SOLUTIONS OF KINETIC-TYPE EQUATIONS WITH PERTURBED COLLISIONS

DARIUSZ BURACZEWSKI, PIOTR DYSZEWSKI AND ALEXANDER MARYNYCH

ABSTRACT. We study a class of kinetic-type differential equations $\partial\phi_t/\partial t+\phi_t=\widehat{\mathscr{Q}}\phi_t$, where $\widehat{\mathscr{Q}}$ is an inhomogeneous smoothing transform and, for every $t\geq 0$, ϕ_t is the Fourier–Stieltjes transform of a probability measure. We show that under mild assumptions on $\widehat{\mathscr{Q}}$ the above differential equation possesses a unique solution and represent this solution as the characteristic function of a certain stochastic process associated with the continuous time branching random walk pertained to $\widehat{\mathscr{Q}}$. Establishing limit theorems for this process allows us to describe asymptotic properties of the solution, as $t\to\infty$.

1. INTRODUCTION

The article is devoted to the study of generalized kinetic-type equations of the form

$$\begin{cases} \partial_t \nu_t = \mathscr{Q}(\nu_t)-\nu_t, \\ \nu_0=\nu; \end{cases} \tag{1.1}$$

where $(\nu_t)_{t\geq 0}$ is a family of probability measures on $\mathbb{R}$. The derivative with respect to the time parameter t is understood in the weak sense, that is, $\partial_t\nu_t$ is a signed measure such that $\frac{\mathrm{d}}{\mathrm{d}t}\int g(y)\nu_t(\mathrm{d}y)=\int g(y)\partial_t\nu_t(\mathrm{d}y)$, for every bounded continuous $g:\mathbb{R}\mapsto\mathbb{C}$. The operator $\mathscr{Q}$ on the right-hand side is a smoothing transform defined in the following fashion. For a given random vector of coefficients $(N,C,A_1,A_2,\ldots)$ taking values in $\{0,1,2,\ldots\}\times\mathbb{R}\times\mathbb{R}^{\mathbb{N}}$ and a probability measure ν on $\mathbb{R}$, take a sequence of independent identically distributed (i.i.d.) random variables $Y_1,Y_2,\ldots$ with common distribution ν which are also independent of the vector $(N,C,A_1,A_2,\ldots)$. The image $\mathscr{Q}(\nu)$ is then defined by

$$\mathscr{Q}(\nu)=\mathrm{Law}\left(\sum_{k=1}^{N}A_kY_k+C\right),$$

where $\mathrm{Law}(X)$ denotes the law of a random variable X.

The equations of the form (1.1) originate from the Boltzmann equation. In the Kac caricature of a Maxwell gas [16], ν_t is the distribution of velocity of a typical particle in a homogeneous gas exposed to elastic, binary collisions. This model corresponds to the choice of parameters $N=2$, $C=0$, $(A_1,A_2)=(\sin\Theta,\cos\Theta)$, with Θ being a random variable with the uniform distribution on $[0,2\pi]$. The term $\mathscr{Q}(\nu_t)-\nu_t$ on the right-hand side of (1.1) describes the change in the velocity after a binary collision. The inelastic version of the caricature was investigated in [5, 8, 11, 18]. Equations of the form (1.1) can be seen as a model of a market economy with wealth redistribution, see [5] and references therein.

Our aim is to provide robust probabilistic tools to study asymptotic properties of the solutions to (1.1). For convenience, we shall formulate and prove our main results in the terms of the characteristic functions of $(\nu_t)_{t\geq 0}$ which will

2020 *Mathematics Subject Classification.* Primary: 60J85; Secondary: 82C40, 60F05.

Key words and phrases. Additive martingale; branching random walk; inhomogeneous smoothing transform; Kac model; kinetic equation; random trees.

DB and PD were supported by the National Science Center, Poland (Opus, grant number 2020/39/B/ST1/00209).

be denoted via

$$\phi_t(\xi) = \phi(t,\xi) = \int_{\mathbb{R}} e^{i\xi x}\, \nu_t(\mathrm{d}x), \quad t \geq 0, \quad \xi \in \mathbb{R}.$$

The relation (1.1) written in terms of $(\phi_t)_{t\geq 0}$ reads

$$\begin{cases} \partial_t \phi_t(\xi) = \widehat{\mathscr{Q}}\phi_t(\xi) - \phi_t(\xi), \\ \phi_0(\xi) = \phi(\xi); \end{cases} \tag{1.2}$$

where $\phi(\xi) = \int_{\mathbb{R}} e^{i\xi x}\, \nu(\mathrm{d}x)$ and $\widehat{\mathscr{Q}}$ is the functional operator associated with the smoothing transform $\mathscr{Q}$. More precisely, for a continuous function $\psi\colon \mathbb{R} \to \mathbb{C}$ taking values in the unit disk $\{z \in \mathbb{C} : \|z\| \leq 1\}$ we define the function $\widehat{\mathscr{Q}}\psi\colon \mathbb{R} \to \mathbb{C}$ via

$$\widehat{\mathscr{Q}}\psi(\xi) = \mathbb{E}\left[e^{i\xi C} \prod_{k=1}^{N} \psi(A_k \xi) \right], \quad \xi \in \mathbb{R}. \tag{1.3}$$

We shall show below, see Theorem 2.1, that under mild assumptions on $(N, C, A_1, A_2, \ldots)$, equation (1.2) possesses a unique solution for an arbitrary initial condition ϕ_0. Having settled uniqueness we then turn our attention to asymptotic analysis of this solution, as $t \to \infty$. Depending on a choice of the distribution of the vector of coefficients $(N, C, A_1, A_2, \ldots)$ we consider the two following scenarios.

In the first regime $\phi_t(\xi) \to \phi_\infty(\xi)$, as $t \to \infty$, for some non-degenerate limit function ϕ_∞ and all $\xi \in \mathbb{R}$. Then one expects that ϕ_∞ satisfies the stationary version of (1.2) that is

$$\widehat{\mathscr{Q}}\phi_\infty(\xi) = \phi_\infty(\xi), \quad \xi \in \mathbb{R}.$$

From the probabilistic point of view one would interpret ϕ_∞ as the Fourier–Stieltjes transform a fixed point of the inhomogeneous smoothing transform $\mathscr{Q}$. As verified by [5, Proposition 3] in the case $\mathbb{P}[N = 2] = 1$ and Theorem 2.5 of the present article for general N, ϕ_∞ is the characteristic function of a random variable X which satisfies

$$\mathrm{Law}(X) = \mathrm{Law}\left(\sum_{k=1}^{N} A_k X_k + C \right),$$

with $(X_k)_{k\geq 1}$ denoting independent copies of X which are also independent of the vector $(N, C, A_1, A_2, \ldots)$.

In the second scenario ϕ_t has a nondegenerate pointwise limit only after a suitable rescaling. For some fixed $\alpha, \beta \in \mathbb{R}$, consider a rescaled function

$$w_t(\xi) = \phi_t\left((t+1)^\beta e^{\alpha t} \xi \right), \quad t \geq 0, \quad \xi \in \mathbb{R}.$$

Note that the factor $(t+1)^\beta$ instead of a more natural and asymptotically equivalent t^β is taken for convenience, since in this case $w_0(\xi) = \phi_0(\xi) = \phi(\xi)$. For the sake of simplicity assume for the time being that $\int_{\mathbb{R}} x \nu_t(\mathrm{d}x) < \infty$, for all $t \geq 0$. This condition ensures that the function $w_t(\xi)$ is differentiable with respect to ξ for all $t \geq 0$, and equation (1.2) can be recast in terms of $(w_t)_{t\geq 0}$ as follows:

$$\begin{cases} \partial_t w_t(\xi) - \xi\left(\alpha + \beta \frac{1}{t+1}\right) \partial_\xi w_t(\xi) = \widehat{\mathscr{Q}}_{(t+1)^\beta e^{\alpha t}} w_t(\xi) - w_t(\xi), \\ w_0(\xi) = \phi(\xi). \end{cases}$$

Here $\widehat{\mathscr{Q}}_s$, for $s \geq 0$, denotes a rescaled smoothing transform obtained by replacing C with sC, that is,

$$\widehat{\mathscr{Q}}_s\psi(\xi) = \mathbb{E}\left[e^{i\xi sC} \prod_{k=1}^{N} \psi(A_k \xi) \right], \quad \xi \in \mathbb{R}.$$

Suppose now that $w_t(\xi) \to w_\infty(\xi)$, as $t \to \infty$, for an appropriate choice of α and β and a nondegenerate limit $w_\infty(\xi)$. Note that if $C \neq 0$ this is possible only if $(t+1)^\beta e^{\alpha t} \to 0$, as $t \to \infty$. Following [4] we can find a probabilistic interpretation of w_∞. Using the partial differential equation for $(w_t)_{t\geq 0}$ we see that w_∞ ought to satisfy

$$-\alpha\xi \frac{\partial}{\partial \xi} w_\infty(\xi) = \widehat{\mathscr{Q}}_0 w_\infty(\xi) - w_\infty(\xi). \tag{1.4}$$

It is worth noting that this equation does not depend on the value of β. The probabilistic interpretation becomes evident after rewriting the above relation as

$$w_\infty(\xi) = \int_0^1 \widehat{\mathscr{Q}}_0 w_\infty(s^{-\alpha}\xi)\,\mathrm{d}s = \mathbb{E}\left[\prod_{k=1}^N w_\infty(L^{-\alpha}A_k\xi)\right],$$

where L is distributed uniformly on $(0,1)$ and is independent of $(N, A_1, A_2, \ldots)$. As verified by our Theorems 2.7 and 2.8, w_∞ is the characteristic function of a fixed point of the modified *homogeneous* smoothing transform

$$\kappa \mapsto \mathrm{Law}\left(L^{-\alpha}\sum_{k=1}^N A_k X_k\right),$$

where $(X_k)_{k\in\mathbb{N}}$ is a sequence of independent random variables with the common distribution κ, which are also independent of the vector $(N, A_1, A_2, \ldots)$. It is worth noting that the equation (1.4) is homogeneous in the argument in the sense that if w_∞ satisfies (1.4), then any function of the form $\xi \mapsto w_\infty(c\xi)$ also satisfies it, for $c \in \mathbb{R}$. In order to characterize w_∞ it is therefore important to find an appropriate boundary condition. As it is verified by our main results the boundary condition for w_∞, that is, the value of $\partial_\xi w_\infty(0)$ depends on the interplay between the initial condition ϕ_0, the expectation $\mathbb{E}[C]$ and the generalized Laplace transform $s \mapsto \mathbb{E}\left[\sum_{k=1}^N A_k^s\right]$.

The main results of the present article show the convergence $w_t \to w_\infty$ for an appropriate choice of α and β. We achieve that by finding a stochastic process $W = (W_t)_{t\geq 0}$ such that ϕ_t is the characteristic function of W_t, for every fixed $t \geq 0$. The dynamics of such process can be guessed from the integral form of (1.2) which reads

$$\phi_t(\xi) = e^{-t}\phi_0(\xi) + \int_0^t e^{-s}\widehat{\mathscr{Q}}\phi_{t-s}(\xi)\mathrm{d}s, \quad \xi \in \mathbb{R}, \quad t \geq 0.$$

Upon plugging the definition of $\widehat{\mathscr{Q}}$ we can write, with E denoting a unit mean exponential random variable independent of everything else,

$$\phi_t(\xi) = \mathbb{E}\left[\mathbb{1}_{\{E>t\}}\phi_0(\xi) + \mathbb{1}_{\{E\leq t\}} e^{i\xi C}\prod_{k=1}^N \phi_{t-E}(A_k\xi)\right], \quad \xi \in \mathbb{R}. \tag{1.5}$$

Therefore, the initial position W_0 of the process W must have the characteristic function ϕ_0, and after unit mean exponential amount of time the process is replaced by a sum of N independent copies of itself rescaled by A_k's and shifted by C. Naturally, the independent copies follow the same dynamics. This in turn gives a hint that W should be driven by a continuous-time branching process. We provide an in-depth construction of the process W in Section 2 very much in the spirit of [9] where the case $C = 0$ has been considered. Note, however, that even in this case our construction is new, since, unlike in [9], we allow N to be random.

The article is organized as follows. In Section 2 we formulate our assumptions and present the main results. In particular, we shall prove an existence and uniqueness result for the solutions to (1.2) by constructing a process W

which yields a probabilistic representation of the solution to (1.2). The corresponding proof is given in Section 3. Many-to-one lemmas, recalled in Section 4, will be utilized for showing the convergence of the stochastic process W in Section 5. This in turn allows us to give short proofs of our main results in Section 6. Throughout the paper we use the following notation. The Dirac measure at $x \in \mathbb{R}$ is denoted by δ_x. Convergence (respectively, equality) in distribution is denoted by $\xrightarrow{d}$ (respectively, $\stackrel{d}{=}$). The notation $\xrightarrow{\mathbb{P}}$ stands for convergence in probability.

2. Main results

We shall first provide some further necessary notation and formulate precisely our assumptions. Then we proceed by giving the probabilistic representation for solutions to (1.2) using a continuous-time branching random walk. In particular, this result yields that such solution exists and is unique. Next, we shall formulate our main results on the asymptotic behaviour of ϕ_t, as $t \to \infty$.

2.1. **Existence and uniqueness.** As mentioned in the introduction, ϕ_t solving (1.2) can be linked with a continuous-time branching process which we shall now describe. In order to avoid an explosion in the branching process, and thus in ϕ_t, we shall impose the classical non-explosion hypothesis on the distribution of N, see [15, Theorem V.9.1]:

$$\int_{1-\varepsilon}^{1} \frac{\mathrm{d}s}{s - \mathbb{E}[s^N]} = \infty \quad \text{for all sufficiently small } \varepsilon > 0. \tag{2.1}$$

Note that $\mathbb{E}[N] < \infty$ is sufficient for (2.1). In what follows we also assume that, given N, the weights A_1, A_2 are a.s. positive and are not degenerate at 1. More precisely, we suppose that for every $m \in \mathbb{N}$

$$\mathbb{P}[A_1 > 0, A_2 > 0, \ldots, A_m > 0 | N = m] = 1 \text{ and } \mathbb{P}[A_1 = 1, A_2 = 1, \ldots, A_m = 1 | N = m] < 1. \tag{2.2}$$

For the sake of transparency we shall further suppose that

$$\mathbb{E}[N] \in (1, \infty], \tag{2.3}$$

which is the standard supercriticality condition and which ensures that our branching process survives with a positive probability. Note that if $\mathbb{E}[N] \leq 1$ then either $\mathbb{P}[N = 1] = 1$ and the branching is degenerate, or $\mathbb{P}[N = 1] < 1$, $\mathbb{E}[N] \leq 1$ and the process dies out with probability one. The equation (1.1) in both cases can be treated using our methods.

We now turn our attention to the probabilistic interpretation of the solution to (1.2) which we shall use to establish the existence and uniqueness result. We hinge on ideas presented in [9]. More precisely we shall work with a (marked) continuous-time branching random walk which can be described in the following fashion. At time $t = 0$ one particle is placed at the origin of the real line $\mathbb{R}$. After a random amount of time, distributed according to the unit mean exponential law, the particle splits into a random number N of new particles which are placed at random points of $\mathbb{R}$ given by a point process

$$\zeta = \sum_{k=1}^{N} \delta_{Z_k}, \qquad Z_k = \log A_k.$$

From here each particle reproduces in exactly the same way independently from other particles. In particular, the relative positions of particles with respect to their mother are distributed according to a copy of ζ. Denote by $\mathscr{T}_\infty \subseteq \bigcup_{n \geq 0} \mathbb{N}^n$, where $\mathbb{N}^0 := \{\varnothing\}$, the (full) Galton-Watson tree of the underlying population with the Ulam-Harris labelling, that is, we label the root with $\varnothing$ and for each $x \in \mathscr{T}_\infty$ we label its children with xi where $1 \leq i \leq N(x)$, with

$N(x)$ denoting the number of children of x. To model the dynamics of the process suppose that each vertex $x \in \mathscr{T}_\infty$ is equipped with $(E(x),U(x),C(x),Z_1(x),Z_2(x),\ldots)$ which is an independent copy of $(E,U,C,Z_1,Z_2,\ldots)$, where E denotes a unit mean exponential random variable independent of $(U,C,Z_1,Z_2,\ldots)$ and U is a random variable independent of $(E,C,Z_1,Z_2,\ldots)$ with a law that will be specified later. Suppose that the triple $(E(x),U(x),C(x))$ is attached to the vertex $x \in \mathscr{T}_\infty$ and $Z_i(x)$ is attached to the edge between x and xi. For every particle $x \in \mathscr{T}_\infty$ we define its time of birth $B(x)$ and the time of death $D(x)$ by the formulas

$$B(\varnothing) = 0, \qquad B(x) = \sum_{k=0}^{|x|-1} E(x_{|k}), \quad |x| \geq 1, \quad D(x) = \sum_{k=0}^{|x|} E(x_{|k}) = B(x) + E(x), \quad x \in \mathscr{T}_\infty,$$

where $x_{|k}$ is the k-th vertex in the unique path from the root $\varnothing$ to x and $|x|$ denotes the length of this path. Note that $E(x)$ is therefore interpreted as the lifetime of particle $x \in \mathscr{T}_\infty$. Denote by $\mathscr{T}_t \subseteq \mathscr{T}_\infty$ the tree discovered up to time $t > 0$, given by

$$\mathscr{T}_t = \{x \in \mathscr{T}_\infty : B(x) \leq t\}.$$

The set of particles alive at time t is given by

$$\partial\mathscr{T}_t = \{x \in \mathscr{T}_\infty : B(x) \leq t < D(x)\}.$$

The relative complement of $\partial\mathscr{T}_t$, that is

$$\mathscr{T}_t^o = \mathscr{T}_t \setminus (\partial\mathscr{T}_t) = \{x \in \mathscr{T}_\infty : D(x) \leq t\},$$

represents the particles removed from the system by time t. Both collections $\partial\mathscr{T}_t$ and $\mathscr{T}_t^o$ will play a significant role in the forthcoming construction of the probabilistic representation of the solutions to (1.2). A possible realization of the marked branching process described above is depicted on Figure 1.

A continuous-time branching random walk is a branching process with a spatial component. The position $V(x)$ of $x \in \mathscr{T}_\infty$ is obtained by summing the edge weights along the unique path from the root $\varnothing$ to x. Thus, for $x \in \mathscr{T}_\infty$,

$$V(x) = \sum_{k=1}^{|x|} Z_{x_{|k}}(x_{|k-1}).$$

Therefore, the positions of particles present in the system at time $t \geq 0$ are given by the point process

$$\mathscr{Z}_t = \sum_{x \in \partial\mathscr{T}_t} \delta_{V(x)}.$$

The measure-valued process $\mathscr{Z} = (\mathscr{Z}_t)_{t\geq 0}$ is called a continuous time branching random walk.

Note that the variables $(U(x),C(x))_{x\in\mathscr{T}_\infty}$ did not play a role so far in the construction of $\mathscr{Z}$. We shall now make use of them for two markings. The first one is given via an independent marking of $\mathscr{Z}$ with independent copies of U, namely

$$\mathscr{X}_{U,t} = \sum_{x \in \partial\mathscr{T}_t} \delta_{V(x)} \otimes \delta_{U(x)}, \quad t \geq 0.$$

The point process $(\mathscr{X}_{U,t})_{t\geq 0}$ will be mainly used to control the contribution of the initial value (by taking $U = R$ with R distributed according to ν_0), but it will be also applied in a slightly different aspect (with a differently distributed U) as well.

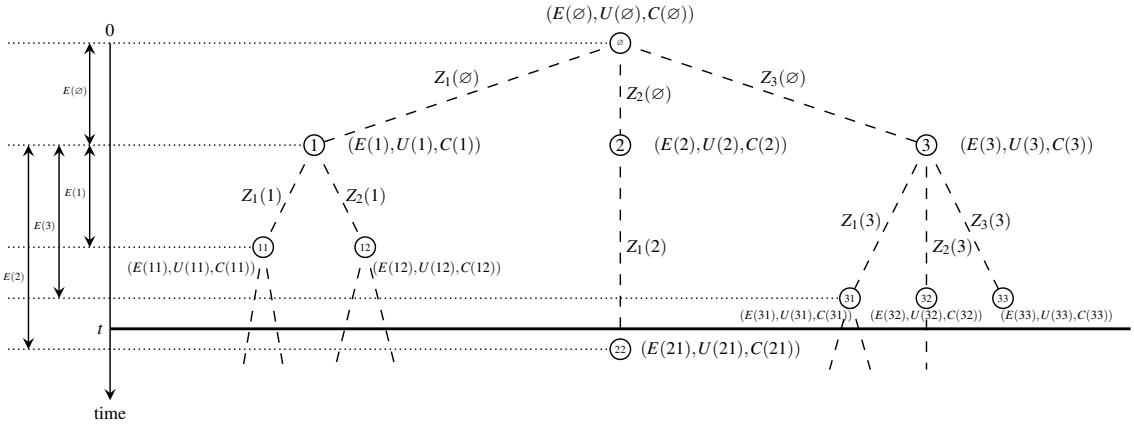

FIGURE 1. A realization of a marked continuous-time branching random walk. The horizontal thick line represents a fixed time $t > 0$. Note that no dashed lines emanating from the vertex 33 downward means that $N(33) = 0$. On the figure it is also assumed that $E(\varnothing) + E(3) + E(33) > t$, that is, the particle 33 dies after time t. The tree discovered up to time t is $\mathscr{T}_t = \{\varnothing, 1, 2, 3, 11, 12, 31, 32, 33\}$ and consists of all vertices above the thick line. The set of particles alive at time t is $\partial\mathscr{T}_t = \{2, 11, 12, 31, 32, 33\}$ and consists of vertices above the thick line which have not split above this line. The set of particles removed from the system by time t is $\mathscr{T}_t^\circ = \{\varnothing, 1, 3\}$.

The second component needed to construct ϕ_t controls the contribution of the inhomogeneous terms (shifts) $C(x)$. Put

$$\mathscr{C}_t = \sum_{x \in \mathscr{T}_t^o} \delta_{V(x)} \otimes \delta_{C(x)}, \quad t \geq 0.$$

Finally, define $\mathscr{W} = (\mathscr{W}_t)_{t \geq 0}$ via

$$\mathscr{W}_t = \mathscr{X}_{R,t} + \mathscr{C}_t, \quad t \geq 0, \tag{2.4}$$

where R is distributed according to ν_0.

Our first main result provides an explicit form of a unique solution to (1.2) which turns out to be the characteristic function of a certain integral with respect to the point process $\mathscr{W}_t$.

Theorem 2.1. *Assume* (2.1), (2.2) *and* (2.3). *Then there exists a unique solution to* (1.2) *given by* $\phi_t(\xi) = \mathbb{E}\exp\{i\xi W_t\}$, *where*

$$W_t = \int e^v z \, \mathscr{W}_t(\mathrm{d}v, \mathrm{d}z)$$

and $\mathscr{W}_t$ *is defined by* (2.4).

The proof of Theorem 2.1 given in Section 3 utilizes (1.5) and the branching properties of $\mathscr{X}_{U,t}$ and $\mathscr{C}_t$ which will also be discussed in Section 3.

2.2. **Asymptotics.** We shall now show how one can use the asymptotic behaviour of $\mathscr{W}_t$ to show that ϕ_t, appropriately scaled, possesses a non-trivial limit as $t \to \infty$. It turns out that the underlying branching process, by a change of

measure argument, is related to a compound Poisson process. The asymptotic of the latter is described in terms of the function $\Phi : [0,\infty) \to [-1,+\infty]$ defined by

$$\Phi(s) = \mathbb{E}\left[\sum_{k=1}^{N} A_k^s\right] - 1, \quad s \geq 0.$$

Let $\mathscr{D} = \left\{t > 0 : \mathbb{E}\left[\sum_{k=1}^{N} A_k^t\right] < \infty\right\}$ and assume that $s_\infty = \sup \mathscr{D}$ is strictly positive (possibly, equal to $+\infty$). Note that $\Phi(0) = \mathbb{E}[N] - 1$ and, particularly, $\mathbb{E}[N] < \infty$ iff $\Phi(0) < \infty$. Moreover, by the merit of Hölder's inequality, $\mathscr{D}$ is a convex set. The function Φ is continuous and strictly convex on the open interval $\mathrm{int}(\mathscr{D})$.

The fundamental object in our context is the so-called additive martingale

$$M_t(\gamma) = e^{-\Phi(\gamma)t} \int e^{\gamma v}\, \mathscr{Z}_t(\mathrm{d}v), \qquad \gamma \in (0, s_\infty). \tag{2.5}$$

If $\Phi(\gamma) > \gamma\Phi'(\gamma)$, then under some mild moment assumptions, see Lemma 5.1 below, $M_t(\gamma)$ converges almost surely and in L_1 to a random variable $M_\infty(\gamma)$ which is almost surely positive on the set $\{|\mathscr{T}_\infty| = \infty\}$. If, on the other hand, $\Phi(\gamma) \leq \gamma\Phi'(\gamma)$ then $M_t(\gamma)$ still has a nontrivial limit in a weaker sense (in probability or distribution) after a polynomial rescaling. The martingale limits $M_\infty(\gamma)$ will appear naturally in our arguments.

As $\mathscr{W}_t = \mathscr{X}_{R,t} + \mathscr{C}_t$ it is convenient to express W_t as a sum of two terms first of which being of the form $X_{R,t}$, for

$$X_{U,t} = \int e^{v} u\, \mathscr{X}_{U,t}(\mathrm{d}v, \mathrm{d}u), \quad t \geq 0, \tag{2.6}$$

where U denotes some random variable (we use here U instead of R, because this notation will be applied below in various contexts). In our settings we shall scale this processes using the so-called spectral function given by

$$\mu(s) = \Phi(s)/s, \quad s \in (0, s_\infty).$$

Since Φ is strictly convex, the spectral function μ can possess at most one minimizer γ^*. If it exists, it is then the unique solution to the equation $\mu'(\gamma^*) = 0$, or equivalently $\Phi(\gamma^*) = \gamma^*\Phi'(\gamma^*)$. We shall mainly consider values of γ smaller than γ^*. However, since we do not assume existence of γ^*, instead of referring to the inequality $\gamma < \gamma^*$, we shall use the condition $\mu'(\gamma) < 0$, or equivalently $\Phi(\gamma) > \gamma\Phi'(\gamma)$. The latter condition will allow us to control the dynamics of $X_{R,t}$ provided (roughly speaking) that the distribution of R lies in the domain of normal attraction of a stable law of index $\gamma \in (0,2]$. More specifically (and similarly to [5, 9]) in some cases we shall work under the following assumptions imposed on the initial condition ν_0. We say that condition (H_γ) is satisfied if:

- for $\gamma = 1$ one of the following conditions (a) or (b) is fulfilled:
  (a) $\int |x|\nu_0(\mathrm{d}x) < \infty$ and in this case we put $m_0 = \int x\nu_0(\mathrm{d}x)$;
  (b) we have

$$\lim_{x\to+\infty} x\nu_0(x,+\infty) = \lim_{x\to-\infty} |x|\nu_0(-\infty,x) = c_0 \in (0,\infty),$$

  and

$$\lim_{T\to+\infty} \int_{-T}^{T} x\nu_0(\mathrm{d}x) = m_0 \in (-\infty,\infty);$$

- for $\gamma = 2$, $\sigma_0^2 = \int x^2\nu_0(\mathrm{d}x) < \infty$ and $\int x\nu_0(\mathrm{d}x) = 0$;

- for $\gamma \in (0,1)\cup(1,2)$ it holds

$$\lim_{x\to+\infty} x^{\gamma}\nu_0(x,+\infty) = c_0^+, \quad \lim_{x\to-\infty} |x|^{\gamma}\nu_0(-\infty,x) = c_0^-$$

with $c_0^+ + c_0^- > 0$. If $\gamma > 1$, then furthermore $\int x\nu_0(\mathrm{d}x) = 0$.

The process $(X_{R,t})_{t\geq 0}$ is related to the homogeneous kinetic equation. More precisely, consider the equation

$$\begin{cases} \partial_t \tilde{\phi}_t = \tilde{\mathscr{Q}}\tilde{\phi}_t - \tilde{\phi}_t, \\ \tilde{\phi}_0(\xi) = \phi_0(\xi) = \mathbb{E}[\exp\{i\xi R\}]; \end{cases} \tag{2.7}$$

where $\tilde{\mathscr{Q}}$ is the homogeneous smoothing transform obtained from (1.3) by putting $C := 0$, that is,

$$\tilde{\mathscr{Q}}\phi(\xi) = \mathbb{E}\left[\prod_{k=1}^{N} \phi(A_k\xi)\right].$$

Under mild hypotheses, see Proposition 2.5 in [9], $\tilde{\phi}_t(\xi) = \mathbb{E}[\exp\{i\xi X_{R,t}\}]$ is a unique solution to (2.7) and its asymptotic depends on the sign of $\mu'(\gamma)$ (or in other words on the mutual position of γ and γ^*, if γ^* exists). We state below results proved[1] in [4, 9, 10]. Assume that condition (H_γ) holds.

- If $\mu'(\gamma) < 0$, then

$$e^{-\mu(\gamma)t} X_{R,t} \xrightarrow{d} X_\infty, \qquad t\to\infty. \tag{2.8}$$

- If $\mu'(\gamma) = 0$, then

$$t^{\frac{1}{2\gamma}} e^{-\mu(\gamma)t} X_{R,t} \xrightarrow{d} X_\infty, \qquad t\to\infty. \tag{2.9}$$

- If $\mu'(\gamma) > 0$, and γ^* exists, then

$$t^{\frac{3}{2\gamma^*}} e^{-\mu(\gamma^*)t} X_{R,t} \xrightarrow{d} X_\infty, \qquad t\to\infty. \tag{2.10}$$

Let us emphasize here that in the sequel we shall use only (2.8), but we present here all possible results for $X_{R,t}$ to provide the reader with a general overview. In all the cases the limit satisfies the following stochastic fixed-point equation

$$X_\infty \stackrel{d}{=} L^{\mu(\gamma\wedge\gamma^*)} \sum_{k=1}^{N} A_k X_{k,\infty},$$

where $(X_{k,\infty})_{k\in\mathbb{N}}$ are independent copies of X_∞, L is uniformly distributed on $(0,1)$ and $(N,A_1,A_2,\ldots)$, L, $(X_{k,\infty})_{k\in\mathbb{N}}$ are mutually independent. All the above results can be stated in terms of the characteristic functions and since the most relevant for us will be the regime when (2.8) holds, we provide further details only in this case. Let

$$\hat{g}_\gamma(\xi) = \begin{cases} e^{im_0\xi}, & \text{if } \gamma = 1 \text{ and } (H_1(a)) \text{ holds}, \\ e^{im_0\xi} - \pi c_0|\xi|, & \text{if } \gamma = 1 \text{ and } (H_1(b)) \text{ holds}, \\ e^{-\sigma_0^2\xi^2/2}, & \text{if } \gamma = 2 \text{ and } (H_2) \text{ holds}, \\ e^{-k_0|\xi|^\gamma(1-ib_0\tan(\pi\gamma/2))\mathrm{sgn}(\xi)}, & \text{if } \gamma\in(0,1)\cup(1,2) \text{ and } (H_\gamma) \text{ holds}; \end{cases}$$

where

$$k_0 = (c_0^+ + c_0^-)\frac{\pi}{2\Gamma(\gamma)\sin(\pi\gamma/2)}, \quad b_0 = \frac{c_0^+ - c_0^-}{c_0^+ + c_0^-}.$$

[1]Note that in [4, 9] the authors assume that N is a constant, however this hypothesis is superfluous and can be easily replaced by (2.1) and (2.3).

Then convergence (2.8) is equivalent to

$$\tilde{\phi}_t(e^{-\mu(\gamma)t}\xi) = \mathbb{E}\big[e^{i\xi e^{-\mu(\gamma)t}X_t}\big] \to \tilde{\phi}_\infty(\xi) = \mathbb{E}\big[e^{i\xi X_\infty}\big], \qquad t\to\infty,$$

where the limiting function has a representation in terms of the martingale limit

$$\tilde{\phi}_\infty(\xi) = \mathbb{E}\big[e^{i\xi X_\infty}\big] = \mathbb{E}\big[\hat{g}_\gamma(\xi M_\infty^{1/\gamma}(\gamma))\big], \quad \xi\in\mathbb{R}, \tag{2.11}$$

see Theorem 2.2 in [4].

Recalling the decomposition $\mathscr{W}_t = \mathscr{X}_{R,t} + \mathscr{C}_t$ we shall now turn to the second term of W_t, that is, $C_t = W_t - X_{R,t}$ which is given by

$$C_t = \int e^v c\,\mathscr{C}_t(\mathrm{d}v,\mathrm{d}c), \quad t\geq 0. \tag{2.12}$$

Taking into account the above results for $X_{R,t}$, to describe the behaviour of W_t for large t, it is necessary to understand the asymptotic behaviour C_t. We summarize the results in the following two theorems. The first one treats the case when C has a non-zero mean. In this case we assume

$$\mathbb{E}[C]\neq 0,\ \mathbb{E}|C|^p<\infty \text{ for some } p>1,\ s_\infty>1 \text{ and } \mu'(1)<0. \tag{2.13}$$

We shall consider three subcases:

(A) $\Phi(1)>0$ and (2.13) holds;
(B) $\Phi(1)=0$ and (2.13) holds;
(C) $\Phi(1)<0$ and (2.13) holds.

Theorem 2.2. *Assume* (2.1)*,* (2.2) *and* (2.3)*. The following limit relations hold true:*

(*a*) *under condition* (*A*)*,*

$$e^{-\mu(1)t}C_t \xrightarrow{\mathbb{P}} \frac{\mathbb{E}[C]}{\Phi(1)}M_\infty(1), \quad t\to\infty;$$

(*b*) *under condition* (*B*)*,*

$$\frac{1}{t}C_t \xrightarrow{\mathbb{P}} \mathbb{E}[C]M_\infty(1), \quad t\to\infty;$$

(*c*) *under condition* (*C*)*,*

$$C_t \xrightarrow{\mathbb{P}} C_\infty := \sum_{x\in\mathscr{T}_\infty} e^{V(x)}C(x), \qquad t\to\infty,$$

where C_∞ is a.s. finite and satisfies the equality in distribution

$$C_\infty \overset{d}{=} \sum_{k=1}^N A_k C_{k,\infty} + C, \tag{2.14}$$

with $(C_{k,\infty})_{k\in\mathbb{N}}$ being independent copies of C_∞ also independent of $(C,N,A_1,A_2,\ldots)$.

To state the results in the zero-mean case we need the following sets of conditions:

(D) $\mathbb{E}[C]=0$, $\mathbb{E}[|C|^p]<\infty$ and $\Phi(p)<0$ for some $p\in(1,2]$, and $\mu'(1)<0$;
(E) $\mathbb{E}[C]=0$, $s_\infty>2$, $\Phi(2)>0$, $\mu'(2)<0$ and $\mathbb{E}[C^2]\in(0,\infty)$.

Theorem 2.3. *Suppose that* (2.1)*,* (2.2) *and* (2.3) *are satisfied. Then,*

(*d*) *under condition* (*D*)*,*

$$C_t \xrightarrow{\mathbb{P}} C_\infty, \qquad t\to\infty.$$

(*e*) *under condition* (E),

$$e^{-\mu(2)t}C_t \xrightarrow{d} \sqrt{M_\infty(2)(\mathbb{E}[C^2])/\Phi(2)}\,\mathrm{N}, \quad t\to\infty,$$

where N *is a standard normal random variable independent of the martingale limit* $M_\infty(2)$.

Remark 2.4. The inhomogeneous stochastic fixed-point equation of the form (2.14) has received considerable attention in the past decades. For example, the set of solutions has been characterized in [1] (case of nonnegative C) and [2] (two-sided case). In particular, under the assumptions (C) or (D), Theorem 2.3 in [2] implies that the set of solutions is indeed nonempty and provides a description of all the solutions. The variable C_∞ appearing in Theorems 2.2 and 2.3 is a particular solution.

Theorems 2.2 and 2.3 will be proved in Section 5 with the only exception being the case (E) in Theorem 2.3. The latter will follow from case (E) in Theorem 2.5 by taking $R=0$, that is $\nu_0=\delta_0$.

In order to formulate our results concerning asymptotic behavior of W_t one needs to compare the influence of C_t (Theorems 2.2 and 2.3) with $X_{R,t}$ (formulas (2.8) – (2.10)). Observe that our results concerning the asymptotic behaviour of C_t are stated either under the assumption $\mu'(1)<0$ or under the assumption $\mu'(2)<0$. In both cases $\mu(\gamma^*)$ (assuming that γ^* exists) is strictly smaller than $\mu(1)$ or $\mu(2)$. Thus, in the regimes (2.9) and (2.10) C_t always grows faster than $X_{R,t}$. Furthermore, this still holds true if we replace the condition (H_γ) (needed, for (2.9) and (2.10)) with an appropriate moment assumption on ν_0. Summarizing, we shall only consider below $X_{R,t}$ either in the regime (2.8), that is, when $\mu'(\gamma)<0$ (Theorems 2.7 and 2.8) or under some moment assumptions on ν_0 ensuring that C_t grows faster than $X_{R,t}$ (Theorem 2.8).

Recall that Theorem 2.1 implies $\phi_t(\xi)=\mathbb{E}\exp\{i\xi W_t\}=\mathbb{E}\exp\{i\xi(X_{R,t}+C_t)\}$. We start with the case when influence of both ingredients C_t and $X_{R,t}$ is comparable. The following result will be proved in Section 6.

Theorem 2.5. *Assume that* (2.1), (2.2) *and* (2.3) *are satisfied.*

- *If conditions* (A) *and* (H_1) *are satisfied, then*

$$\phi_t(e^{-\mu(1)t}\xi)=\mathbb{E}\big[e^{i\xi e^{-\mu(1)t}W_t}\big]\to\mathbb{E}\big[\hat g_1(\xi M_\infty(1))e^{i\xi\mathbb{E}[C]/\Phi(1)\cdot M_\infty(1)}\big], \qquad t\to\infty.$$

- *If conditions* (C) *and* (H_γ) *are satisfied for some* $\gamma<1$ *such that* $\Phi(\gamma)=0$, *then*

$$\phi_t(\xi)=\mathbb{E}\big[e^{i\xi W_t}\big]\to\mathbb{E}\big[\hat g_\gamma(\xi M_\infty^{1/\gamma}(\gamma))e^{i\xi C_\infty}\big], \quad t\to\infty,$$

- *If conditions* (D) *and* (H_γ) *are satisfied for some* $\gamma<p$ *such that* $\Phi(\gamma)=0$, *then*

$$\phi_t(\xi)=\mathbb{E}\big[e^{i\xi W_t}\big]\to\mathbb{E}\big[\hat g_\gamma(\xi M_\infty^{1/\gamma}(\gamma))e^{i\xi C_\infty}\big], \quad t\to\infty,$$

- *If conditions* (E) *and* (H_2) *are satisfied, then*

$$\phi_t(e^{-\mu(2)t}\xi)=\mathbb{E}[e^{i\xi e^{-\mu(2)t}W_t}]\to\mathbb{E}\exp\left\{-\xi^2\left(M_\infty(2)(\sigma_0^2+\mathbb{E}[C^2]/\Phi(2)\right)/2\right\}, \quad t\to\infty.$$

Remark 2.6. Note that if condition (B) holds, then C_t/t converges in probability and there is no regime in which $X_{R,t}$ also grows linearly meaning that C_t and $X_{R,t}$ cannot be of the same magnitude. This explains why the case (B) is excluded in Theorem 2.5.

Now we consider the case when $X_{R,t}$ dominates over C_t and thus determines asymptotic behavior of W_t. Our next result is a direct consequence of the Slutsky theorem.

Theorem 2.7. *Assume* (2.1)*,* (2.2) *and* (2.3)*. Suppose further that that one of the following set of conditions holds:*

- (A) *and* (H_γ) *for* $\gamma < 1$*;*
- (B) *and* (H_γ) *for* $\gamma < 1$*;*
- (C) *and* (H_γ) *for* $\gamma < 1$ *and* $\Phi(\gamma) > 0$*;*
- (D) *and* (H_γ) *for* $\gamma < p$ *and* $\Phi(\gamma) > 0$*;*
- (E) *and* (H_γ) *for* $\gamma < 2$*.*

Then

$$\phi_t(\xi e^{-\mu(\gamma)t}) \to \mathbb{E}\exp(i\xi X_\infty), \qquad t \to \infty,$$

where the characteristic function on the right-hand side is given by (2.11)*.*

Finally, when C_t dominates, we do not need to refer to conditions (H_γ) and, instead, finiteness of certain moments of R is sufficient.

Theorem 2.8. *Assume that* (2.1)*,* (2.2) *and* (2.3) *are satisfied. Then,*

- *If (A) holds and, further,* $\mathbb{E}[R] = 0$ *and* $\mathbb{E}[|R|^{1+\delta}] < \infty$ *for some* $\delta > 0$*, then*

$$e^{-\mu(1)t}W_t \xrightarrow{\mathbb{P}} \frac{\mathbb{E}[C]}{\Phi(1)}M_\infty(1) \quad \text{and} \quad \phi_t(e^{-\mu(1)t}\xi) \to \mathbb{E}\exp(i\xi(\mathbb{E}[C])M_\infty(1)/\Phi(1)), \quad t \to \infty.$$

- *If (B) holds and, further, either condition* $(H_1(b))$ *holds, or* $\mathbb{E}[R] = 0$ *and* $\mathbb{E}[|R|^{1+\delta}] < \infty$ *for some* $\delta > 0$*, then*

$$\frac{1}{t}W_t \xrightarrow{\mathbb{P}} \mathbb{E}[C]M_\infty(1) \quad \text{and} \quad \phi_t(t^{-1}\xi) \to \mathbb{E}\exp(i\xi(\mathbb{E}[C])M_\infty(1)), \quad t \to \infty.$$

- *If (C) holds and, further, there exists* $\gamma < 1$ *such that* $\Phi(\gamma) < 0$ *and* $\mathbb{E}[|R|^\gamma] < \infty$*, then*

$$W_t \xrightarrow{\mathbb{P}} C_\infty \quad \text{and} \quad \phi_t(\xi) \to \mathbb{E}\exp(i\xi C_\infty), \quad t \to \infty.$$

- *If (D) holds and, further, there exists* $\gamma < p$ *such that* $\Phi(\gamma) < 0$*,* $\mathbb{E}|R|^\gamma < \infty$ *and* $\mathbb{E}[R] = 0$*, if* $\gamma > 1$*, then*

$$W_t \xrightarrow{\mathbb{P}} C_\infty \quad \text{and} \quad \phi_t(\xi) \to \mathbb{E}\exp(i\xi C_\infty), \quad t \to \infty.$$

Note that there is no case (E) in Theorem 2.8 since it is essentially contained in the fourth part of Theorem 2.5. We provide a short proof of Theorem 2.8 in Section 6.

3. Utilizing the branching property: proof of Theorem 2.1

In this section we shall demonstrate how one can use the branching property of continuous-time branching random walks to prove the existence and uniqueness of solutions to (1.2). The key fact that we are going to use throughout this section is that given $\mathscr{T}_t$ the collections $((E(x),U(x),C(x),Z_1(x),Z_2(x),\ldots))_{x\in\mathscr{T}_t,x\geq y}$ for $y \in \partial\mathscr{T}_t$ are conditionally i.i.d.

The process $Y_t = |\partial\mathscr{T}_t|$ which counts the number of particles in $\partial\mathscr{T}_t$ is usually called a continuous-time branching process or a Yule process, see [3, Chapter III]. Note that, according to (2.1), Y_t is finite a.s. for each $t \geq 0$. Moreover, by the memoryless property of the exponential distribution, $Y = (Y_t)_{t\geq 0}$ is a Markov process with the jump rates

$$\mathbb{P}[Y(t+\mathrm{d}t) = k \,|\, Y(t) = j] = j\mathbb{P}[N = k-j+1]\mathrm{d}t + o(\mathrm{d}t) \tag{3.1}$$

for $k \geq j-1$.

We shall now state the branching property for $(\mathscr{X}_{U,t})_{t\geq 0}$ and $(\mathscr{C}_t)_{t\geq 0}$ that will be utilized in the proof of Theorem 2.1. First, note that for $t,s>0$, $\mathscr{X}_{U,t+s}$ can be written in the following fashion

$$\mathscr{X}_{U,t+s}=\sum_{y\in\partial\mathscr{T}_s}\sum_{x\in\partial\mathscr{T}_{t+s},y\leq x}\delta_{V(x)-V(y)+V(y)}\otimes\delta_{U(x)}.$$

Thus, if we denote for $y\in\partial\mathscr{T}_s$

$$\mathscr{X}^{(y)}_{U,s,s+t}=\sum_{x\in\partial\mathscr{T}_{t+s},y\leq x}\delta_{V(x)-V(y)}\otimes\delta_{U(x)},$$

then the previous relation takes the form

$$\mathscr{X}_{U,t+s}(\cdot,\cdot)=\sum_{y\in\partial\mathscr{T}_s}\mathscr{X}^{(y)}_{U,s,s+t}(\cdot-V(y),\cdot). \tag{3.2}$$

The key feature of this representation is the following property. For every fixed $s\geq 0$, given $\partial\mathscr{T}_s$, the point processes $(\mathscr{X}^{(y)}_{U,s,s+t}(\cdot,\cdot))_{t\geq 0}$, $y\in\partial\mathscr{T}_s$ are independent copies of $\mathscr{X}_{U,t}$. In other words, $\mathscr{X}_{U,t+s}$ is a sum of $|\partial\mathscr{T}_s|$ (conditionally) independent copies of $\mathscr{X}_{U,t}$ appropriately shifted. The process $\mathscr{C}=(\mathscr{C}_t)_{t\geq 0}$ also enjoys a version of the branching property. For $s,t\geq 0$ it holds

$$\mathscr{C}_{t+s}=\mathscr{C}_s(\cdot,\cdot)+\sum_{y\in\partial\mathscr{T}_s}\mathscr{C}^{(y)}_{s,s+t}(\cdot-V(y),\cdot) \tag{3.3}$$

with

$$\mathscr{C}^{(y)}_{s,s+t}=\sum_{x\in\partial\mathscr{T}_{t+s},y\leq x}\delta_{V(x)-V(y)}\otimes\delta_{C(x)},\qquad y\in\partial\mathscr{T}_s.$$

Once again, for every fixed $s\geq 0$, given $\partial\mathscr{T}_s$, $\mathscr{C}^{(y)}_{s,s+t}$ is an independent copy of $\mathscr{C}_t$, for every $y\in\partial\mathscr{T}_s$.

Using (3.2) and (3.3) we can write, with $\mathscr{W}^{(y)}_{s,s+t}=\mathscr{X}^{(y)}_{R,s,s+t}+\mathscr{C}^{(y)}_{s,s+t}$,

$$\mathscr{W}_{t+s}(\cdot,\cdot)=\mathscr{C}_s(\cdot,\cdot)+\sum_{y\in\partial\mathscr{T}_s}\mathscr{W}^{(y)}_{s,s+t}(\cdot-V(y),\cdot). \tag{3.4}$$

The notation introduced so far allows for a probabilistic representation of the solutions to (1.2).

Proof of Theorem 2.1. We intend to prove that the function

$$\psi_t(\xi)=\mathbb{E}\big[e^{i\xi W_t}\big],\quad t\geq 0,\quad \xi\in\mathbb{R},$$

is the unique solution to equation (1.2). Recall also that ϕ_0 is the characteristic function of R. First we prove that the function ψ_t satisfies equation (1.2). The argument follows the proof of [9, Proposition 2.5] with some slight changes. We first establish an auxiliary continuity estimate for ψ_t. Take $t>s$ and recall that under our standing assumptions Y_s is a.s. finite. Using (3.1) we write

$$|\psi_t(\xi)-\psi_s(\xi)|\leq 2\mathbb{P}[\text{there are splits during }[s,t]]\leq 2\mathbb{E}\big[1-e^{-(t-s)Y_s}\big]\to 0,\qquad s\to t,$$

by an appeal to the dominated convergence theorem. Now define the filtration $(\mathscr{F}_s)_{s\geq 0}$ via

$$\mathscr{F}_s=\sigma\left((E(x),C(x),(Z_k(x))_{k\geq 0}:x\in\mathscr{T}_s\right)$$

and use (3.4) to write for $s<t$,

$$\psi_t(\xi)=\mathbb{E}\left[\mathbb{E}\left[\exp\{i\xi W_t\}\,|\,\mathscr{F}_s\right]\right]=\mathbb{E}\left[\mathbb{E}\left[\exp\left\{i\xi C_s+\sum_{y\in\partial\mathscr{T}_s}i\xi e^{V(y)}\int e^{vz}\mathscr{W}^{(y)}_{s,t}(\mathrm{d}v,\mathrm{d}z)\right\}\Big|\mathscr{F}_s\right]\right].$$

The measurability of C_s with respect to $\mathscr{F}_s$ yields

$$\psi_t(\xi) = \mathbb{E}\left[e^{i\xi C_s}\prod_{y\in\partial\mathscr{T}_s}\psi_{t-s}\big(\xi e^{V(y)}\big)\right]. \tag{3.5}$$

We shall use this formula to study left and right derivatives of ψ_t with respect to t. For $h\in(0,t)$, by considering the number of splits that occur within the time interval $[0,h]$, we arrive at

$$\begin{aligned}\psi_t(\xi) =&\mathbb{E}\left[e^{i\xi C_h}\prod_{y\in\partial\mathscr{T}_h}\psi_{t-h}\big(\xi e^{V(y)}\big)\mathbb{1}\big\{\text{no splits during } [0,h]\big\}\right]\\ &+\mathbb{E}\left[e^{i\xi C_h}\prod_{y\in\partial\mathscr{T}_h}\psi_{t-h}\big(\xi e^{V(y)}\big)\mathbb{1}\big\{\text{one split during } [0,h]\big\}\right]+o(h)\\ =&\psi_{t-h}(\xi)e^{-h}+\mathbb{E}\left[e^{i\xi C}\prod_{k=1}^{N}\psi_{t-h}(\xi A_k)\right]h+o(h).\end{aligned}$$

Dividing both sides by h, sending $h\to 0^+$ and using the continuity estimate established at the beginning of the proof yields

$$\partial_t^-\psi_t(\xi)+\psi_t(\xi) = \mathbb{E}\left[e^{i\xi C}\prod_{k=1}^{N}\psi_t(\xi A_k)\right] = \widehat{\mathscr{Q}}\psi_t(\xi).$$

If we now replace t with $t+h$, put $s=h$ in (3.5), and then send $h\to 0^+$ we conclude that $\partial_t^+\psi_t = \partial_t^-\psi_t$. Thus, ψ is indeed a solution of (1.2) as claimed.

To prove uniqueness of the solution we adopt to our settings the arguments provided in the proof of [10, Proposition 1.4]. Let ϕ_t be any solution to (1.2). Going back to (1.5) write

$$\begin{aligned}\phi_t(\xi) &= \mathbb{P}[E(\varnothing)>t]\phi_0(\xi)+\mathbb{E}\left[\mathbb{1}_{\{E(\varnothing)\le t\}}\widehat{\mathscr{Q}}\phi_{t-E(\varnothing)}(\xi)\right]\\ &= \mathbb{E}\left[\prod_{x\in\partial\mathscr{T},|x|\le 1}\phi_0\left(e^{V(x)}\xi\right)\prod_{x\in\mathscr{T}_t^o,|x|\le 0}e^{i\xi e^{V(x)}C(x)}\prod_{x\in\mathscr{T}_t^o,|x|=1}\phi_{t-D(x)}\left(e^{V(x)}\xi\right)\right].\end{aligned}$$

Iterating the above equation n times gives

$$\phi_t(\xi) = \mathbb{E}\left[\prod_{x\in\partial\mathscr{T},|x|\le n}\phi_0\left(e^{V(x)}\xi\right)\prod_{x\in\mathscr{T}_t^o,|x|\le n-1}e^{i\xi e^{V(x)}C(x)}\prod_{x\in\mathscr{T}_t^o,|x|=n}\phi_{t-D(x)}\left(e^{V(x)}\xi\right)\right].$$

Under assumption (2.1), $\partial\mathscr{T}_t$ and $\mathscr{T}_t^o$ are both finite a.s. and therefore all three products under the expectation have a.s. finite limits as $n\to\infty$. Furthermore, the limit of the third product is one. An appeal to the dominated convergence theorem gives

$$\phi_t(\xi) = \mathbb{E}\left[\prod_{x\in\mathscr{T}_t^o}e^{i\xi e^{V(x)}C(x)}\prod_{x\in\partial\mathscr{T}_t}\phi_0\left(e^{V(x)}\xi\right)\right],\quad t\ge 0,\quad \xi\in\mathbb{R}.$$

Recalling that ϕ_0 is the characteristic function of the random variable R we obtain

$$\phi_t(\xi) = \mathbb{E}\left[\prod_{x\in\mathscr{T}_t^o}e^{i\xi e^{V(x)}C(x)}\prod_{x\in\partial\mathscr{T}_t}e^{i\xi e^{V(x)}R(x)}\right] = \mathbb{E}\big[e^{i\xi C_t}e^{i\xi X_{R,t}}\big] = \mathbb{E}\big[e^{i\xi W_t}\big] = \psi_t(\xi)$$

for $t\ge 0,\xi\in\mathbb{R}$. The proof is complete. □

4. MANY-TO-ONE LEMMAS

One of the most useful tools in the analysis of branching random walks are many-to-one lemmas. It turns out that one can express the expectation of a statistic, which depends on the entire collection of particles, in terms of a trajectory of one, tagged, particle after an appropriate change of measure. The claims of these lemmas follow from general theory (see [14, Theorem 8.2]). We shall however present the details for the readers convenience. For $\alpha \in (0, s_\infty)$ denote

$$\lambda(\alpha) = \log \mathbb{E}\Big[\sum_{|x|=1} e^{\alpha V(x)}\Big] = \log\big(\Phi(\alpha)+1\big)$$

and consider a standard zero-delayed random walk $S^{(\alpha)} = \big(S_n^{(\alpha)}\big)_{n\geq 0}$ with increments having distribution defined by the formula

$$\mathbb{E}\big[h\big(S_1^{(\alpha)}\big)\big] = \mathbb{E}\Big[\sum_{|x|=1} e^{\alpha V(x)-\lambda(\alpha)} h(V(x))\Big] \tag{4.1}$$

for any bounded measurable function $h : \mathbb{R} \to \mathbb{R}$. Furthermore, let $\eta = (\eta_t)_{t\geq 0}$ be a homogeneous Poisson process on $[0,\infty)$ given via

$$\eta_t = \#\{k \geq 1 \,:\, T_k \leq t\}, \quad t \geq 0,$$

where $T_k = \sum_{j=1}^k E_j$, $k \in \mathbb{N}$, and $(E_j)_{j\geq 1}$ is a sequence of i.i.d. random variables with the unit mean exponential distribution. Next lemma is the aforementioned many-to-one formula for Markov branching process specialized to the continuous time random walk $\big(S_{\eta_t}^{(\alpha)}\big)_{t\geq 0}$, see [14, Theorem 8.2]. In what follows $S^{(\alpha)}$, η and U are assumed to be independent.

Lemma 4.1 (Many-to-one formula I)**.** *Assume* (2.1)*,* (2.2) *and* $s_\infty > 0$*. Under the introduced notation, for any* $\alpha \in (0, s_\infty)$*,* $t \geq 0$ *and any measurable function* $g\colon \mathbb{R}^2 \to [0,\infty)$*,*

$$\mathbb{E}\left[\int g(v,x)\, \mathscr{X}_{U,t}(\mathrm{d}v,\mathrm{d}x)\right] = \mathbb{E}\Big[e^{-\alpha S_{\eta_t}^{(\alpha)}+\lambda(\alpha)\eta_t}\, g\big(S_{\eta_t}^{(\alpha)}, U\big)\Big]. \tag{4.2}$$

Proof. Define for $n \in \mathbb{N}\cup\{0,\infty\}$,

$$F_n(t,z) = \begin{cases} \mathbb{E}\big[\int g(z+v,x)\, \mathscr{X}_{U,t}^n(\mathrm{d}v,\mathrm{d}x)\big], & t \geq 0,\\ \mathbb{E}\big[g(z,U)\big], & t < 0,\end{cases}$$

where

$$\mathscr{X}_{U,t}^n = \sum_{x\in\partial\mathscr{T}_t, |x|<n} \delta_{V(x)} \otimes \delta_{U(x)}, \quad t \geq 0.$$

Then $\mathscr{X}_{U,t}^\infty = \mathscr{X}_{U,t}$ and thus $F_\infty(t,0)$ is equal to the left-hand side of (4.2). Note also that $F_0(t,z) = 0$ for $t \geq 0$. Utilizing the branching property of $\mathscr{X}_{U,t}^n$ and recalling (4.1) we obtain, for finite $n \geq 1$,

$$\begin{aligned} F_n(t,z) &= \mathbb{E}\Big[\mathbb{1}_{\{E(\varnothing)\leq t\}} \sum_{|y|=1} F_{n-1}(t-E(\varnothing), z+V(y)) + \mathbb{1}_{\{E(\varnothing)>t\}} g(z,U)\Big]\\ &= \mathbb{E}\Big[e^{-\alpha S_1^{(\alpha)}+\lambda(\alpha)} \mathbb{1}_{\{E_1\leq t\}} F_{n-1}\big(t-E_1, z+S_1^{(\alpha)}\big) + \mathbb{1}_{\{E_1>t\}} g(z,U)\Big]\\ &= \mathbb{E}\Big[\exp\big\{-\alpha S_{\eta_t^{(1)}}^{(\alpha)} + \lambda(\alpha)\eta_t^{(1)}\big\} F_{n-1}\big(t-T_1, z+S_{\eta_t^{(1)}}^{(\alpha)}\big)\Big], \end{aligned}$$

where

$$\eta_t^{(k)} = \#\big\{1 \leq j \leq k \,:\, T_j \leq t\big\}, \quad t \geq 0, \quad k \in \mathbb{N}.$$

Upon iterating n times the above formula for $F_n(t,z)$ we deduce

$$\begin{aligned} F_n(t,z) &= \mathbb{E}\Big[\exp\big\{-\alpha S^{(\alpha)}_{\eta^{(n)}_t} + \lambda(\alpha)\eta^{(n)}_t\big\} F_0\big(t-T_n, z+S^{(\alpha)}_{\eta^{(n)}_t}\big)\Big] \\ &= \mathbb{E}\Big[\exp\big\{-\alpha S^{(\alpha)}_{\eta_t} + \lambda(\alpha)\eta_t\big\} g\big(z+S^{(\alpha)}_{\eta_t}, U\big)\mathbb{1}_{\{T_n>t\}}\Big]. \end{aligned}$$

Putting $z=0$ in the above equality and sending $n\to\infty$ yields (4.2) in view of the monotone convergence theorem. □

For $t\geq 0$ and $\gamma\in\mathbb{R}$ define

$$Z_t(\gamma) = \int e^{\gamma v}\,\mathscr{Z}_t(\mathrm{d}v) = \sum_{x\in\partial\mathscr{T}_t} e^{\gamma V(x)}. \tag{4.3}$$

A direct consequence of Lemma 4.1 is the following formula, valid for $\gamma\in(0,s_\infty)$,

$$\mathbb{E}\big[Z_t(\gamma)\big] = \mathbb{E}\left[\int e^{\gamma v}\,\mathscr{Z}_t(\mathrm{d}v)\right] = e^{t\Phi(\gamma)}, \quad t\geq 0. \tag{4.4}$$

We mention in passing that the above formula combined with the branching property of $\mathscr{X}$ stated in (3.2) can be used to check that $M_t(\gamma)$ defined in (2.5) is indeed a martingale. Differentiating both sides of the above equation with respect to γ yields

$$\mathbb{E}\left[\int v e^{\gamma v}\,\mathscr{Z}_t(\mathrm{d}v)\right] = t\Phi'(\gamma)e^{t\Phi(\gamma)} \quad\text{and}\quad \mathbb{E}\left[\int v^2 e^{\gamma v}\,\mathscr{Z}_t(\mathrm{d}v)\right] = \big(t\Phi''(\gamma)+t^2(\Phi'(\gamma))^2\big)\,e^{t\Phi(\gamma)}.$$

We shall also use a simple consequence of the Marcinkiewicz–Zygmund inequality, see Theorem 10.3.2 in [12]). If $\mathbb{E}[U]=0$, then, for any $p\in[1,2]$ and $\gamma>0$ such that $p\gamma<s_\infty$, there exists a universal constant $c_p>0$ that depends only on p such that

$$\begin{aligned} \mathbb{E}[|X_{U,t}|^p] &= \mathbb{E}\left[\left|\int e^{\gamma v} u\,\mathscr{X}_{U,t}(\mathrm{d}v,\mathrm{d}u)\right|^p\right] \leq c_p\mathbb{E}\left[\left(\int e^{2\gamma v}u^2\,\mathscr{X}_{U,t}(\mathrm{d}v,\mathrm{d}u)\right)^{\frac{p}{2}}\right] \\ &\leq c_p\mathbb{E}\left[\int e^{p\gamma v}|u|^p\,\mathscr{X}_{U,t}(\mathrm{d}v,\mathrm{d}u)\right] = c_p\mathbb{E}\left[\int e^{p\gamma v}\,\mathscr{Z}_t(\mathrm{d}v)\right]\mathbb{E}\left[|U|^p\right] = c_p e^{t\Phi(p\gamma)}\mathbb{E}\left[|U|^p\right]. \end{aligned} \tag{4.5}$$

In the above chain of estimates the first one is the aforementioned Marcinkiewicz–Zygmund inequality, the second one follows from subadditivity of $x\mapsto|x|^{p/2}$, for $p\in[1,2]$, and the final equality follows from (4.4).

Our next lemma is a less standard one. In view of similarities with the previous result we shall also refer to it as a many-to-one formula.

Lemma 4.2 (Many-to-one formula II)**.** *Assume* (2.1)*,* (2.2) *and* $s_\infty>0$*. For every* $\alpha\in(0,s_\infty)$*, every measurable function* $g\colon\mathbb{R}^2\to[0,\infty)$ *and every* $t\geq 0$*,*

$$\mathbb{E}\left[\int g(v,c)\,\mathscr{C}_t(\mathrm{d}v,\mathrm{d}c)\right] = \mathbb{E}\left[\sum_{k=0}^{\eta_t-1} e^{-\alpha S^{(\alpha)}_k+\lambda(\alpha)k} g\big(S^{(\alpha)}_k, C\big)\right], \tag{4.6}$$

where C *is independent of* $S^{(\alpha)}$ *and* η *on the right-hand side.*

Proof. We shall use the same procedure as in the proof of Lemma 4.1. Define, for $n\in\mathbb{N}\cup\{0,\infty\}$,

$$H_n(t,z) = \begin{cases} \mathbb{E}\left[\int g(z+v,c)\,\mathscr{C}^n_t(\mathrm{d}v,\mathrm{d}c)\right], & t\geq 0,\\ 0, & t<0, \end{cases}$$

where

$$\mathscr{C}_t^n := \sum_{x\in\mathscr{T}_t^o,|x|<n} \delta_{V(x)}\otimes\delta_{C(x)}, \quad t\geq 0.$$

In particular, $H_0(t,z)=0$. As in the previous lemma we shall use the branching property of $\mathscr{C}_t^n$. For a finite $n\in\mathbb{N}$, it holds

$$\begin{aligned} H_n(t,z) &= \mathbb{E}\Big[\sum_{|y|=1} H_{n-1}(t-E(\varnothing),z+V(y))+g(z,C)\mathbb{1}_{\{E(\varnothing)\leq t\}}\Big] \\ &= \mathbb{E}\Big[e^{-\alpha S_1^{(\alpha)}+\lambda(\alpha)}H_{n-1}\big(t-E_1,z+S_1^{(\alpha)}\big)+\mathbb{1}_{\{E_1\leq t\}}g(z,C)\Big]. \end{aligned}$$

Iterating the above equality yields

$$\begin{aligned} H_n(t,z) &= \mathbb{E}\Big[\exp\big\{-\alpha S^{(\alpha)}_{\eta_t^{(n)}}+\lambda(\alpha)\eta_t^{(n)}\big\}H_0\big(t-T_n,z+S_n^{(\alpha)}\big)+\sum_{k=0}^{\eta_t^{(n)}-1} e^{-\alpha S_k^{(\alpha)}+\lambda(\alpha)k}g\big(z+S_k^{(\alpha)},C\big)\Big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{\eta_t^{(n)}-1} e^{-\alpha S_k^{(\alpha)}+\lambda(\alpha)k}g\big(z+S_k^{(\alpha)},C\big)\Big]. \end{aligned}$$

Taking $z=0$ and sending $n\to\infty$ yields the desired formula by an appeal to the monotone convergence theorem. □

Assume that $\mathbb{E}[|C|^p]<\infty$ for some $p>0$. Then formula (4.6) entails for $\gamma<s_\infty$,

$$\mathbb{E}\left[\int e^{\gamma v}|c|^p\,\mathscr{C}_t(\mathrm{d}v,\mathrm{d}c)\right] = \mathbb{E}\left[|C|^p\right]\cdot\begin{cases}\frac{e^{t\Phi(\gamma)}-1}{\Phi(\gamma)}, & \Phi(\gamma)\neq 0,\\ t, & \Phi(\gamma)=0,\end{cases} \tag{4.7}$$

for every $\gamma<s_\infty$. Recall that C_t is defined via (2.12) and note that, given $\mathscr{T}_t^o$ and $(V(x))_{x\in\mathscr{T}_t^o}$, the variables $(C(x))_{x\in\mathscr{T}_t^o}$ are independent copies of C. Thus, if $\mathbb{E}[C]=0$, we can apply the Marcinkiewicz–Zygmund inequality as we did in (4.5). This entails, for $q\in[1,2]$,

$$\mathbb{E}\big[|C_t|^q\big]\leq c_q\mathbb{E}\left[\int e^{qv}|c|^q\,\mathscr{C}_t(\mathrm{d}v,\mathrm{d}c)\right]. \tag{4.8}$$

5. Limits of the associated stochastic processes: proofs of Theorems 2.2 and 2.3

Having represented the solution to (1.2) as the characteristic function of the process $(W_t)_{t\geq 0}$ we shall analyse the asymptotic behaviour of the latter. Recall the decomposition $W_t=C_t+X_{R,t}$ with $X_{R,t}$ and C_t given via (2.6) and (2.12), respectively. The asymptotics of $(X_{R,t})_{t\geq 0}$ is essentially determined by the condition (H_γ) and the behaviour of the process $(M_t(\gamma))_{t\geq 0}$, defined in (2.5), which is well-known from the literature, since

$$M_t(\gamma)=e^{-\Phi(\gamma)t}\int e^{\gamma v}\,\mathscr{Z}_t(\mathrm{d}v)=e^{-\Phi(\gamma)t}Z_t(\gamma)$$

forms a so-called continuous-time additive martingale. Since it is positive, it converges a.s. to a limit as $t\to\infty$. It turns out that the limit is nondegenerate if and only if $\mu'(\gamma)<0$. Whenever $\mu'(\gamma)\geq 0$, the martingale $M_t(\gamma)$ converges to 0 a.s. and a polynomial correction term is necessary to obtain a nondegenerate limit for a rescaled $M_t(\gamma)$. We sum up this discussion in the following lemma, which follows from [6, Theorem 1.1].

Lemma 5.1. *Assume* (2.1)*,* (2.2) *and* (2.3)*. Suppose further that* $s_\infty > 0$ *and*

$$\mathbb{E}\left[\left(\sum_{k=1}^{N} A_k^\gamma\right)\log^+\left(\sum_{k=1}^{N} A_k^\gamma\right)\right] < \infty,$$

for some $\gamma \in (0, s_\infty)$*. If* $\mu'(\gamma) < 0$*, then*

$$\lim_{t\to+\infty} M_t(\gamma) = M_\infty(\gamma) \quad \text{a.s. and in } L_1.$$

Furthermore, $\mathbb{P}[M_\infty(\gamma) > 0 \mid |\mathscr{T}_\infty| = \infty] = 1$ *and also*

$$\lim_{t\to+\infty}\left(\max_{x\in\partial\mathscr{T}_t} \gamma V(x) - \Phi(\gamma)t\right) = -\infty \quad \text{a.s.} \tag{5.1}$$

Note that (5.1) holds a.s. conditionally on the event $\{|\mathscr{T}_\infty| = \infty\}$. However, on the complementary event $\{|\mathscr{T}_\infty| < \infty\}$, the set $\partial\mathscr{T}_t$ is empty for all large enough $t > 0$, and (5.1) is still true if we interpret the maximum over the empty set as $-\infty$.

We aim now to prove Theorem 2.2. To this end we apply a similar procedure as in the proofs of Proposition 3.2 in [9] and Theorem 2.3 in [13]. First, we fix some $\theta > 0$ and analyse the asymptotic behavior of the discrete skeleton $(C_{\theta n})_{n\geq 0}$. At the second step we pass to a continuous parameter by using an appropriate approximation argument. Note that for $\theta > 0$ and $n \in \mathbb{N}$, branching property (3.3) yields

$$C_{\theta(n+1)} = C_{\theta n} + \sum_{x\in\partial\mathscr{T}_{\theta n}} e^{V(x)} \int e^{v} c\, \mathscr{C}^{(x)}_{\theta n,\theta(n+1)}(\mathrm{d}v, \mathrm{d}c),$$

and $C_\theta^{(n)}(x) := \int e^v c\, \mathscr{C}^{(x)}_{\theta n,\theta(n+1)}(\mathrm{d}v,\mathrm{d}c)$, $x \in \partial\mathscr{T}_{\theta n}$, are independent copies of C_θ. Thus, recalling (4.3),

$$\begin{aligned} C_{\theta(n+1)} &= C_{\theta n} + \sum_{x\in\partial\mathscr{T}_{\theta n}} e^{V(x)} C_\theta^{(n)}(x) \\ &= C_{\theta n} + (\mathbb{E}[C_\theta])\cdot Z_{\theta n}(1) + \sum_{x\in\partial\mathscr{T}_{\theta n}} e^{V(x)}\big(C_\theta^{(n)}(x) - \mathbb{E}[C_\theta]\big). \end{aligned} \tag{5.2}$$

Denote

$$\mathbb{X}_k := \sum_{x\in\partial\mathscr{T}_{\theta k}} e^{V(x)}\big(C_\theta^{(k)}(x) - \mathbb{E}[C_\theta]\big),$$

so that

$$C_{\theta(n+1)} = \mathbb{E}[C_\theta]\sum_{k=0}^{n} Z_{\theta k}(1) + \sum_{k=0}^{n}\mathbb{X}_k. \tag{5.3}$$

Proof of Theorem 2.2 in cases (a) and (b). We start with decomposition (5.3). The first term on the right-hand side of (5.3) can be treated using Lemma 5.1. Indeed, since $\mu'(1) < 0$,

$$\lim_{n\to\infty} e^{-\theta n\Phi(1)} Z_{\theta n}(1) = \lim_{n\to\infty} M_{\theta n}(1) = M_\infty(1) \quad \text{a.s.} \tag{5.4}$$

To estimate the second term in (5.3), pick a constant $q \in (1,2] \cap \mathrm{int}(\mathscr{D})$ such that $\mathbb{E}|C|^q < \infty$ and $\mu'(q) < 0$, then $\mu(1) > \mu(q)$. Notice that $\mathbb{X}_k$ has the same distribution as $X_{U^{(k)},\theta k}$ with the vertices marks $U^{(k)}(x) = C_\theta^{(k)}(x) - \mathbb{E}[C_\theta]$, $x \in \partial\mathscr{T}_{\theta k}$. Inequalities (4.7), (4.8) yield

$$\mathbb{E}\big[\big|C_\theta^{(n)}(x) - \mathbb{E}[C_\theta]\big|^q\big] < c_{q,\theta},$$

for some constant $c_{q,\theta}$. Thus, (4.5) entails

$$\mathbb{E}[|\mathbb{X}_n|^q] \leq c_{q,\theta} e^{\theta n\Phi(q)}.$$

For any fixed $\varepsilon > 0$ we can write using Markov's inequality

$$\mathbb{P}\left[\left|\mathbb{X}_n\right| > \varepsilon e^{n\theta\Phi(1)}\right] \le \varepsilon^{-q} e^{-qn\theta\Phi(1)} \mathbb{E}[|\mathbb{X}_n|^q] \le c_{q,\theta}\varepsilon^{-q} \cdot e^{qn\theta(\mu(q)-\mu(1))}.$$

A standard application of the Borel-Cantelli lemma entails

$$e^{-n\theta\Phi(1)}\mathbb{X}_n \to 0 \quad \text{a.s.}, \tag{5.5}$$

as $n \to \infty$. We now argue case by case.

CASE (a): If $\Phi(1) > 0$, then $\mathbb{E}[C_\theta] = \mathbb{E}[C](e^{\theta\Phi(1)} - 1)/\Phi(1)$ by (4.7). Using the Stolz–Cesáro lemma it is easy to check that the limit relation $e^{-\beta k} a_k \to a_\infty$, $k \to \infty$, for some sequence of real numbers $(a_n)_{n\in\mathbb{N}}$, $\beta > 0$ and $a_\infty \in \mathbb{R}$, implies

$$e^{-\beta n} \sum_{k=1}^{n} a_k \to \frac{e^\beta}{e^\beta - 1} a_\infty, \qquad n \to \infty.$$

Therefore, invoking (5.3) together with (5.4) and (5.5),

$$e^{-\theta n\Phi(1)} C_{\theta(n+1)} = e^{-\theta n\Phi(1)} \sum_{k=0}^{n} \left(\mathbb{E}[C] \frac{e^{\theta\Phi(1)} - 1}{\Phi(1)} Z_{\theta k}(1) + \mathbb{X}_k \right) \to \frac{e^{\theta\Phi(1)}\mathbb{E}[C]}{\Phi(1)} M_\infty(1) \quad \text{a.s.}$$

CASE (b): If $\Phi(1) = 0$, then $\mathbb{E}[C_\theta] = \theta\mathbb{E}[C]$ by (4.7). Moreover, since $Z_{\theta n}(1) \to M_\infty(1)$ a.s. by (5.4) and $\mathbb{X}_n \to 0$ a.s. by (5.5), another appeal to the Stolz–Cesáro lemma yields

$$\frac{1}{n\theta} C_{(n+1)\theta} \to \mathbb{E}[C] M_\infty(1) \quad \text{a.s.}$$

To conclude the proof of Theorem 2.2 in cases (a) and (b) it remains to pass from the discrete parameter $n\theta$ to the continuous parameter t in both cases. Take $t > 0$ and for $\theta > 0$ let $n = \lfloor t/\theta \rfloor$. Utilizing (3.3), we conclude

$$\mathbb{E}[|C_t - C_{\theta n}|] = \mathbb{E}[|X_{C_{t-\theta n},\theta n}|].$$

Since, invoking (4.7),

$$\mathbb{E}[|C_{t-\theta n}|] \le \mathbb{E}\left[\int e^v |c| \, \mathscr{C}_\theta(\mathrm{d}v, \mathrm{d}c)\right] \le \text{const} \cdot \theta\mathbb{E}[|C|] < \infty,$$

we can bound the L_1 norm of $C_t - C_{\theta n}$ as follows

$$\mathbb{E}\left[|C_t - C_{\theta n}|\right] \le \mathbb{E}[Z_{\theta n}(1)]\mathbb{E}|C_{t-\theta n}| \le \text{const} \cdot \theta e^{\theta n\Phi(1)} \mathbb{E}[|C|].$$

This implies

$$\lim_{\theta\to 0^+} \limsup_{t\to\infty} e^{-\theta n\Phi(1)} \mathbb{E}\left[|C_t - C_{\theta n}|\right] = 0,$$

which secures our claim. □

Proofs of Theorem 2.2 in case (c) and of Theorem 2.3 in case (d). Fix $\theta > 0$. As in the previous part, we shall first show the convergence along the sequence $(\theta n)_{n\ge 0}$. We shall argue case by case.

CASE (d): If $\mathbb{E}[C] = 0$, then (5.2) entails that $(C_{\theta n})_{n\ge 0}$ is a martingale. It is uniformly integrable since for $p \in [1,2] \cap [1, s_\infty)$ and $\Phi(p) < 0$, invoking consecutively (5.2) and (4.5), we obtain

$$\mathbb{E}\left[|C_{\theta(n+1)} - C_{\theta n}|^p\right] = \mathbb{E}\left[|\mathbb{X}_n|^p\right] \le c_p \mathbb{E}[|C_\theta|^p] e^{n\theta\Phi(p)}.$$

As a consequence, the martingale $(C_{\theta n})_{n\ge 0}$ is bounded in L_p and whence uniformly integrable and convergent.

CASE (c): Recall decomposition (5.3). By an appeal to (5.4) the series $\sum_{k=0}^{\infty} Z_{k\theta}(1)$ is absolutely convergent a.s. Moreover, the process $\sum_{k=0}^{n} \mathbb{X}_k$ forms a uniformly integrable martingale, which can be verified in the same way as in case (d).

In both cases we have shown

$$C_{n\theta} \to C_\infty = \int e^v c\, \mathscr{C}_\infty(\mathrm{d}v, \mathrm{d}c) = \sum_{x \in \mathscr{T}_\infty} e^{V(x)} C(x).$$

The convergence $C_t \to C_\infty$, $t \to \infty$, can be checked in a similar fashion as in the proof of Theorem 2.2. □

Recall that case (e) of Theorem 2.3 follows from the proof of part (e) in Theorem 2.5 given in the next section.

6. PROOF OF THEOREM 2.5 AND THEOREM 2.8

The convergence results obtained in the previous section allow us to prove Theorem 2.5. We shall use an inhomogeneous version of Lemma 4.1 in [9].

Lemma 6.1. *Assume that $(r_t)_{t\geq 0}$ is an integer-valued stochastic process such that $r_t \xrightarrow{\mathbb{P}} \infty$, as $t \to \infty$. Suppose additionally that for every $t \geq 0$ there is an array $(a_{k,t})_{k=1}^{r_t}$ of a.s. positive random weights and a stochastic process $(b_t)_{t\geq 0}$ such that*

$$\sum_{k=1}^{r_t} a_{k,t}^{\gamma} \xrightarrow{\mathbb{P}} a_\infty, \quad \max_{k=1,\ldots,r_t} a_{k,t} \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad b_t \xrightarrow{\mathbb{P}} b_\infty, \quad t \to \infty,$$

for some $\gamma \in (0,2]$, a nonnegative random variable a_∞ and a random variable b_∞. Assume that ν_0 satisfies the condition (H_γ) with the same γ as above. For a sequence $(R_k)_{k\in\mathbb{N}}$ of i.i.d. random variables distributed according to ν_0 and which is independent of $(r_t)_{t\geq 0}$, $\big((a_{k,t})_{k=1}^{r_t} : t \geq 0\big)$ and $(b_t)_{t\geq 0}$, put

$$S_t = \sum_{k=1}^{r_t} a_{k,t} R_k + b_t.$$

Then, as $t \to \infty$,

$$\mathbb{E}\left[e^{i\xi S_t}\right] \to \mathbb{E}\left[\hat{g}_\gamma(\xi a_\infty^{1/\gamma}) e^{i\xi b_\infty}\right], \quad \xi \in \mathbb{R}. \tag{6.1}$$

Proof. Note that, for every fixed $\xi \in \mathbb{R}$,

$$\mathbb{E}\exp(i\xi S_t) = \mathbb{E}\exp\left(i\xi b_t + \sum_{k=1}^{r_t} \log \phi_0(a_{k,t}\xi)\right),$$

where $\log\phi_0$ is the principal branch of the logarithm of the characteristic function of R_1. In Lemma 4.1 in [9] it was proved that

$$\exp\left(\sum_{k=1}^{r_t} \log \phi_0(a_{k,t}\xi)\right) \xrightarrow{\mathbb{P}} \hat{g}_\gamma(\xi a_\infty^{1/\gamma}), \quad t \to \infty.$$

Since $b_t \xrightarrow{\mathbb{P}} b_\infty$, $t \to \infty$, we also have

$$\exp\left(\sum_{k=1}^{r_t} \log \phi_0(a_{k,t}\xi) + i\xi b_t\right) \xrightarrow{\mathbb{P}} \hat{g}_\gamma(\xi a_\infty^{1/\gamma}) e^{i\xi b_\infty}, \quad t \to \infty.$$

By the Lebesgue dominated convergence theorem, which is applicable because

$$\left|\exp\left(\sum_{k=1}^{r_t} \log \phi_0(a_{k,t}\xi) + i\xi b_t\right)\right| \leq 1,$$

we obtain (6.1). □

Proof of Theorem 2.5. The first three cases (A), (C) and (D) follows immediately from the above lemma. For the reader's convenience we present here the proof of (C). By Lemma 5.1 and using that $\Phi(\gamma)=0$, we obtain

$$\sum_{y\in\partial\mathscr{T}_t} e^{\gamma V(y)} \to M_\infty(\gamma) \quad \text{a.s.} \quad \text{and} \quad \max_{y\in\partial\mathscr{T}_t} e^{V(y)} \to 0 \quad \text{a.s.}, \qquad t\to\infty.$$

Further, by Theorem 2.2(c), $C_t \overset{\mathbb{P}}{\to} C_\infty$, $t\to\infty$. Therefore, by Lemma 6.1 applied with $r_t = Y_t = |\partial\mathscr{T}_t|$, $a_{y,t} = e^{\gamma V(y)}$, $b_t = C_t$, it holds

$$\mathbb{E}\big[e^{i\xi W_t}\big] \to \mathbb{E}\big[\hat{g}_\gamma(\xi M_\infty^{1/\gamma}(\gamma)) e^{i\xi C_\infty}\big], \quad t\to\infty.$$

Let us now consider case (E). Fix $\theta>0$ and let us show that

$$e^{-\mu(2)\theta n} W_{\theta n} \overset{d}{\to} \sqrt{M_\infty(2)(\sigma_0^2 + \mathbb{E}[C]^2/\Phi(2))}\mathrm{N}, \quad n\to\infty, \tag{6.2}$$

where N is a standard normal random variable which is independent of $M_\infty(2)$.

Fix $l\in\mathbb{N}$. Using the decomposition $W_{\theta n} = X_{R,\theta n} + C_{\theta n}$ and the branching properties (3.2) and (3.3), we can write

$$\begin{aligned} e^{-\mu(2)\theta n} W_{\theta n} &= e^{-\mu(2)\theta n}(X_{R,\theta n} + C_{\theta n}) \\ &= \left(e^{-\mu(2)\theta l} \sum_{y\in\partial\mathscr{T}_{\theta(n-l)}} e^{V(y)-\mu(2)\theta(n-l)} \left(X^{(y)}_{R,\theta l} + C^{(y)}_{\theta(n-l),\theta n} \right) \right) + e^{-\mu(2)\theta n} C_{\theta(n-l)}, \end{aligned}$$

where

$$C^{(y)}_{\theta(n-l),\theta n} := \int e^v c \mathscr{C}^{(y)}_{\theta(n-l),\theta n}(\mathrm{d}v,\mathrm{d}c) \quad \text{and} \quad X^{(y)}_{R,\theta l} := \int e^v r \mathscr{X}^{(y)}_{R,\theta l}(\mathrm{d}v,\mathrm{d}r), \quad y\in\partial\mathscr{T}_{\theta(n-l)}.$$

Note that, for every $y\in\partial\mathscr{T}_{\theta(n-l)}$, $X^{(y)}_{R,\theta l} + C^{(y)}_{\theta(n-l),\theta n}$ are independent copies of a random variable

$$W_{\theta,l} := X_{R,\theta l} + C_{\theta l},$$

and are also independent of $\partial\mathscr{T}_{\theta(n-l)}$ and $(e^{V(y)})_{y\in\partial\mathscr{T}_{\theta(n-l)}}$. Since

$$\mathbb{E}[W_{\theta,l}] = 0 \quad \text{and} \quad \mathbb{E}[W_{\theta,l}^2] < \infty,$$

we can apply Lemma 6.1 with $\gamma=2$, $t=\theta(n-l)$, $a_{y,t} := e^{V(y)-\mu(2)t}$, $r_t = |\partial\mathscr{T}_t|$, $a_\infty = M_\infty(2)$ and $b_t = 0$, to conclude that

$$\sum_{y\in\partial\mathscr{T}_{\theta(n-l)}} e^{V(y)-\mu(2)\theta(n-l)} \left(X^{(y)}_{R,\theta l} + C^{(y)}_{\theta(n-l),\theta n} \right) \overset{d}{\to} \sqrt{M_\infty(2)\mathbb{E}[W_{\theta,l}]^2}\mathrm{N}, \quad n\to\infty,$$

for every fixed $l\in\mathbb{N}$. Thus, according to Theorem 3.2 in [7], in order to deduce (6.2) it remains to prove that

$$\lim_{l\to\infty} e^{-\Phi(2)\theta l}\mathbb{E}[W_{\theta,l}^2] = \sigma_0^2 + \mathbb{E}[C^2]/\Phi(2), \tag{6.3}$$

and that, for every $\varepsilon>0$,

$$\lim_{l\to\infty} \limsup_{n\to\infty} \mathbb{P}\big\{ \big| e^{-\mu(2)\theta n} C_{\theta(n-l)} \big| > \varepsilon \big\} = 0. \tag{6.4}$$

By Chebyshev's inequality in order to prove (6.4) it suffices to check that

$$\lim_{l\to\infty} \limsup_{n\to\infty} e^{-\theta n\Phi(2)}\mathbb{E}[C^2_{\theta(n-l)}] = 0.$$

But this follows from the chain of estimates

$$e^{-\theta n\Phi(2)}\mathbb{E}[C^2_{\theta(n-l)}] \overset{(4.8)}{\leq} c_2 e^{-\theta n\Phi(2)}\mathbb{E}\left[\int e^{2v}c^2\mathscr{C}_{\theta(n-l)}(\mathrm{d}v,\mathrm{d}c)\right] \overset{(4.7)}{=} c_2\mathbb{E}[C^2]e^{-\theta n\Phi(2)}\frac{e^{\theta(n-l)\Phi(2)}-1}{\Phi(2)},$$

since $\Phi(2)>0$. Limit relation (6.3) can be checked by direct calculations:

$$\begin{aligned}
\mathbb{E}W^2_{\theta,l} = \mathbb{E}(X_{R,\theta l}+C_{\theta l})^2 &= \mathbb{E}\left(\sum_{x\in\partial\mathscr{T}_{\theta l}} e^{V(x)}R(x)+\sum_{y\in\mathscr{T}^o_{\theta l}} e^{V(y)}C(y)\right)^2\\
&= \mathbb{E}\left(\sum_{x\in\partial\mathscr{T}_{\theta l}} e^{V(x)}R(x)\right)^2+\mathbb{E}\left(\sum_{y\in\mathscr{T}^o_{\theta l}} e^{V(y)}C(y)\right)^2\\
&= \mathbb{E}[R^2]\mathbb{E}\left[\sum_{x\in\partial\mathscr{T}_{\theta l}} e^{2V(x)}\right]+\mathbb{E}[C^2]\mathbb{E}\left[\sum_{y\in\mathscr{T}^o_{\theta l}} e^{2V(y)}\right]\\
&\overset{(4.7)}{=} \sigma_0^2 e^{\theta l\Phi(2)}+\mathbb{E}[C^2]\frac{e^{\theta l\Phi(2)}-1}{\Phi(2)}.
\end{aligned}$$

This completes the proof of (6.3) as well as of (6.2). Fix $\xi\in\mathbb{R}$ and put $h_\xi(t):=\mathbb{E}\exp(i\xi e^{-\mu(2)t}W_t)$. From (6.2) we know that, for every fixed $\theta>0$, it holds

$$\lim_{n\to\infty} h_\xi(n\theta) = \mathbb{E}\exp\left\{-\xi^2\left(M_\infty(2)(\sigma_0^2+\mathbb{E}[C^2]/\Phi(2)\right)/2\right\}.$$

In order to conclude that the limit $\lim_{n\to\infty}h_\xi(n\theta)=\lim_{t\to\infty,t\in\mathbb{R}}h_\xi(t)$ it remains to apply the Croft-Kingman lemma, see Corollary 2 in [17]. The fact that $t\mapsto h_\xi(t)$ is right-continuous follows from the Lebesgue dominated convergence theorem and the fact that $(W_t)_{t\geq 0}$ has a version with a.s. right-continuous paths. □

Proof of Theorem 2.8. The result is a consequence of the Marcinkiewicz-Zygmund inequality (4.5). Let us provide the details in the first case when (A) holds, $\mathbb{E}[R]=0$ and $\mathbb{E}[|R|^{1+\delta}]<\infty$ for some $\delta>0$. Invoking Theorem 2.2, it is sufficient to ensure that $e^{-\mu(1)t}X_{R,t}$ converges to 0 in probability. The latter follows from (4.5) applied with $\gamma=1$, because for any $\varepsilon>0$, choosing $p\in(1,\min\{1+\delta,s_\infty\})$ such that $\mu(p)<\mu(1)$, we obtain

$$\mathbb{P}\left[e^{-\mu(1)t}|X_{R,t}|>\varepsilon\right] \leq (\varepsilon e^{\mu(1)t})^{-p}\mathbb{E}[|X_{R,t}|^p] \overset{(4.5)}{\leq} c_p\varepsilon^{-p}\mathbb{E}[|R|^p]\cdot e^{pt(\mu(p)-\mu(1))}\to 0,\quad t\to\infty.$$

All the remaining cases can be treated in exactly the same way. □

REFERENCES

1. G. Alsmeyer and M. Meiners, *Fixed points of inhomogeneous smoothing transforms*, J. Diff. Equation Appl. **18** (2012), no. 8, 1287–1304.
2. ———, *Fixed points of the smoothing transform: two-sided solutions*, Probab. Theory Rel. Fields **155** (2013), no. 1, 165–199.
3. K. B. Athreya and P. E. Ney, *Branching Processes*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
4. F. Bassetti and L. Ladelli, *Self-similar solutions in one-dimensional kinetic models: A probabilistic view*, Ann. Appl. Probab. **22** (2012), no. 5, 1928–1961.
5. F. Bassetti, L. Ladelli, and G. Toscani, *Kinetic Models with Randomly Perturbed Binary Collisions*, J. Stat. Physics **142** (2011), no. 4, 686–709.

6. J. Bertoin and B. Mallein, *Biggins' martingale convergence for branching Lévy processes*, Electron. Comm. Probab. **23** (2017), no. 83, 1–12.
7. P. Billingsley, *Convergence of probability measures*, second ed., John Wiley & Sons, Inc., New York, 1999.
8. V. Bobylev, A, J. A. Carrillo, and I. M. Gamba, *On some properties of kinetic and hydrodynamic equations for inelastic interactions*, J. Stat. Physics **98** (2000), no. 3, 743–773.
9. K. Bogus, D. Buraczewski, and A. Marynych, *Self-similar solutions of kinetic-type equations: The boundary case*, Stoch. Proc. Appl. **130** (2020), no. 2, 677–693.
10. D. Buraczewski, K. Kolesko, and M. Meiners, *Self-similar solutions to kinetic-type evolution equations: beyond the boundary case*, Electron. J. Probab. **26** (2021), 1 – 18.
11. J. A. Carrillo, S. Cordier, and G. Toscani, *Over-populated tails for conservative-in-the-mean inelastic Maxwell models*, Discrete and Continuous Dynamical Systems **24** (2009), no. 1, 59–81.
12. Y. S. Chow and H. Teicher, *Probability theory: Independence, interchangeability, martingales*, second ed., Springer-Verlag, New York, 1988.
13. B. Dadoun, *Asymptotics of self-similar growth-fragmentation processes*, Electron. J. Probab. **22** (2017), 1–30.
14. R. Hardy and C. H. Simon, *A spine approach to branching diffusions with applications to L_p-convergence of martingales*, Séminaire de probabilités XLII, Springer, 2009, pp. 281–330.
15. T. E. Harris, *The theory of branching processes*, vol. 6, Springer Berlin, 1963.
16. M. Kac, *Foundations of Kinetic Theory*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability **3** (1956), no. 600, 171–197.
17. J. F. C. Kingman, *Ergodic properties of continuous-time markov processes and their discrete skeletons*, Proc. London Math. Soc. **s3-13** (1963), no. 1, 593–604.
18. A. Pulvirenti and G. Toscani, *Asymptotic properties of the inelastic Kac model*, J. Stat. Physics **114** (2004), no. 5-6, 1453–1480.

DARIUSZ BURACZEWSKI AND PIOTR DYSZEWSKI: Mathematical Institute, University of Wroclaw, 50-384 Wroclaw, Poland
E-mail: dariusz.buraczewski@math.uni.wroc.pl; piotr.dyszewski@math.uni.wroc.pl

ALEXANDER MARYNYCH: Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, 01601 Kyiv, Ukraine
E-mail: marynych@unicyb.kiev.ua