) \in \mathrm{Nor}(K)\}$ is called the normal bundle cone of K, see [3].

Note that the set $\mathfrak{Z}_K$ is almost surely convex, closed and unbounded. Thus, it is natural to consider the recession cone of $\mathfrak{Z}_K$ which is formally defined as

$$\mathrm{rec}(\mathfrak{Z}_K) := \left\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : (x,C) + \mathfrak{Z}_K \subseteq \mathfrak{Z}_K\right\}.$$

For instance, since $\mathfrak{Z}_K$ always contains $(0,-\lambda I)$ with $\lambda \le 0$, the recession cone also contains $\{(0,-\lambda I) : \lambda \le 0\}$.

Proposition 5.1. *The set R_K defined at* (3.6) *is contained in the following set*

$$T_K := \bigcap_{y\in K} \left\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : -Cy - x \in S(K,y)\right\}, \tag{5.2}$$

which is a convex closed cone in $\mathbb{R}^d \times \mathsf{M}_d$. Furthermore, with probability one

$$\check{T}_K \subseteq \mathrm{rec}(\mathfrak{Z}_K). \tag{5.3}$$

Moreover, if K is smooth, then $\check{T}_K = \mathrm{rec}(\mathfrak{Z}_K)$, and, with $\mathbb{H}$ satisfying (3.5)*,*

$$\mathrm{rec}(\mathfrak{Z}_K \cap \mathfrak{C}_{\mathbb{H}}) = \check{T}_K \cap \mathfrak{C}_{\mathbb{H}}. \tag{5.4}$$

In particular, the limit $\check{\mathfrak{Z}}_K \cap \mathfrak{C}_{\mathbb{H}}$ of $(n\widetilde{\log}\left(\mathbb{X}_{K,\mathbb{H}}(\Xi_n) \cap (\mathbb{R}^d \times \mathbb{V})\right)) \cap \mathfrak{C}_{\mathbb{H}}$ is a random compact set with probability one if and only if $\check{T}_K \cap \mathfrak{C}_{\mathbb{H}} = \{(0,0)\}$.

Proof. It is clear that $R_K \subseteq \cap_{y\in K} R_{K,y}$, where

$$R_{K,y} := \bigcup_{k\ge 1} \bigcap_{n\ge k} \left\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : y \in \exp(C/n)(K + x/n)\right\}.$$

A pair $(x,C) \in \mathbb{R}^d \times \mathsf{M}_d$ lies in $R_{K,y}$ if and only if there exists a $k \in \mathbb{N}$ such that $\exp(-C/n)y - x/n \in K$ for all $n \ge k$, equivalently,

$$n(\exp(-C/n) - I)y - x \in n(K - y) \quad \text{for all} \quad n \ge k.$$

Sending $n \to \infty$ and using that $\limsup_{n\to\infty} n(K-y) = S(K,y)$ show that $-Cy - x \in S(K,y)$, which means that $(x,C) \in T_{K,y}$. Thus,

$$R_{K,y} \subseteq \left\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : -Cy - x \in S(K,y)\right\} =: T_{K,y},$$

so that $R_K \subseteq T_K$. Since $T_{K,y}$ is a convex closed cone for all $y \in K$, the set T_K is a convex closed cone as well.

In order to check (5.3) note that $(N(K,y))^o = S(K,y)$. Hence, $-Cy - x \in S(K,y)$ if and only if $\langle Cy + x, u\rangle \ge 0$ for all $u \in N(K,y)$. Therefore,

$$T_K = \bigcap_{y\in K} \bigcap_{u\in N(K,y)} \left\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : \langle Cy + x, u\rangle \ge 0\right\}$$

$$= \bigcap_{(y,u)\in \mathrm{Nor}(K)} \Big\{(x,C)\in\mathbb{R}^d\times \mathsf{M}_d : \langle Cy+x,u\rangle \geq 0\Big\},$$

where we have used that $N(K,y)=\{0\}$ if $y\in \mathrm{Int}\,K$.

It follows from well-known results on recession cones, see p. 62 in [13], that

$$\mathrm{rec}(\mathfrak{Z}_K) = \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,C)\in\mathbb{R}^d\times \mathsf{M}_d : \langle (x,C),(u,\eta\otimes u)\rangle_1 \leq 0\Big\}$$
$$= \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,C)\in\mathbb{R}^d\times \mathsf{M}_d : \langle C\eta+x,u\rangle \leq 0\Big\}.$$

This immediately yields that $\check{T}_K \subseteq \mathrm{rec}(\mathfrak{Z}_K)$. To see the converse inclusion for smooth K note that the set

$$\{(\eta,u)\in \mathrm{Nor}(K) : (t,\eta,u)\in\mathscr{P}_K \text{ for some } t>0\}$$

is a.s. dense in $\mathrm{Nor}(K)=\{(x,u_K(x)) : x\in\partial K\}$, where $u_K(x)$ is the unique unit outer normal to K at x, see Lemma 4.2.2 and Theorem 4.5.1 in [14]. Thus, with probability one, for every $(x,C)\in\mathrm{rec}(\mathfrak{Z}_K)$ and $(y,u)\in\mathrm{Nor}(K)$ there exists a sequence (η_n,u_n) such that $(\eta_n,u_n)\to(y,u)$, as $n\to\infty$, and $\langle C\eta_n+x,u_n\rangle\leq 0$. Thus, $\langle Cy+x,u\rangle \leq 0$ and $(x,C)\in\check{T}_K$. Finally, relation (5.4) follows from Corollary 8.3.3 in [13] since

$$\mathrm{rec}(\mathfrak{Z}_k\cap\mathfrak{C}_{\mathbb{H}}) = \mathrm{rec}(\mathfrak{Z}_k)\cap\mathrm{rec}(\mathfrak{C}_{\mathbb{H}}) = \mathrm{rec}(\mathfrak{Z}_k)\cap\mathfrak{C}_{\mathbb{H}}. \qquad \square$$

Further information on the properties of $\mathfrak{Z}_K$ is encoded in its polar set which takes the following rather simple form

$$\mathfrak{Z}_K^o = \mathrm{conv}\Big(\bigcup_{(t,\eta,u)\in\mathscr{P}_K} [0,t^{-1}(u,(\eta\otimes u))]\Big),$$

as easily follows from (4.4). Since $\mathfrak{Z}_K$ a.s. contains the origin in the interior, $\mathfrak{Z}_K^o$ is a.s. compact. Note that $\mathfrak{Z}_K^o$ is a subset of the Cartesian product of $\mathbb{R}^d$ and Gruber's normal bundle cone, see [3]. The projection of $\mathfrak{Z}_K^o$ on the first factor $\mathbb{R}^d$ is a random polytope with probability one, which was recently studied in [10], see Section 5.1 therein.

5.2. **Affine transformations of** K**.** Let us now derive various properties of $\mathfrak{Z}_K$ with respect to transformations of K. First of all, it is easy to see that $\mathfrak{Z}_{rK}$ coincides in distribution with $r^{-1}\mathfrak{Z}_K$, for every fixed $r>0$. Let $A\in\mathbb{O}_{\mathrm{d}}$ be a fixed orthogonal matrix. Note that the point process $\mathscr{P}_{AK}$ has the same distribution as the image of $\mathscr{P}_K$ under the map $(t,\eta,u)\mapsto(t,A\eta,Au)$. Then, with $\overset{\mathrm{d}}{=}$ denoting equality of distributions,

$$\mathfrak{Z}_{AK} \overset{\mathrm{d}}{=} \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,C)\in\mathbb{R}^d\times \mathsf{M}_d : \langle CA\eta+x,Au\rangle \leq t\Big\}$$
$$= \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,C)\in\mathbb{R}^d\times \mathsf{M}_d : \langle A^\top CA\eta+A^\top x,u\rangle \leq t\Big\}$$

$$= \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(Ay, ABA^\top) \in \mathbb{R}^d \times \mathsf{M}_d : \langle B\eta + y, u\rangle \le t\Big\},$$

so that $\mathfrak{Z}_{AK}$ has the same distribution as $\mathfrak{Z}_K$ transformed using the map $\mathscr{O}_A : (x,C) \to (Ax, ACA^\top)$, which is orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_1$ in $\mathbb{R}^d \times \mathsf{M}_d$. In particular, if K is invariant under A, then the distribution of $\mathfrak{Z}_K$ is invariant under $\mathscr{O}_A$. Most importantly, if K is a ball, then the distribution of $\mathfrak{Z}_K$ is invariant under $\mathscr{O}_A$ for any $A \in \mathbb{O}_\mathrm{d}$.

If K is translated by $v \in \mathbb{R}^d$, then

$$\begin{aligned}\mathfrak{Z}_{K+v} &\overset{\mathrm{d}}{=} \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : \langle C\eta, u\rangle + \langle Cv, u\rangle + \langle x, u\rangle \le t\Big\} \\ &= \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x - Cv, C) \in \mathbb{R}^d \times \mathsf{M}_d : \langle C\eta, u\rangle + \langle x, u\rangle \le t\Big\},\end{aligned}$$

meaning that $\mathfrak{Z}_{K+v}$ is the image of $\mathfrak{Z}_K$ under the linear operator $(x,C) \mapsto (x - Cv, C)$.

6. Examples

Throughout this section $A = \Xi_n$ is a sample from the uniform distribution on K. If $\mathbb{G}$ consists of the unit matrix and $\mathbb{T} = \mathbb{R}^d$, then Theorem 4.1 turns into Theorem 5.1 of [10]. Another object which has been recently treated in the literature is given in Example 6.6 below.

Consider further examples involving nontrivial matrix groups.

Example 6.1 (General linear group). Let $\mathbb{G}$ be the general linear group, so that $\mathfrak{G}$ is the family M_d. If $\mathbb{T} = \mathbb{R}^d$, Proposition 2.5 shows that $Q_n = \operatorname{conv}(\Xi_n)$ for every choice of $K \in \mathscr{K}^d_{(0)}$.

Assume that $\mathbb{T} = \{0\}$ and let K be the unit ball B_1. Then Q_n is strictly larger than $\operatorname{conv}(\Xi_n)$ with probability 1. Indeed, it is clear that $Q_n \supseteq \operatorname{conv}(\Xi_n)$, and the inclusion is strict because the set Q_n is symmetric with respect to the origin, while the set $\operatorname{conv}(\Xi_n)$ is almost surely not. Choose $\mathfrak{C}_\mathbb{H} = \{0\} \times \mathsf{M}_d$. Then

$$\mathfrak{Z}_{B_1} \cap (\{0\} \times \mathsf{M}_d) = \{0\} \times \bigcap_{(t,u)\in\mathscr{P}} \{C \in \mathsf{M}_d : \langle Cu, u\rangle \le t\},$$

where $\mathscr{P}$ is the Poisson process on $\mathbb{R}_+ \times \mathbb{S}^{d-1}$ with intensity being the product of the Lebesgue measure multiplied by d and the uniform probability measure on $\mathbb{S}^{d-1}$. The factor d results from taking the ratio of the surface area of the unit sphere and the volume of the unit ball.

By Proposition 5.1, since $S(B_1, y) = H_y^-(0) = \{x \in \mathbb{R}^d : \langle x, y\rangle \le 0\}$

$$\operatorname{rec}(\mathfrak{Z}_{B_1} \cap (\{0\} \times \mathsf{M}_d)) = \{0\} \times \bigcap_{y\in B_1} \{C \in \mathsf{M}_d : \langle Cy, y\rangle \le 0\}.$$

Thus,

$$\operatorname{rec}\big(\check{\mathfrak{Z}}_{B_1} \cap (\{0\} \times \mathsf{M}_d)\big) = \{0\} \times \{C \in \mathsf{M}_d : C \text{ is positive semi-definite}\}.$$

In particular, the second factor contains the subspace $\mathsf{M}_d^{\mathrm{SSym}}$ of all skew-symmetric matrices, as well as all real symmetric matrices with nonnegative eigenvalues. The former reflects the fact that

B_1 is invariant with respect to $\mathbb{O}_{\mathrm{d}}$ for which $\mathsf{M}_d^{\mathrm{SSym}}$ is the Lie algebra. The latter is a consequence of the fact that $\mathbb{X}_{B_1,\mathbb{GL}_{\mathrm{d}}}(\Xi_n)$ contains all scalings with scaling factors (possibly different along pairwise orthogonal directions) larger or equal than 1.

Example 6.2 (Special linear group). Let $\mathbb{G}$ be the special linear group $\mathbb{SL}_d$, which consists of all $d\times d$ real-valued matrices with determinant one and assume again that $\mathbb{T}=\{0\}$. The elements of the corresponding Lie algebra $\mathfrak{G}=\{C\in\mathsf{M}_d : \operatorname{Tr} C=0\}$ are matrices with zero trace. Thus, we can set $\mathfrak{C}_{\mathbb{H}}=\{0\}\times\mathfrak{G}$. If $K=B_1$, then

$$\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})=\{0\}\times\bigcap_{(t,u)\in\mathscr{P}}\big\{C\in\mathfrak{G}:\langle Cu,u\rangle\le t\big\}.$$

By Proposition 5.1

$$\begin{aligned}\mathfrak{R}:=\operatorname{rec}\big(\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})\big)&=\{0\}\times\bigcap_{y\in B_1}\big\{C\in\mathfrak{G}:\langle Cy,y\rangle\le 0\big\}\\&=\{0\}\times\bigcap_{y\in B_1}\big\{C\in\mathsf{M}_d:\operatorname{Tr} C=0,\langle Cy,y\rangle\le 0\big\}.\end{aligned}$$

The intersection of $\mathfrak{R}$ and $\check{\mathfrak{R}}$ is called the lineality space of $\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})$; it consists of all vectors that are parallel to a line contained in $\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})$, see p. 16 in [14]. Clearly, the lineality space of $\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})$ is a.s. equal to

$$\mathfrak{R}\cap\check{\mathfrak{R}}=\{0\}\times\bigcap_{y\in B_1}\big\{C\in\mathsf{M}_d:\operatorname{Tr} C=0,\langle Cy,y\rangle=0\big\}=\{0\}\times\mathsf{M}_d^{\mathrm{SSym}}.$$

The vector space of square matrices M_d is the direct sum of the vector spaces of symmetric and skew-symmetric matrices:

$$\mathsf{M}_d=\mathsf{M}_d^{\mathrm{Sym}}\oplus\mathsf{M}_d^{\mathrm{SSym}}.$$

Furthermore, with respect to the inner product $\langle A,B\rangle:=\operatorname{Tr}(AB^\top)$ this direct sum decomposition is orthogonal. Similarly, the space $\mathfrak{G}$ is a direct sum of two vector spaces $\mathsf{M}_d^{\mathrm{SSym}}$ and $\mathfrak{G}_+$, where $\mathfrak{G}_+:=\{C\in\mathsf{M}_d^{\mathrm{Sym}}:\operatorname{Tr} C=0\}$. By Lemma 1.4.2 in [14] we a.s. have the orthogonal decomposition

$$\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})=\{0\}\times\Big(\mathsf{M}_d^{\mathrm{SSym}}\oplus\big(\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})\big)_+\Big),$$

where

$$\big(\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})\big)_+:=\bigcap_{(t,u)\in\mathscr{P}}\big\{C\in\mathsf{M}_d^{\mathrm{Sym}}:\operatorname{Tr} C=0,\langle Cu,u\rangle\le t\big\}.$$

If a matrix $C\in\mathfrak{G}_+$ does not vanish, then at least one of its eigenvalues is strictly positive (because all eigenvalues are real by symmetry and their sum is 0). If we denote by v the corresponding unit eigenvector, then $\langle Cv,v\rangle>0$. Since the set of u_i's for which $(t_i,u_i)\in\mathscr{P}$ is a.s. dense on the unit sphere in $\mathbb{R}^d$, it follows that $\langle Cu_i,u_i\rangle>0$ for some i. Thus, $sC\notin\big(\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})\big)_+$ if $s>0$ is sufficiently large. Therefore, the convex set $\big(\mathfrak{Z}_{B_1}\cap(\{0\}\times\mathfrak{G})\big)_+$ is a.s. bounded, hence, is a compact subset of $\mathsf{M}_d^{\mathrm{Sym}}$.

As in the previous example, the unbounded component $\{0\} \times \mathsf{M}_d^{\mathrm{SSym}}$ is present in $\mathfrak{Z}_{B_1} \cap (\{0\} \times \mathfrak{G})$ due to the fact that B_1 is invariant with respect to the orthogonal group $\mathbb{O}_{\mathrm{d}}$ which is a Lie subgroup of $\mathbb{SL}_d$. Since arbitrarily large scalings are not allowed in $\mathbb{SL}_d$, the random closed set $\mathfrak{Z}_{B_1} \cap (\{0\} \times \mathfrak{G})$ is a.s. bounded on the complement to $\{0\} \times \mathsf{M}_d^{\mathrm{SSym}}$.

Example 6.3. Let $\mathbb{G} = \mathbb{O}_{\mathrm{d}}$ be the orthogonal group. As has already been mentioned, the corresponding Lie algebra $\mathfrak{G} = \mathsf{M}_d^{\mathrm{SSym}}$ is the $d(d-1)/2$-dimensional subspace of M_d, consisting of all skew symmetric matrices. If $\mathbb{T} = \mathbb{R}^d$, then $\mathfrak{C}_{\mathbb{H}} = \mathbb{R}^d \times \mathfrak{G}$ and

$$\mathfrak{Z}_K \cap (\mathbb{R}^d \times \mathfrak{G}) = \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d^{\mathrm{SSym}} : \langle C\eta, u\rangle + \langle x,u\rangle \le t\Big\}.$$

In the special case $d = 2$ the Lie algebra $\mathfrak{G}$ is one-dimensional and is represented by the matrices

$$C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \quad c \in \mathbb{R}.$$

Write $\eta := (\eta', \eta'')$ and $u := (u', u'')$. Then, with $\cong$ denoting the isomorphism of $\mathbb{R}^2 \times \mathsf{M}_2^{\mathrm{SSym}}$ and $\mathbb{R}^2 \times \mathbb{R}$, we can write

$$\begin{aligned}\mathfrak{Z}_K \cap \big(\mathbb{R}^2 \times \mathsf{M}_2^{\mathrm{SSym}}\big) &\cong \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,c) \in \mathbb{R}^2 \times \mathbb{R} : c(\eta''u' - \eta'u'') + \langle x,u\rangle \le t\Big\} \\ &= \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \Big\{(x,c) \in \mathbb{R}^2 \times \mathbb{R} : \langle x,u\rangle \le t - c[u,\eta]\Big\},\end{aligned}$$

where $[u,\eta]$ is the (signed) area of the parallelepiped spanned by u and η. Therefore, $\mathfrak{Z}_K \cap \big(\mathbb{R}^2 \times \mathsf{M}_2^{\mathrm{SSym}}\big)$ is (isomorphic to) the zero cell of a hyperplane tessellation $H_{(u,[u,\eta])}(t)$, $(t,\eta,u) \in \mathscr{P}_K$ in $\mathbb{R}^2 \times \mathbb{R}$.

Let $K = [-1,1]^2$ be a square in $\mathbb{R}^2$. Then $\Theta_{d-1}(K,\cdot)$ is the sum of four terms, being products of the 1-dimensional Hausdorff measures supported by the sides of K and the Dirac measure at unit normal vectors $u_1,\dots,u_4$ to these sides. The push-forward of each of these four measures is the product of the Lebesgue measure on $[-1,1]$ and the Dirac measure at $u_1,\dots,u_4$. Hence,

$$\mathfrak{Z}_K \cap (\mathbb{R}^d \times \mathfrak{G}) \cong \bigcap_{i=1}^{4} \bigcap_{(t,y)\in\mathscr{P}_i} \Big\{(x,c) \in \mathbb{R}^2 \times \mathbb{R} : cy + \langle x,u_i\rangle \le t\Big\},$$

where $\mathscr{P}_1,\dots,\mathscr{P}_4$ are independent homogeneous Poisson processes on $\mathbb{R}_+ \times [-1,1]$ of intensity $1/4$ (recall that the area of K is 4).

In the special case $\mathbb{T} = \{0\}$ we can set $x = 0$, so that

$$\mathfrak{Z}_K \cap (\{0\} \times \mathsf{M}_2^{\mathrm{SSym}}) \cong \{0\} \times \bigcap_{i=1}^{4} \bigcap_{(t,y)\in\mathscr{P}_i} \big\{c \in \mathbb{R} : cy \le t\big\}.$$

An easy calculation (based on considering separately $\mathscr{P}_i$ restricted on $[0,1]\times\mathbb{R}$ and $[-1,0]\times\mathbb{R}$) shows that the double intersection above is a segment $[-\zeta',\zeta'']$, where ζ' and ζ'' are two exponentially distributed random variables of mean one.

Example 6.4 (Scaling by constants). Let $\mathbb{H}$ be the product of $\mathbb{T}=\mathbb{R}^d$ and the family $\mathbb{G}=\{e^r I : r\in\mathbb{R}\}$ of scaling transformations, so that $\mathfrak{C}_{\mathbb{H}}=\mathbb{R}^d\times\{rI : r\in\mathbb{R}\}$. Then, with $\cong$ denoting the natural isomorphism between $\mathfrak{C}_{\mathbb{H}}$ and $\mathbb{R}^d\times\mathbb{R}$,

$$\begin{aligned}
\mathfrak{Z}_K\cap\mathfrak{C}_{\mathbb{H}} &\cong \bigcap_{(t,\eta,u)\in\mathscr{P}_K}\Big\{(x,r)\in\mathbb{R}^d\times\mathbb{R} : r\langle\eta,u\rangle+\langle x,u\rangle\le t\Big\}\\
&= \bigcap_{(t,\eta,u)\in\mathscr{P}_K}\Big\{(x,r)\in\mathbb{R}^d\times\mathbb{R} : rh(K,u)+\langle x,u\rangle\le t\Big\}\\
&= \bigcap_{(t,\eta,u)\in\mathscr{P}_K}\Big\{(x,r)\in\mathbb{R}^d\times\mathbb{R} : h(rK+x,t^{-1}u)\le 1\Big\}\\
&= \bigcap_{(t,\eta,u)\in\mathscr{P}_K}\Big\{(x,r)\in\mathbb{R}^d\times\mathbb{R} : t^{-1}u\in(rK+x)^o\Big\}\\
&= \Big\{(x,r)\in\mathbb{R}^d\times\mathbb{R} : \operatorname{conv}\left(\{t^{-1}u : (t,\eta,u)\in\mathscr{P}_K\}\right)\subseteq(rK+x)^o\Big\}.
\end{aligned}$$

The set $\operatorname{conv}\left(\{t^{-1}u : (t,\eta,u)\in\mathscr{P}_K\}\right)$ has been studied in [10], in particular, the polar set to this hull is the zero cell Z_K of the Poisson hyperplane tessellation in $\mathbb{R}^d$, whose intensity measure is the product of the Lebesgue measure (scaled by $V_d^{-1}(K)$) and the surface area measure $S_{d-1}(K,\cdot)=\Theta_{d-1}(K,\mathbb{R}^d\times\cdot)$ of K. Thus, we can write

$$\mathfrak{Z}_K\cap\mathfrak{C}_{\mathbb{H}}\cong\{(x,r)\in\mathbb{R}^d\times\mathbb{R} : rK+x\subseteq Z_K\}. \tag{6.1}$$

If K is the unit Euclidean ball B_1, then (6.1) can be recast as

$$\mathfrak{Z}_{B_1}\cap\mathfrak{C}_{\mathbb{H}}\cong\Big\{(x,r)\in\mathbb{R}^d\times\mathbb{R} : x\in\cap_{i\ge1}H^-_{u_i}(t_i-r)\Big\},$$

where $\{H_{u_i}(t_i), i\ge1\}$ is a stationary Poisson hyperplane tessellation in $\mathbb{R}^d$. For every $r_0\ge0$, the section of $\mathfrak{Z}_{B_1}\cap\mathfrak{C}_{\mathbb{H}}$ by the hyperplane $\{r=r_0\}$ is the set $\{x\in\mathbb{R}^d : x+B_{r_0}\subseteq Z\}$, where Z is the zero cell of the aforementioned tessellation. If $r_0<0$, then the section by $\{r=r_0\}$ is the Minkowski sum $Z+B_{-r_0}$. The set $\mathfrak{Z}_{B_1}\cap\mathfrak{C}_{\mathbb{H}}$ can also be considered as the zero cell of the Poisson hyperplane tessellation in $\mathbb{R}^{d+1}$ with the directional measure being the product of the surface area measure of the unit ball (scaled by its volume) and the Dirac measure δ_{-1}.

Example 6.5 (Diagonal matrices). Let $\mathbb{G}$ be the group of diagonal matrices with positive entries given by $\operatorname{diag}(e^z)$ for $z\in\mathbb{R}^d$. If $\mathbb{T}=\mathbb{R}^d$, then $\mathfrak{C}_{\mathbb{H}}\cong\mathbb{R}^d\times\mathbb{R}^d$. If K is the unit ball, we get

$$\mathfrak{Z}_{B_1}\cap\mathfrak{C}_{\mathbb{R}^d\times\mathbb{G}}\cong\Big\{(x,z)\in\mathbb{R}^d\times\mathbb{R}^d : x\in\bigcap_{i\ge1}H^-_{u_i}\Big(t_i-\sum_{j=1}^d z_j u_{i,j}^2\Big)\Big\}.$$

If $\mathbb{T} = \{0\}$ and K is arbitrary, then

$$\begin{aligned}\mathfrak{Z}_K \cap \mathfrak{C}_{\mathbb{R}^d \times \mathbb{G}} &= \{0\} \times \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \big\{z \in \mathbb{R}^d : \langle \operatorname{diag}(z)\eta, u\rangle \le t\big\} \\ &= \{0\} \times \bigcap_{(t,\eta,u)\in\mathscr{P}_K} \big\{z \in \mathbb{R}^d : \langle z, (\operatorname{diag}(\eta)u)\rangle \le t\big\},\end{aligned}$$

where $\operatorname{diag}(\eta)u$ is the vector given by componentwise products of η and u. Thus, the above intersection is the zero cell of a Poisson tessellation in $\mathbb{R}^d$ whose directional measure is obtained as the push-forward of $\Theta_{d-1}(K,\cdot)$ under the map $(x,u) \mapsto \operatorname{diag}(x)u$. If K is the unit ball, then this directional measure is the push-forward of the uniform distribution on the unit sphere under the map which transforms x to the vector composed of the squares of its components. By Proposition 5.1

$$\operatorname{rec}(\mathfrak{Z}_{B_1} \cap \mathfrak{C}_{\{0\}\times\mathbb{G}}) \cong \{0\} \times \bigcap_{u\in B_1} \big\{z \in \mathbb{R}^d : \langle z, \operatorname{diag}(u)u\rangle \le 0\big\} = \{0\} \times (-\infty, 0]^d.$$

Example 6.6 (Random cones and spherical polytopes). Assume that K is the closed unit upper half-ball B_1^+ defined at (2.3), $\mathbb{T} = \{0\}$ and $\mathbb{G} = \mathbb{SO}_d$, so that $\mathfrak{C}_{\mathbb{H}} = \{0\} \times \mathsf{M}_d^{\mathrm{SSym}}$. If Ξ_n is a sample from the uniform distribution in K, then $\pi(\Xi_n)$ is a sample from the uniform distribution on the half-sphere $\mathbb{S}_+^{d-1}$, where $\pi(x) = x/\|x\|$ for $x \ne 0$. Indeed, for a Borel set $A \subseteq \mathbb{S}_+^{d-1}$,

$$\mathbf{P}\{\pi(\xi_1) \in A\} = \mathbf{P}\{\xi_1/\|\xi_1\| \in A\} = \mathbf{P}\{\xi_1 \in \operatorname{pos}(A)\} = \frac{V_d(\operatorname{pos}(A) \cap B_1^+)}{V_d(B_1^+)} = \frac{\mathscr{H}_{d-1}(A)}{\mathscr{H}_{d-1}(\mathbb{S}_+^{d-1})},$$

where $\mathscr{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. According to (2.5),

$$Q_n \cap \mathbb{S}^{d-1} = \operatorname{conv}_{K,\mathbb{H}}(\Xi_n) \cap \mathbb{S}^{d-1} = \operatorname{pos}(\Xi_n) \cap \mathbb{S}^{d-1} = \operatorname{pos}(\pi(\Xi_n)) \cap \mathbb{S}^{d-1}$$

is a closed random spherical polytope obtained as the spherical hull of n independent points, uniformly distributed on $\mathbb{S}_+^{d-1}$. This object has been intensively studied in [7]. Let $T_n : \mathbb{R}^d \to \mathbb{R}^d$ be a linear mapping

$$T_n(x_1, x_2, \dots, x_d) = (nx_1, x_2, \dots, x_d), \quad n \in \mathbb{N}.$$

Theorem 2.1 in [7] implies that the sequence of random closed cones $(T_n(\operatorname{pos}(\Xi_n)))_{n\in\mathbb{N}}$ converges in distribution in the space of closed subsets of $\mathbb{R}^d$ endowed with the Fell topology to a closed random cone whose intersection with affine hyperplane $\{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_1 = 1\}$ is the convex set $(\operatorname{conv}(\widetilde{\mathscr{P}}), 1)$, where $\widetilde{\mathscr{P}}$ is a Poisson point process on $\mathbb{R}^{d-1}$ with the intensity measure

$$x \mapsto c_d \|x\|^{-d}, \quad x \in \mathbb{R}^{d-1} \setminus \{0\}, \tag{6.2}$$

with an explicit positive constant c_d. The following arguments show that it is possible to establish an isomorphism between the positive dual cone $\{x \in \mathbb{R}^d : \langle x, \xi_k\rangle \ge 0, k = 1, \dots, n\}$ to the cone $\operatorname{pos}(\Xi_n)$ and the set $\mathbb{X}_{K,\mathbb{H}}(\Xi_n)$ defined at (3.1), so that our limit theorem yields the limit for this normalised dual cone.

Denote by $e_1,\dots,e_d$ standard basis vectors. Since $\langle C\eta,u\rangle=0$ for all $(t,\eta,u)\in\mathscr{P}_K$ with $u\in\mathbb{S}_+^{d-1}$ and $C\in\mathsf{M}_d^{\mathrm{SSym}}$, we need only to consider $(t,\eta,u)\in\mathscr{P}_K$ such that $u=-e_1$, meaning that η lies on the flat boundary part of B_1^+, so that

$$\begin{aligned}\mathfrak{Z}_{B_1^+}\cap(\{0\}\times\mathsf{M}_d^{\mathrm{SSym}}) &= \{0\}\times\bigcap_{(t,\eta,u)\in\mathscr{P}_{B_1^+}}\Big\{C\in\mathsf{M}_d^{\mathrm{SSym}}:\langle C\eta,u\rangle\le t\Big\}\\ &= \{0\}\times\bigcap_{(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}}\Big\{C\in\mathsf{M}_d^{\mathrm{SSym}}:\langle C\eta,-e_1\rangle\le t\Big\}\\ &= \{0\}\times\bigcap_{(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}}\Big\{C\in\mathsf{M}_d^{\mathrm{SSym}}:\langle \eta,Ce_1\rangle\le t\Big\}.\end{aligned}$$

Note that every skew-symmetric matrix can be uniquely decomposed into a sum of a skew-symmetric matrix with zeros in the first row and the first column and a skew symmetric matrix with zeros everywhere except the first row and the first column. This corresponds to the direct sum decomposition of the space of skew symmetric matrices $\mathsf{M}_d^{\mathrm{SSym}}:=V_1\oplus V_2$, where $V_1\cong\mathscr{M}_{d-1}^{\mathrm{SSym}}$. For every $(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}$ and $C\in V_1$ we obviously have $\langle\eta,Ce_1\rangle=0$. Thus,

$$\mathfrak{Z}_{B_1^+}\cap\big(\{0\}\times\mathsf{M}_d^{\mathrm{SSym}}\big)\cong\{0\}\times\Big(\mathscr{M}_{d-1}^{\mathrm{SSym}}\oplus\bigcap_{(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}}\big\{C\in V_2:\langle\eta,Ce_1\rangle\le t\big\}\Big).$$

The fact that $\mathfrak{Z}_{B_1^+}\cap(\{0\}\times\mathsf{M}_d^{\mathrm{SSym}})$ contains the subspace V_1 has the following interpretation. It is known that the exponential map from $\mathsf{M}_d^{\mathrm{SSym}}$ to $\mathbb{SO}_d$ is surjective, that is, every orthogonal matrix with determinant one can be represented as the exponent of a skew-symmetric matrix, see Corollary 11.10 in [4]. The image $\exp(V_1)$ is precisely the set of orthogonal matrices with determinant one and for which e_1 is a fixed point. This set is a subgroup of $\mathbb{SO}_d$ which is isomorphic to $\mathbb{SO}_{d-1}$, and B_1^+ is invariant with respect to all transformations from $\exp(V_1)$. The set $\exp(V_2)$ is not a subgroup of $\mathbb{SO}_d$ but is a smooth manifold of dimension $d-1$. Note that the above construction is the particular case of the well-known general concept of quotient manifolds in Lie groups, see Chapter 11.4 in [4].

There is a natural isomorphism $\phi:\{0\}\times V_2\to\mathbb{R}^{d-1}$ which sends $(0,C)\in\{0\}\times V_2$ to the vector $\phi(C)\in\mathbb{R}^{d-1}$ which is the first column of C with the first component (it is always zero) deleted. Moreover, if $(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}$, then η is necessarily of the form $\eta=(0,\eta')$, where $\eta'\in B_1'$ and B_1' is a $(d-1)$-dimensional centred unit ball. It can be checked that the Poisson process $\{(t^{-1}\eta')\in\mathbb{R}^{d-1}\setminus\{0\}:(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}\}$ has intensity (6.2). Summarising,

$$\phi\big(\mathfrak{Z}_{B_1^+}\cap(\{0\}\times V_2)\big)=\bigcap_{(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}}\big\{x\in\mathbb{R}^{d-1}:\langle\eta',x\rangle\le t\big\}=:\widetilde{Z}_0,$$

is the zero cell of the Poisson hyperplane tessellation $\{H_{\eta'}^-(t):(t,\eta,-e_1)\in\mathscr{P}_{B_1^+}\}$ of $\mathbb{R}^{d-1}$. Remarkably, the polar set to $\widetilde{Z}_0$ is the convex hull of $\widetilde{\mathscr{P}}$.

7. APPENDIX

The subsequent presentation concerns random sets in Euclidean space $\mathbb{R}^d$ of generic dimension d. These results are applied in the main part of this paper to random sets of affine transformations, which are subsets of the space $\mathbb{R}^d \times \mathsf{M}_d$. This latter space can be considered an Euclidean space of dimension $d+d^2$.

Let $\mathscr{F}^d$ be the family of closed sets in $\mathbb{R}^d$. Denote by $\mathscr{C}^d$ the family of (nonempty) compact sets and by $\mathscr{K}^d$ the family of convex compact sets. The family of convex compact sets containing the origin is denoted by $\mathscr{K}_0^d$, while $\mathscr{K}_{(0)}^d$ is the family of convex compact sets which contain the origin in their interiors. Each set from $\mathscr{K}_{(0)}^d$ is a convex body (a convex compact set with nonempty interior).

The family $\mathscr{F}^d$ is endowed with the Fell topology, whose base consists of finite intersections of the sets $\{F : F \cap G \neq \varnothing\}$ and $\{F : F \cap L = \varnothing\}$ for all open G and compact L. The definition of the Fell topology and its basic properties can be found in Section 12.2 of [15] or Appendix C in [11]. It is well known that $F_n \to F$ in the Fell topology (this will be denoted by $F_n \xrightarrow{\text{Fell}} F$) if and only if F_n converges to F in the Painlevé–Kuratowski sense, that is, $\limsup F_n = \liminf F_n = F$. Recall that $\limsup F_n$ is the set of all limits of convergent subsequences $x_{n_k} \in F_{n_k}$, $k \geq 1$, and $\liminf F_n$ is the set of limits of convergent sequences $x_n \in F_n$, $n \geq 1$. The space $\mathscr{F}^d$ is compact in the Fell topology, see Theorem 12.2.1 in [15].

The family $\mathscr{C}^d$ is endowed with the topology generated by the Hausdorff metric which we denote by d_{H}. The topology induced by d_{H} on $\mathscr{K}^d$ is exactly the Painlevé–Kuratowski topology, that is, the topology induced on $\mathscr{K}^d \subseteq \mathscr{F}^d$ by the Fell topology on $\mathscr{F}^d$, see Theorem 1.8.8 in [14]. In comparison, the topology induced by d_{H} on $\mathscr{C}^d$ is strictly finer, then the topology induced on $\mathscr{C}^d$ by the Fell topology, see Theorem 12.3.2 in [15].

It is easy to see that the convergence $(F_n \cap L) \xrightarrow{d_{\mathrm{H}}} (F \cap L)$, as $n \to \infty$, for each compact set L implies the Fell convergence $F_n \xrightarrow{\text{Fell}} F$. The inverse implication is false in general, since the intersection operation is not continuous, see Theorem 12.2.6 in [15]. The following result establishes a kind of continuity property for the intersection map. A closed set F is said to be *regular closed* if it coincides with the closure of its interior. The empty set is also considered regular closed. A nonempty convex closed set is regular closed if and only if its interior is not empty.

Lemma 7.1. *Let $(F_n)_{n\in\mathbb{N}}$ and F be closed sets such that $F_n \xrightarrow{\text{Fell}} F$, $n \to \infty$, and let L be a closed set in $\mathbb{R}^d$. Assume that one of the following conditions hold:*

(i) $F \cap L$ is regular closed;

(ii) the sets F and L are convex, $0 \in \operatorname{Int} F$ and $0 \in L$.

Then $(F_n \cap L) \xrightarrow{\text{Fell}} (F \cap L)$, as $n \to \infty$.

Proof. By Theorem 12.2.6(a) in [15], we have

$$\limsup(F_n \cap L) \subseteq (F \cap L).$$

If F is empty, this finishes the proof. Otherwise, it suffices to show that $(F\cap L)\subseteq \liminf(F_n\cap L)$, assuming that F is not empty, so that F_n is also nonempty for all sufficiently large n.

(i) For every $x\in \mathrm{Int}(F\cap L)$, there exists a sequence $x_n\in F_n$, $n\geq 1$, such that $x_n\to x$ and $x_n\in L$ for all sufficiently large n. Thus, $\mathrm{Int}(F\cap L)\subseteq \liminf(F_n\cap L)$ and therefore

$$F\cap L=\mathrm{cl}\,(\mathrm{Int}(F\cap L))\subseteq \liminf(F_n\cap L),$$

where for the equality we have used that $F\cap L$ is regular closed, and for the inclusion that the lower limit is always a closed set.

(ii) First of all, note that

$$\mathrm{cl}\,((\mathrm{Int}\,F)\cap L)=F\cap L. \tag{7.1}$$

Indeed, if $x\in (F\cap L)\setminus\{0\}$, then convexity of $F\cap L$ and $0\in F\cap L$ imply that $x_n:=(1-\frac{1}{n})x\in (\mathrm{Int}\,F)\cap L$, for all $n\in\mathbb{N}$. Since $x_n\to x$, we obtain $x\in \mathrm{cl}\,((\mathrm{Int}\,F)\cap L)$. Obviously, $\{0\}\in \mathrm{cl}\,((\mathrm{Int}\,F)\cap L)$. Thus, $F\cap L\subseteq \mathrm{cl}\,((\mathrm{Int}\,F)\cap L)$. The inverse inclusion holds trivially. Taking into account (7.1) and that $\liminf$ is a closed set it suffices to show that

$$(\mathrm{Int}\,F)\cap L\subseteq \liminf(F_n\cap L). \tag{7.2}$$

Assume that $x\in(\mathrm{Int}\,F)\cap L$. Pick a small enough $\varepsilon>0$ and a sufficiently large $R>0$ such that $x+B_\varepsilon\subseteq F\cap B_R$. Since $F\cap B_R$ is convex and contains the origin in the interior, it is regular closed. Thus, by part (i), $F_n\cap B_R\xrightarrow{\text{Fell}} F\cap B_R$. By Theorem 12.3.2 in [15], we also have $F_n\cap B_R\xrightarrow{d_{\mathrm{H}}} F\cap B_R$. In particular, there exists $n_0\in\mathbb{N}$ such that $F\cap B_R\subseteq (F_n\cap B_R)+B_{\varepsilon/2}$, for $n\geq n_0$, and, thereupon, $x+B_{\varepsilon/2}\subseteq F_n$. Hence $x\in F_n\cap L$ for all $n\geq n_0$. Thus, (7.2) holds. □

The following result establishes continuity properties of the polar transform $L\mapsto L^o$ defined by (4.3) on various subfamilies of convex closed sets which contain the origin. It follows from Theorem 4.2 in [11] that the polar map $L\mapsto L^o$ is continuous on $\mathscr{K}^d_{(0)}$ in the Hausdorff metric, equivalently, in the Fell topology. While L^o is compact if L contains the origin in its interior, L^o is not necessarily bounded for $L\in\mathscr{K}^d_0\setminus\mathscr{K}^d_{(0)}$. Recall that $\mathrm{dom}(L)$ denotes the set of $u\in\mathbb{R}^d$ such that $h(L,u)<\infty$.

Lemma 7.2. *Let L and L_n, $n\in\mathbb{N}$, be convex closed sets which contain the origin.*

- *(i) Assume that $\mathrm{dom}(L_n)=\mathrm{dom}(L)$ is closed for all $n\in\mathbb{N}$, and $h(L_n,u)\to h(L,u)$, as $n\to\infty$, uniformly over $u\in\mathrm{dom}(L)\cap\mathbb{S}^{d-1}$. Then $L_n^o\to L^o$ in the Fell topology.*
- *(ii) The polar transform is continuous as a map from $\mathscr{K}^d_0$ with the Hausdorff metric to $\mathscr{F}^d$ with the Fell topology.*
- *(iii) The polar transform is continuous as the map from the family of convex closed sets which contain the origin in their interior with the Fell topology to $\mathscr{K}^d_0$ with the Hausdorff metric.*
- *(iv) The polar transform is continuous as the map from $\mathscr{K}^d_{(0)}$ to $\mathscr{K}^d_{(0)}$, where both spaces are equipped with the Hausdorff metric.*

Proof. (i) Consider a sequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \in L^o_{n_k}$, $k \in \mathbb{N}$, and $x_{n_k} \to x$, as $k \to \infty$. Assume that $x \notin L^o$. If $h(L,x) = \infty$, that is $x \in (\mathrm{dom}(L))^c$, then also $x_{n_k} \in (\mathrm{dom}(L))^c$ for all sufficiently large k, since the complement to $\mathrm{dom}(L)$ is open. Hence, $x_{n_k} \in (\mathrm{dom}(L_{n_k}))^c$ and $h(L_{n_k}, x_{n_k}) = \infty$, meaning that $x_{n_k} \notin L^o_{n_k}$. Assume now that $h(L,x) < \infty$ and $h(L_{n_k}, x_{n_k}) < \infty$ for all k. If $u, v \in \mathrm{dom}(L) \cap \mathbb{S}^{d-1}$, then $h(L,u) = \langle x, u\rangle$ for some $x \in L$, so that

$$h(L,u) = \langle x, u-v\rangle + \langle x, v\rangle \le \|x\|\|u-v\| + h(L,v).$$

Hence, the support function of L is Lipschitz on $\mathrm{dom}(L) \cap \mathbb{S}^{d-1}$ with the Lipschitz constant at most $c_L := \sup_{u\in\mathrm{dom}(L)\cap\mathbb{S}^{d-1}} h(L,u) < \infty$. Since we assume $x \notin L^o$, we have $h(L,x) \ge 1+\varepsilon$ for some $\varepsilon > 0$. The uniform convergence assumption yields that, for all sufficiently large k,

$$h(L_{n_k}, x_{n_k}) \ge h(L, x_{n_k}) - \varepsilon/4 \ge h(L,x) - \varepsilon/4 - c_L\|x_{n_k} - x\| \ge 1 + \varepsilon/2$$

for all sufficiently large k, meaning that $x_{n_k} \notin L^o_{n_k}$, which is a contradiction. Hence, $\limsup L^o_n \subseteq L^o$.

Let $x \in L^o$. Then $h(L,x) \le 1$, so that $h(L_n, x) \le 1 + \varepsilon_n$, where $\varepsilon_n \downarrow 0$ as $n \to \infty$. Letting $x_n := x/(1+\varepsilon_n)$, we have that $x_n \in L^o_n$ and $x_n \to x$. Thus, $L^o \subseteq \liminf L^o_n$.

(ii) If all sets (L_n) and L are compact, then $\mathrm{dom}(L) = \mathbb{R}^d$, the convergence in the Hausdorff metric is equivalent to the uniform convergence of support functions on $\mathbb{S}^{d-1}$, see Lemma 1.8.14 in [14]. Thus, $L^o_n \xrightarrow{\text{Fell}} L^o$ by part (i).

(iii) Assume that $L_n \xrightarrow{\text{Fell}} L$. In view of Lemma 7.1(i), $L_n \cap B_R \xrightarrow{\text{Fell}} L \cap B_R$, for every fixed $R > 0$, and, therefore, $L_n \cap B_R \xrightarrow{d_{\text{H}}} L \cap B_R$ by Theorem 1.8.8 in [14]. Fix a sufficiently small $\varepsilon > 0$ such that $B_\varepsilon \subseteq L$. Then $B_{\varepsilon/2} \subseteq L_n$, for all sufficiently large n. By part (ii), $(L_n \cap B_R)^o \xrightarrow{\text{Fell}} (L \cap B_R)^o$. Since $(L_n \cap B_R)^o \subseteq B_{(\varepsilon/2)^{-1}}$, for all sufficiently large n, $(L_n \cap B_R)^o \xrightarrow{d_{\text{H}}} (L \cap B_R)^o$, again by Theorem 1.8.8 in [14]. Finally, note that

$$\begin{aligned} d_H(L^o_n, L^o) &\le d_H(L^o_n, (L_n \cap B_R)^o) + d_H((L_n \cap B_R)^o, (L \cap B_R)^o) + d_H((L \cap B_R)^o, L^o) \\ &= d_H(L^o_n, \mathrm{conv}\,(L^o_n \cup B_{R^{-1}})) + d_H((L_n \cap B_R)^o, (L \cap B_R)^o) + d_H(\mathrm{conv}\,(L^o \cup B_{R^{-1}}), L^o) \\ &\le R^{-1} + d_H((L_n \cap B_R)^o, (L \cap B_R)^o) + R^{-1}, \end{aligned}$$

where we have used that $(A_1 \cap A_2)^o = \mathrm{conv}\,(A^o_1 \cup A^o_2)$ and $B^o_R = B_{R^{-1}}$.

(iv) Follows from (iii), since the Fell topology on $\mathscr{K}^d_{(0)}$ coincides with the topology induced by the Hausdorff metric. $\square$

A random closed set X is a measurable map from a probability space to $\mathscr{F}^d$ endowed with the Borel σ-algebra generated by the Fell topology. This is equivalent to the assumption that $\{X \cap L \neq \varnothing\}$ is a measurable event for all compact sets L. The distribution of X is uniquely determined by its capacity functional

$$T_X(L) = \mathbf{P}\{X \cap L \neq \varnothing\}, \quad L \in \mathscr{C}^d.$$

A sequence $(X_n)_{n\in\mathbb{N}}$ of random closed sets in $\mathbb{R}^d$ converges in distribution to a random closed set X (notation $X_n \xrightarrow{\text{d}} X$) if the corresponding probability measures on $\mathscr{F}^d$ (with the Fell topology) weakly converge. By Theorem 1.7.7 in [12], this is equivalent to the pointwise convergence of capacity functionals

$$T_{X_n}(L) \to T_X(L) \quad \text{as } n \to \infty \tag{7.3}$$

for all $L \in \mathscr{C}^d$ which satisfy

$$\mathbf{P}\{X \cap L \neq \varnothing\} = \mathbf{P}\{X \cap \operatorname{Int} L \neq \varnothing\}, \tag{7.4}$$

that is, $T_X(L) = T_X(\operatorname{Int} L)$. The latter condition means that the family $\{F \in \mathscr{F}^d : F \cap L \neq \varnothing\}$ is a continuity set for the distribution of X, and we also say that L itself is a continuity set. It suffices to impose (7.3) for sets L which are regular closed or which are finite unions of balls of positive radii; these families constitute so called convergence determining classes, see Corollary 1.7.14 in [12].

Lemma 7.3. *A sequence of random closed sets $(X_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ converges in distribution to a random closed set X if there exists a sequence $(L_m)_{m\in\mathbb{N}}$ of compact sets such that $\operatorname{Int} L_m \uparrow \mathbb{R}^d$ and $(X_n \cap L_m) \xrightarrow{\text{d}} (X \cap L_m)$ as $n \to \infty$ for each $m \in \mathbb{N}$.*

Proof. We will check (7.3). Fix an $L \in \mathscr{C}^d$ such that $T_X(L) = T_X(\operatorname{Int} L)$. Pick $m \in \mathbb{N}$ so large that L_m contains L in its interior. We have that

$$T_{X\cap L_m}(L) = \mathbf{P}\{X \cap L_m \cap L \neq \varnothing\} = \mathbf{P}\{X \cap L_m \cap \operatorname{Int} L \neq \varnothing\} = T_{X\cap L_m}(\operatorname{Int} L).$$

Since $(X_n \cap L_m) \xrightarrow{\text{d}} (X \cap L_m)$, as $n \to \infty$, we have that

$$\begin{aligned} T_{X_n}(L) = \mathbf{P}\{X_n \cap L \neq \varnothing\} &= \mathbf{P}\{X_n \cap L_m \cap L \neq \varnothing\} \\ &\to \mathbf{P}\{X \cap L_m \cap L \neq \varnothing\} = \mathbf{P}\{X \cap L \neq \varnothing\} = T_X(L), \end{aligned}$$

meaning that $X_n \xrightarrow{\text{d}} X$. □

For a random closed set X, the functional

$$I_X(L) = \mathbf{P}\{L \subseteq X\}, \quad L \in \mathscr{B}(\mathbb{R}^d),$$

is called the inclusion functional of X. While the capacity functional determines uniquely the distribution of X, this is not the case for the inclusion functional, e.g., if X is a singleton with a nonatomic distribution.

Let $\mathscr{E}$ be the family of all convex regular closed subsets of $\mathbb{R}^d$ (including the empty set), and let $\mathscr{E}'$ denote the family of closed complements to all sets from $\mathscr{E}$. The whole space also belongs to the family $\mathscr{E}$. Recall that a nonempty convex closed set is regular closed if and only if its interior is not empty.

Lemma 7.4. *The map $F \mapsto \operatorname{cl}(F^c)$ is a bicontinuous (in the Fell topology) bijection between $\mathscr{E}$ and $\mathscr{E}'$.*

Proof. The map $F \mapsto \mathrm{cl}(F^c)$ is self-inverse on $\mathscr{E}$, hence a bijection. Let us prove continuity.

Assume that $F_n \xrightarrow{\text{Fell}} F$. Let $\mathrm{cl}(F^c) \cap G \neq \varnothing$ for an open set G. Then $F \neq \mathbb{R}^d$ and $F^c \cap G \neq \varnothing$, meaning that G is not a subset of F. Take $x \in G \setminus F$. There exists an $\varepsilon > 0$ such that $x + B_\varepsilon \subseteq G$ and $(x + B_\varepsilon) \cap F = \varnothing$. Since F_n converges to F, we have that $(x + B_\varepsilon) \cap F_n = \varnothing$, for sufficiently large n. Thus, $F_n^c \cap G \neq \varnothing$, which means that $\mathrm{cl}(F_n^c) \cap G \neq \varnothing$, for sufficiently large n. This argument also applies if F_n converges to the empty set. Suppose that $\mathrm{cl}(F^c) \cap L = \varnothing$ for a compact nonempty set L (in this case F is necessarily nonempty). By compactness, $\mathrm{cl}(F^c) \cap (L + B_\varepsilon) = \varnothing$, for a sufficiently small $\varepsilon > 0$. Therefore,

$$L + B_\varepsilon \subseteq (\mathrm{cl}(F^c))^c = \mathrm{Int} F. \tag{7.5}$$

By convexity of F, it is possible to replace L with its convex hull, so assume that L is convex. Pick a large $R > 0$ such that $L + B_\varepsilon \subseteq \mathrm{Int} B_R$. From Lemma 7.1(i) and using the same reasoning as in the proof of part (iii) of Lemma 7.2 we conclude that

$$F_n \cap B_R \xrightarrow{d_{\mathrm{H}}} F \cap B_R, \quad n \to \infty.$$

Thus, $(F \cap B_R) \subseteq (F_n \cap B_R) + B_{\varepsilon/2}$ for all sufficiently large n. In conjunction with (7.5) this yields $L + B_\varepsilon \subseteq (F_n \cap B_R) + B_{\varepsilon/2}$ for sufficiently large n. Since L and $F_n \cap B_R$ are convex, we conclude that $L \subseteq \mathrm{Int}(F_n \cap B_R) \subseteq \mathrm{Int} F_n$. Hence, $L \cap \mathrm{cl}(F_n^c) = \varnothing$ for all sufficiently large n. This observation completes the proof of continuity of the direct mapping.

It remains to prove continuity of the inverse mapping. Assume that $\mathrm{cl}(F_n^c) \xrightarrow{\text{Fell}} \mathrm{cl}(F^c)$, as $n \to \infty$, with $F_n, F \in \mathscr{E}$. If $F \cap G \neq \varnothing$ for an open set G, then $(\mathrm{Int} F) \cap G \neq \varnothing$ and also $\mathrm{cl}(F^c) \neq \mathbb{R}^d$. Take a point $x \in (\mathrm{Int} F) \cap G$. Then $x \notin \mathrm{cl}(F^c)$, so that $x \notin \mathrm{cl}(F_n^c)$, for all sufficiently large n, meaning that $x \in F_n$ and $G \cap F_n \neq \varnothing$, for all sufficiently large n.

Now assume that $F \cap L = \varnothing$ for a nonempty compact set L. We aim to show that $F_n \cap L = \varnothing$, for all sufficiently large n (note that $F = \varnothing$ is allowed). By compactness of L, for sufficiently small $\varepsilon > 0$, we have $F \cap (L + B_\varepsilon) = \varnothing$. Thus,

$$L + B_\varepsilon \subseteq F^c \subseteq \mathrm{cl}(F^c).$$

Pick again a sufficiently large $R > 0$ such that $L + B_\varepsilon \subseteq B_R$. By Lemma 7.1(i) $(\mathrm{cl}(F_n^c) \cap B_R) \xrightarrow{d_{\mathrm{H}}} (\mathrm{cl}(F^c) \cap B_R)$, and therefore $(\mathrm{cl}(F^c) \cap B_R) \subseteq (\mathrm{cl}(F_n^c) \cap B_R) + B_{\varepsilon/2}$ for all sufficiently large n. Thus, $L + B_\varepsilon \subseteq (\mathrm{cl}(F_n^c) \cap B_R) + B_{\varepsilon/2}$, which implies $L \subseteq F_n^c$. Therefore, $L \cap F_n = \varnothing$ and the proof is complete. □

The following result establishes the convergence in distribution of random convex closed sets with values in $\mathscr{E}$ from the convergence of their inclusion functionals. It provides an alternative proof and an extension of Proposition 1.8.16 in [12], which establishes this fact for random sets with values in $\mathscr{K}^d_{(0)}$.

Theorem 7.5. *Let X and X_n, $n \in \mathbb{N}$ be random closed sets in $\mathbb{R}^d$ which almost surely take values from the family $\mathscr{E}$ of regular convex closed sets (including the empty set). If*

$$\mathbf{P}\{L \subseteq X_n\} \to \mathbf{P}\{L \subseteq X\} \quad \textit{as} \quad n \to \infty \tag{7.6}$$

for all regular closed $L \in \mathscr{K}^d$ such that

$$\mathbf{P}\{L \subseteq X\} = \mathbf{P}\{L \subseteq \operatorname{Int} X\}, \tag{7.7}$$

then $X_n \xrightarrow{\text{d}} X$, as $n \to \infty$.

Proof. In view of Lemma 7.4 it suffices to prove that $\operatorname{cl}(X_n^c) \xrightarrow{\text{d}} \operatorname{cl}(X^c)$, as $n \to \infty$. Furthermore, since regular closed compact sets constitute a convergence determining class, see Corollary 1.7.14 in [12], it suffices to check that

$$\mathbf{P}\{\operatorname{cl}(X_n^c) \cap L \neq \varnothing\} \to \mathbf{P}\{\operatorname{cl}(X^c) \cap L \neq \varnothing\} \quad \text{as} \quad n \to \infty, \tag{7.8}$$

for all regular closed $L \in \mathscr{C}^d$, which are continuity sets for $\operatorname{cl}(X^c)$. The latter means that

$$\mathbf{P}\{\operatorname{cl}(X^c) \cap L = \varnothing\} = \mathbf{P}\{\operatorname{cl}(X^c) \cap \operatorname{Int} L = \varnothing\}. \tag{7.9}$$

Fix a regular closed set $L \in \mathscr{C}^d$ such that (7.9) holds. Since

$$\mathbf{P}\{\operatorname{cl}(X^c) \cap L = \varnothing\} = \mathbf{P}\{L \subseteq \operatorname{Int} X\} \quad \text{and} \quad \mathbf{P}\{\operatorname{cl}(X^c) \cap \operatorname{Int} L = \varnothing\} = \mathbf{P}\{\operatorname{Int} L \subseteq \operatorname{Int} X\},$$

we conclude that

$$\mathbf{P}\{L \subseteq X\} \leq \mathbf{P}\{\operatorname{Int} L \subseteq \operatorname{Int} X\} = \mathbf{P}\{L \subseteq \operatorname{Int} X\} \leq \mathbf{P}\{L \subseteq X\}.$$

Thus, L satisfies (7.7).

Let $(\varepsilon_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers such that $\varepsilon_k \downarrow 0$, as $k \to \infty$, and

$$\mathbf{P}\{L + B_{\varepsilon_k} \subseteq X\} = \mathbf{P}\{L + B_{\varepsilon_k} \subseteq \operatorname{Int} X\}, \quad k \in \mathbb{N}.$$

Sending $n \to \infty$ in the chain of inequalities

$$\mathbf{P}\{L + B_{\varepsilon_k} \subseteq X_n\} \leq \mathbf{P}\{L \subseteq \operatorname{Int} X_n\} = \mathbf{P}\{\operatorname{cl}(X_n^c) \cap L = \varnothing\} \leq \mathbf{P}\{L \subseteq X_n\},$$

and using (7.6), we conclude that

$$\mathbf{P}\{L + B_{\varepsilon_k} \subseteq X\} \leq \liminf_{n \to \infty} \mathbf{P}\{\operatorname{cl}(X_n^c) \cap L = \varnothing\} \leq \limsup_{n \to \infty} \mathbf{P}\{\operatorname{cl}(X_n^c) \cap L = \varnothing\} \leq \mathbf{P}\{L \subseteq X\}. \tag{7.10}$$

Since

$$\mathbf{P}\{L + B_{\varepsilon_k} \subseteq X\} \uparrow \mathbf{P}\{L \subseteq \operatorname{Int} X\} = \mathbf{P}\{L \subseteq X\} \quad \text{as} \quad k \to \infty,$$

the desired convergence (7.8) follows upon sending $k \to \infty$ in (7.10). $\square$

If F is an arbitrary closed set, then, in general, the convergence $X_n \xrightarrow{\text{d}} X$ does not imply the convergence of $X_n \cap F$ to $X \cap F$. The latter is equivalent to the convergence of the capacity functionals of $X_n \cap F$ on sets $L \in \mathscr{C}^d$ such that

$$\mathbf{P}\{(X \cap F) \cap L \neq \varnothing\} = \mathbf{P}\{(X \cap F) \cap \operatorname{Int} L \neq \varnothing\}.$$

At a first glance, the aforementioned implication looks plausible since the capacity functional of $X_n \cap F$ on L is just the capacity function of X_n on $F \cap L$. However, from the convergence $X_n \xrightarrow{\text{d}} X$ we can only deduce the convergence of their capacity functionals on sets $F \cap L$ under condition that

$$\mathbf{P}\{X \cap F \cap L \neq \varnothing\} = \mathbf{P}\{X \cap \operatorname{Int}(F \cap L) \neq \varnothing\}.$$

This latter is too restrictive if F has empty interior. The following result relies on an alternative argument in order to establish convergence in distribution of random sets intersected with a deterministic convex closed set containing the origin.

Lemma 7.6. *Let X and X_n, $n \in \mathbb{N}$, be random convex closed sets. Assume that* $\mathbf{P}\{0 \in X\} = \mathbf{P}\{0 \in \operatorname{Int} X\} > 0$ *and* $\mathbf{P}\{0 \in X_n\} = \mathbf{P}\{0 \in \operatorname{Int} X_n\} > 0$ *for all sufficiently large n. Assume that* (7.6) *holds for all $L \in \mathscr{K}^d$ satisfying* (7.7)*. Let F be a convex closed set which contains the origin. Then*

$$\mathbf{P}\{X_n \cap F \cap L \neq \varnothing, 0 \in X_n\} \to \mathbf{P}\{X \cap F \cap L \neq \varnothing, 0 \in X\} \quad \text{as} \quad n \to \infty \tag{7.11}$$

for each compact set L in $\mathbb{R}^d$ such that

$$\mathbf{P}\{(X \cap F) \cap L \neq \varnothing, 0 \in X\} = \mathbf{P}\{(X \cap F) \cap \operatorname{Int} L \neq \varnothing, 0 \in X\}. \tag{7.12}$$

Proof. Define the following auxiliary random closed sets

$$Y_n := \begin{cases} X_n, & \text{if } 0 \in \operatorname{Int} X_n, \\ \varnothing, & \text{if } 0 \notin \operatorname{Int} X_n; \end{cases} \quad \text{and} \quad Y := \begin{cases} X, & \text{if } 0 \in \operatorname{Int} X, \\ \varnothing, & \text{if } 0 \notin \operatorname{Int} X. \end{cases}$$

By construction, random closed sets Y_n and Y are almost surely regular closed. Let us show with the help of Theorem 7.5, that $Y_n \xrightarrow{\text{d}} Y$, as $n \to \infty$. Let L be a nonempty compact set such that

$$\mathbf{P}\{L \subseteq Y\} = \mathbf{P}\{L \subseteq \operatorname{Int} Y\}. \tag{7.13}$$

The latter is equivalent to

$$\mathbf{P}\{L \subseteq X, 0 \in \operatorname{Int} X\} = \mathbf{P}\{L \subseteq \operatorname{Int} X, 0 \in \operatorname{Int} X\},$$

and, since $\mathbf{P}\{0 \in X\} = \mathbf{P}\{0 \in \operatorname{Int} X\}$, to

$$\mathbf{P}\{L \subseteq X, 0 \in X\} = \mathbf{P}\{L \subseteq \operatorname{Int} X, 0 \in \operatorname{Int} X\}.$$

Finally, by convexity of X we see that (7.13) is the same as

$$\mathbf{P}\{\operatorname{conv}(L \cup \{0\}) \subseteq X\} = \mathbf{P}\{\operatorname{conv}(L \cup \{0\}) \subseteq \operatorname{Int} X\}.$$

Thus, if a nonempty compact set L and satisfies (7.13), then $\operatorname{conv}(L\cup\{0\}) \in \mathscr{K}^d$ satisfies (7.7) and we can use (7.6) to conclude that

$$\begin{aligned}\mathbf{P}\{L\subseteq Y_n\} = \mathbf{P}\{L\subseteq X_n, 0\in X_n\} = \mathbf{P}\{\operatorname{conv}(L\cup\{0\})\subseteq X_n\}\\ \to \mathbf{P}\{\operatorname{conv}(L\cup\{0\})\subseteq X\} = \mathbf{P}\{L\subseteq X, 0\in X\} = \mathbf{P}\{L\subseteq Y\} \quad \text{as} \quad n\to\infty.\end{aligned}$$

Theorem 7.5 yields that $Y_n \xrightarrow{\text{d}} Y$, as $n\to\infty$. Note that Y is a.s. either empty or contains 0 in the interior. Thus, $Y\cap F$ is a.s. either empty (thus, regular closed) or Y contains 0 in the interior. In both cases Lemma 7.1 is applicable and by continuous mapping theorem $Y_n\cap F \xrightarrow{\text{d}} Y\cap F$. The latter means that

$$\mathbf{P}\{(Y_n\cap F)\cap L\neq\varnothing\} \to \mathbf{P}\{(Y\cap F)\cap L\neq\varnothing\} \quad \text{as} \quad n\to\infty$$

for all L such that

$$\mathbf{P}\{(Y\cap F)\cap L\neq\varnothing\} = \mathbf{P}\{(Y\cap F)\cap \operatorname{Int} L\neq\varnothing\}.$$

By definition of Y_n and Y, this is the same as (7.11) for L satisfying (7.12). □

The next result follows either from Lemma 7.6 or from Lemma 7.1 and continuous mapping theorem.

Corollary 7.7. *Let X and X_n, $n\in\mathbb{N}$, be random convex closed sets, whose interiors almost surely contain the origin. If $X_n \xrightarrow{\text{d}} X$, as $n\to\infty$, then $X_n\cap F \xrightarrow{\text{d}} X\cap F$, as $n\to\infty$, for each convex closed set F which contains the origin.*

The following result is used in the proof of Theorem 4.1 in order to establish convergence in distribution of (not necessarily convex) random closed sets by approximating them with convex ones.

Lemma 7.8. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random closed sets in $\mathbb{R}^d$. Assume that $Y^-_{m,n}\subseteq X_n\subseteq Y^+_{n,m}$ a.s. for all $n,m\in\mathbb{N}$ and sequences $(Y^-_{n,m})_{n\in\mathbb{N}}$ and $(Y^+_{n,m})_{n\in\mathbb{N}}$ of random closed sets. Further, assume that, for each $m\in\mathbb{N}$:*

(i) *the random closed set $Y^+_{n,m}$ converges in distribution to a random closed set Y^+_m, as $n\to\infty$;*
(ii) *there exists a random closed set Y^-_m such that*

$$\mathbf{P}\left\{Y^-_{m,n}\cap L\neq\varnothing, 0\in Y^-_{m,n}\right\} \to \mathbf{P}\left\{Y^-_m\cap L\neq\varnothing, 0\in Y^-_m\right\} \quad \textit{as} \quad n\to\infty \tag{7.14}$$

for all $L\in\mathscr{C}^d$ which are continuity sets for Y^-_m.

Further assume that $\mathbf{P}\{0\in Y^-_m\}\to 1$, and that $Y^+_m\downarrow Z$, $Y^-_m\uparrow Z$ a.s., as $m\to\infty$, in the Fell topology for a random closed set Z. Then $X_n \xrightarrow{\text{d}} Z$, as $n\to\infty$.

Proof. Since the family of regular closed compact sets constitute a convergence determining class, see Corollary 1.7.14 in [12], it suffices to check that the capacity functional of X_n converges to the capacity functional of Z on all compact sets L, which are regularly closed and are continuity sets for Z. There exist sequences (L_k^-) and (L_k^+) of compact sets, which are continuity sets for Z and all (Y_m^-) and (Y_m^+), respectively, and such that $L_k^- \uparrow \operatorname{Int} L$ and $L_k^+ \downarrow L$ as $k \to \infty$. These sets can be chosen from the families of inner and outer parallel sets to L, see p. 148 in [14].

Then

$$\mathbf{P}\left\{Y_{n,m}^- \cap L_k^- \neq \varnothing, 0 \in Y_{m,n}^-\right\} \leq \mathbf{P}\{X_n \cap L \neq \varnothing\} \leq \mathbf{P}\left\{Y_{n,m}^+ \cap L_k^+ \neq \varnothing\right\}.$$

Passing to the limit, as $n \to \infty$, yields that

$$\begin{aligned}\mathbf{P}\left\{Y_m^- \cap L_k^- \neq \varnothing, 0 \in Y_m^-\right\} &\leq \liminf_{n\to\infty} \mathbf{P}\{X_n \cap L \neq \varnothing\} \\ &\leq \limsup_{n\to\infty} \mathbf{P}\{X_n \cap L \neq \varnothing\} \leq \mathbf{P}\left\{Y_m^+ \cap L_k^+ \neq \varnothing\right\}.\end{aligned}$$

Note that the a.s. convergence of $Y_m^\pm$ to Z implies the convergence in distribution. Sending $m \to \infty$ and using that $\mathbf{P}\{0 \in Y_m^-\} \to 1$, we conclude

$$\mathbf{P}\left\{Z \cap L_k^- \neq \varnothing\right\} \leq \liminf_{n\to\infty} \mathbf{P}\{X_n \cap L \neq \varnothing\} \leq \limsup_{n\to\infty} \mathbf{P}\{X_n \cap L \neq \varnothing\} \leq \mathbf{P}\left\{Z \cap L_k^+ \neq \varnothing\right\}.$$

Finally, sending $k \to \infty$ gives

$$\mathbf{P}\{Z \cap \operatorname{Int} L \neq \varnothing\} \leq \liminf_{n\to\infty} \mathbf{P}\{X_n \cap L \neq \varnothing\} \leq \limsup_{n\to\infty} \mathbf{P}\{X_n \cap L \neq \varnothing\} \leq \mathbf{P}\{Z \cap L \neq \varnothing\},$$

which completes the proof since $\mathbf{P}\{Z \cap L \neq \varnothing\} = \mathbf{P}\{Z \cap \operatorname{Int} L \neq \varnothing\}$. □

Proposition 7.9. *Let $\Psi_n := \{(X_1,\xi_1),\dots,(X_n,\xi_n)\}$, $n \in \mathbb{N}$, be a sequence of binomial point processes on $(\mathscr{K}_0^d \setminus \{0\}) \times \mathbb{R}^d$ obtained by taking independent copies of a pair (X,ξ), where X is a random convex closed set and ξ is a random vector in $\mathbb{R}^d$, which may depend on X. Furthermore, let $\Psi := \{(Y_i,y_i), i \geq 1\}$ be a locally finite Poisson process on $(\mathscr{K}_0^d \setminus \{0\}) \times \mathbb{R}^d$ with the intensity measure μ. Then $n^{-1}\Psi_n := \{(n^{-1}X_i,\xi_i) : i = 1,\dots,n\}$ converges in distribution to Ψ if and only if the following convergence takes place*

$$n\mathbf{P}\left\{n^{-1}X \not\subseteq L, \xi \in B\right\} = n\mathbf{P}\left\{(n^{-1}X,\xi) \in \mathscr{A}_L^c \times B\right\} \to \mu(\mathscr{A}_L^c \times B) \quad \text{as } n \to \infty, \tag{7.15}$$

for every μ-continuous set $\mathscr{A}_L^c \times B \subseteq (\mathscr{K}_0^d \setminus \{0\}) \times \mathbb{R}^d$, where

$$\mathscr{A}_L := \{A \in \mathscr{K}_0^d \setminus \{0\} : A \subseteq L\},$$

and $L \in \mathscr{K}_0^d \setminus \{0\}$ is an arbitrary convex compact set containing the origin and which is distinct from $\{0\}$.

Proof. By a simple version of the Grigelionis theorem for binomial point processes (see, e.g., Proposition 11.1.IX in [2] or Corollary 4.25 in [8] or Theorem 4.2.5 in [12]), $n^{-1}\Psi_n \xrightarrow{\mathrm{d}} \Psi$ if and only if

$$\mu_n(\mathscr{A} \times B) := n\mathbf{P}\left\{(n^{-1}X,\xi) \in \mathscr{A} \times B\right\} \to \mu(\mathscr{A} \times B) \quad \text{as } n \to \infty, \tag{7.16}$$

for all Borel $\mathscr{A}$ in $\mathscr{K}_0^d \setminus \{0\}$ and Borel B in $\mathbb{R}^d$, such that $\mathscr{A} \times B$ is a continuity set for μ.

Thus, we need to show that convergence (7.16) follows from (7.15). In other words, we need to show that the family of sets of the form $\mathscr{A}_L^c \times B$ is the convergence determining class.

Fix some $\varepsilon > 0$ and let $L_0 := B_\varepsilon \subseteq \mathbb{R}^d$ be the closed centred ball of radius ε. It is always possible to ensure that $\mathscr{A}_{L_0}^c \times B$ is a continuity set for μ. For each Borel $\mathscr{A}$ in $\mathscr{K}_0^d \setminus \{0\}$, put

$$\tilde{\mu}_n(\mathscr{A} \times B) := \frac{\mu_n\big((\mathscr{A} \cap \mathscr{A}_{L_0}^c) \times B\big)}{\mu_n\big(\mathscr{A}_{L_0}^c \times \mathbb{R}^d\big)}, \quad n \geq 1,$$

and define $\tilde{\mu}$ by the same transformation applied to μ. Then $\tilde{\mu}_n$ is a probability measure on $(\mathscr{K}_0^d \setminus \{0\}) \times \mathbb{R}^d$, and so on $\mathscr{K}^d \times \mathbb{R}^d$. Thus, μ_n defines the distribution of a random convex closed set $Z_n \times \{\zeta_n\}$ in $\mathscr{K}^d \times \mathbb{R}^d$, which we can regard as a subset of $\mathscr{K}^{d+1}$.

Assume that we have shown that $\tilde{\mu}_n$ converges in distribution to $\tilde{\mu}$, as $n \to \infty$. Then, (7.15) implies (7.16). Indeed, it obviously suffices to assume in (7.16) that $\mathscr{A}$ is closed in the Hausdorff metric and is such that $\mathscr{A} \times B$ is a $\tilde{\mu}$-continuous set. Then there exists an $\varepsilon > 0$ such that each $A \in \mathscr{A}$ is not a subset of $L_0 = B_\varepsilon$. Then $\mathscr{A} \cap \mathscr{A}_{L_0}^c = \mathscr{A}$, so that

$$\frac{\mu_n(\mathscr{A} \times B)}{\mu_n(\mathscr{A}_{L_0}^c \times \mathbb{R}^d)} = \tilde{\mu}_n(\mathscr{A} \times B) \quad \to \quad \tilde{\mu}(\mathscr{A} \times B) = \frac{\mu(\mathscr{A} \times B)}{\mu(\mathscr{A}_{L_0}^c \times \mathbb{R}^d)} \quad \text{as } n \to \infty.$$

Since the denominators also converge in view of (7.15) we obtain (7.16).

In order to check that $\tilde{\mu}_n$ converges in distribution to $\tilde{\mu}$ we shall employ Theorem 1.8.14 from [12]. According to the cited theorem $\tilde{\mu}_n$ converges in distribution to $\tilde{\mu}$ if and only if $\tilde{\mu}_n(\mathscr{A}_L \times B) \to \tilde{\mu}(\mathscr{A}_L \times B)$ for all $L \in \mathscr{K}^d$ and convex compact B in $\mathbb{R}^d$ such that $\mathscr{A}_L \times B$ is a continuity set for $\tilde{\mu}$ and $\tilde{\mu}(\mathscr{A}_L \times B) \uparrow 1$ if L and B increase to the whole space. The latter is clearly the case, since Ψ has a locally finite intensity measure, hence, at most a finite number of its points intersects the complement of the centred ball B_r in $\mathbb{R}^{d+1}$ for any $r > 0$. For the former, note that, for every $L \in \mathscr{K}^d \setminus \{0\}$,

$$\begin{aligned}
\tilde{\mu}_n(\mathscr{A}_L \times B) &= \frac{\mu_n\big(\mathscr{A}_{L_0}^c \times B\big) - \mu_n\big((\mathscr{A}_L^c \cap \mathscr{A}_{L_0}^c) \times B\big)}{\mu_n\big(\mathscr{A}_{L_0}^c \times \mathbb{R}^d\big)} = \frac{\mu_n\big((\mathscr{A}_L^c \cup \mathscr{A}_{L_0}^c) \times B\big) - \mu_n\big(\mathscr{A}_L^c \times B\big)}{\mu_n\big(\mathscr{A}_{L_0}^c \times \mathbb{R}^d\big)} \\
&= \frac{\mu_n\big((\mathscr{A}_L \cap \mathscr{A}_{L_0})^c \times B\big) - \mu_n\big(\mathscr{A}_L^c \times B\big)}{\mu_n\big(\mathscr{A}_{L_0}^c \times \mathbb{R}^d\big)} = \frac{\mu_n\big(\mathscr{A}_{L \cap L_0}^c \times B\big) - \mu_n\big(\mathscr{A}_L^c \times B\big)}{\mu_n\big(\mathscr{A}_{L_0}^c \times \mathbb{R}^d\big)} \\
&\to \frac{\mu\big(\mathscr{A}_{L \cap L_0}^c \times B\big) - \mu\big(\mathscr{A}_L^c \times B\big)}{\mu\big(\mathscr{A}_{L_0}^c \times \mathbb{R}^d\big)} = \tilde{\mu}(\mathscr{A}_L \times B) \quad \text{as } n \to \infty,
\end{aligned}$$

where the convergence in the last line follows from (7.15). The proof is complete. □

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